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ON THE STABILITY, BOUNDEDNESS, AND SQUARE INTEGRABILITY OF SOLUTIONS OF THIRD ORDER NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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Abstract. In this paper, sufficient conditions are established for the stability, boundedness and square integrability of solutions for some nonlinear neutral delay differential equations of third order. Lyapunov's direct method is used to obtain the results.

1. Introduction

In this paper we consider the class of third order neutral delay differential equation of the form

$$(1.1)\quad (x(t)+\beta x(t-r))''' + g(x(t),x'(t))x''(t) + f(x(t-r),x'(t-r)) + h(x(t-r)) = q(t),$$

where $r>0$, $\beta>0$, and the functions $f(x(t-r),x'(t-r))$, $g(x(t),x'(t))$, $h(x(t-r))$, and $q(t)$ are continuous in their respective arguments for all $t \ge t_1 \ge t_0 + r$ and $h(0)=0$. In addition, it is also assumed that the derivatives $g_x(x,y) = \frac{\partial}{\partial x} g(x,y))$, $f_x(x,y) = \frac{\partial}{\partial x} f(x,y)$, $f_y(x,y) = \frac{\partial}{\partial y} f(x,y)$, and $h'(x)$ exist and are continuous. The existence of solutions to nonlinear delay and neutral delay equations is discussed, for example, in El'sgol'ts and Norkin [6, Chapter 1]; also see Burton [5, Chapter 3], Hale [10, Theorem 2.1], and Hale and Lunel [11].

In recent years, the asymptotic behavior of solutions of third-order differential equations has been widely applied in research in the physical sciences. For example, the third order nonlinear equation

$$x'''(t) + x(t)x''(t) = 0$$

known as the Blasius equation arises in boundary layer theory (see [13, 22]). Delay differential equations ($r \neq 0$) arise in a variety of applications such as mechanics, radio technology, transmission in lossless high-speed computer networks, control systems, life sciences, and population dynamics. Many results concerning the theory of neutral functional differential equations ($\beta \neq 0$) are given in the monographs by Hale and Lunel [10, 11]. These equations find numerous applications in natural sciences and technology but, as a rule, they are characterized by specific properties that make their study difficult

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in both concepts and techniques. Neutral equations arise for example as the Euler-Lagrange equation for functionals containing a delay.

There have been a number of papers written over the years on special cases of equation (1.1) especially where $\beta = r = 0$, i.e., ordinary differential equations (no neutral terms and no delays). In the 1960s and 1970s, the names Ezeilo, Hara, Swick, and Tejumola were often associated with these early results (see the brief discussion in [8] and the monograph of Reissig, Sansone, and Conti [18] which surveyed much of what was known at the time up to its publication in 1974). Lyapunov functions and functionals have been successfully used to obtain boundedness, stability, square integrability, and other behavioral properties of solutions of functional differential equations. For example, the asymptotic stability of solutions of equation (1.1) with $\beta = 0$ and $r = 0$ has been discussed by several authors [16, 17, 24]. Some nice results on non-neutral delay equations of the third order can be found, for example, in the recent papers of Ademola and Arawomo [1, 2]. Papers [15, 16, 17] as well as a number of those by Tunç (see, for example, [24] among others) are for third order ordinary equations without delays.

Works on third order neutral equations have been for the most part on two term nonautonomous equations (for example, see [4, 12, 14]).

The results here generalize or complement previous results in the literature. For example, Ademola and Arawomo [1] obtained some boundedness and stability results for the equation

$$x'''(t) + g\left(x(t - r(t)), x'(t - r(t))\right) + h(x(t - r(t))) = p(t, x, x', x''),$$

where $0 \le r(t) \le r_1$. Graef *et al.* [9] considered the equation

$$[x(t) + \beta x(t - \tau)]''' + a(t)\left(Q(x(t))x'(t)\right)' + b(t)\left(R(x(t))x'(t)\right) + c(t)f(x(t - \tau)) = h(t)$$

which can be seen to be a special case of (1.1). On the other hand, Tunç [25] considered the equation

$$x'''(t) + \varphi(x(t), x'(t), x''(t)) + \psi(x(t), x'(t)) + \sum_{i=1}^{N} f_i(x(t), x'(t - r)) + h(x(t)) = p(t)$$

and obtained boundedness and global asymptotic stability of solutions.

For the sake of convenience, we introduce the notation

$$\begin{cases} X(t) = x(t) + \beta x(t - r), \\ Y(t) = x'(t) + \beta x'(t - r), \\ Z(t) = x''(t) + \beta x''(t - r). \end{cases}$$

By a *solution* of (1.1) we mean a continuous function $x : [t_x, \infty) \to \mathbb{R}$ such that $X(t) \in C^3\big([t_x, \infty), \mathbb{R}\big)$ and which satisfies equation (1.1) on $[t_x, \infty)$.

Without further mention, we only consider those solutions $x(t)$ of (1.1) that are continuable to the right and nontrivial, i.e., $x(t)$ is defined on some ray $[t_x, \infty)$ and $\sup_{t \geq T} |x(t)| > 0$ for every $T \geq t_x$. Moreover, we tacitly assume that equation (1.1) possesses such solutions but we are not assuming that all solutions are in fact continuable to the right.

This paper is organized as follows. In Section 2, we give some stability results for the case $q(t) \equiv 0$. In Section 3, the boundedness of solutions of (1.1) is discussed. Finally, in Section 4, sufficient conditions for the square integrability of all solutions and their derivatives are given.

2. Stability

We need to define the Banach space C to be the space of continuous functions $\phi : [-r, 0] \to \mathbb{R}$ with the norm $||\phi|| = \sup_{-r \leq s \leq 0} |\phi(s)|$. Then x_t is the restriction of x to the interval $[t - r, t]$ translated to $[-r, 0]$, i.e., $x_t(s) = x(t + s)$ for $-r \leq s \leq 0$.

In what follows we will assume that there are positive constants a, b, B, μ, δ, η, δ_0, δ_1, ε, L, and K such that the following conditions are satisfied:

(i) $a + \mu \leq g(x, y) \leq a + \eta$ and $y g_x(x, y) \leq 0$;

(ii) $b + \delta \leq \frac{f(x,y)}{y} \leq B$, $-K \leq f_x(x, y) \leq 0$, and $|f_y(x, y)| \leq L$;

(iii) $h'(x) \leq \delta_0$ for all x and $\frac{h(x)}{x} \geq \delta_1$ for $x \neq 0$;

(iv) $a > \frac{\delta_0}{2}$, $\delta_0 + \omega + \beta\delta_0 - a(\delta + b) < 0$ and $-\mu + \frac{\beta\eta}{2} + \omega < -\varepsilon$, where $\omega = \frac{\beta B}{2} + (\beta + 1)^2$.

Let

$$\mu_1 = \frac{1}{2}(1 + \beta + a)(\delta_0 + K) \quad \text{and} \quad \mu_2 = \frac{L}{2}\big(1 + \beta + a\big). \tag{2.1}$$

For the case $q(t) \equiv 0$, that is, for the equation

$$\begin{aligned}(x(t) + \beta x(t - r))''' &+ g(x(t), x'(t))x''(t) \\ &+ f(x(t - r), x'(t - r)) + h(x(t - r)) = 0,\end{aligned} \tag{2.2}$$

our main stability result in this paper is contained in the following theorem.

Theorem 2.1. *In addition to conditions (i)–(iv), assume that*

$$r < \min\left\{\frac{a(\delta + b) - \delta_0 - \beta\delta_0 - \omega}{\frac{a}{2}\sigma + \mu_1}, \frac{(\mu - \frac{\beta\eta}{2} - \omega - \epsilon)}{\frac{1}{2}\sigma + \mu_2}, \frac{2\varepsilon}{\beta\sigma}\right\},$$

where $\sigma = \delta_0 + K + L$. Then the zero solution of (2.2) is uniformly asymptotically stable.

Proof. We will replace (2.2) with the equivalent system

$$\begin{cases} x'(t) = y(t), \\ y'(t) = z(t), \\ (z(t) + \beta z(t-r))' = -g(x(t), y(t))z(t) - f(x(t), y(t)) \\ \qquad\qquad\qquad -h(x(t)) + \Delta_1 + \Delta_2 + \Delta_3, \end{cases} \tag{2.3}$$

where

$$\begin{aligned} \Delta_1 &= \int_{t-r}^{t} f_x(x(s), y(s))y(s)ds, \\ \Delta_2 &= \int_{t-r}^{t} f_y(x(s), y(s))z(s)ds, \\ \Delta_3 &= \int_{t-r}^{t} h'(x(s))y(s)ds. \end{aligned} \tag{2.4}$$

It easy to see from (2.3) that $Y(t) = y(t) + \beta y(t-r)$ and $Z(t) = z(t) + \beta z(t-r)$.

We define the continuously differentiable Lyapunov function $V = V(t, x_t, y_t, z_t) : \mathbb{R} \times C \times C \times C \to \mathbb{R}$ by

$$\begin{aligned} V = {} & a\int_0^x h(u)du + h(x)Y + Y^2 + \frac{1}{2}(ay + Z)^2 + a\int_0^y [g(x,\xi) - a]\,\xi d\xi \\ & + \int_0^y f(x,\xi)d\xi + \mu_1 \int_{-r}^{0}\int_{t+s}^{t} y^2(\xi)d\xi ds + \mu_2 \int_{-r}^{0}\int_{t+s}^{t} z^2(\xi)d\xi ds \\ & + \lambda_1 \int_{t-r}^{t} y^2(s)ds + \lambda_2 \int_{t-r}^{t} z^2(s)ds, \end{aligned} \tag{2.5}$$

where λ_1 and λ_2 are positive constants to be specified below.

Since $h(0) = 0$, we see that

$$2\int_0^x h'(u)h(u)du = h^2(x).$$

From condition (iii), we have

$$\begin{aligned} a\int_0^x h(u)du + h(x)Y + Y^2 &= a\int_0^x h(u)du + \left(Y + \frac{1}{2}h(x)\right)^2 - \frac{1}{4}h^2(x) \\ &\geq a\int_0^x h(u)du - \frac{1}{2}\int_0^x h'(u)h(u)du \\ &\geq \int_0^x \left(a - \frac{\delta_0}{2}\right) h(u)du \\ &\geq \left(a - \frac{\delta_0}{2}\right)\frac{\delta_1}{2}x^2. \end{aligned}$$

From conditions (i)–(ii), we have

$$a\int_0^y [g(x,\xi)-a]\,\xi d\xi \geq \frac{\mu a}{2}y^2 \quad \text{and} \quad \int_0^y f(x,\xi)d\xi \geq \frac{b+\delta}{2}y^2.$$

We see that $V = V(t, x_t, y_t, z_t) \geq 0$ and $V = V(t, x_t, y_t, z_t) = 0$ if and only if $x = y = z = 0$. That is, V is positive definite, and so there exists a positive constant K_0, small enough such that

$$V \geq K_0(x^2(t) + y^2(t) + Z^2(t)). \tag{2.6}$$

Differentiating V along the solutions of system (2.3), we obtain

$$\begin{aligned} V'_{(2.3)} = V_1 + V_2 + V_3 - \mu_1\int_{t-r}^t y^2(\theta)d\theta - \mu_2\int_{t-r}^t z^2(\theta)d\theta - \lambda_1 y^2(t-r) \\ - \lambda_2 z^2(t-r) + ay(t)\int_0^y g_x(x,\xi)\xi d\xi + y(t)\int_0^y f_x(x,\xi)d\xi, \end{aligned}$$

where

$$\begin{aligned} V_1 &= \left(h'(x) + \lambda_1\right)y^2(t) - ay(t)f(x(t),y(t)) + \mu_1 r y^2(t) \\ &\quad + [a - g(x(t),y(t)) + \lambda_2 + \mu_2 r]\,z^2(t), \\ V_2 &= \beta\,(a - g(x(t),y(t)))\,z(t)z(t-r) - \beta z(t-r)f(x(t),y(t)) \\ &\quad + \beta y(t)y(t-r)h'(x) + 2y(t)z(t) + 2\beta z(t)y(t-r) \\ &\quad + 2\beta y(t)z(t-r) + 2\beta^2 y(t-r)z(t-r), \\ V_3 &= \Big(z(t) + \beta z(t-r) + ay(t)\Big)\Big(\Delta_1 + \Delta_2 + \Delta_3\Big). \end{aligned}$$

From conditions (i)–(ii), we have

$$ay(t)\int_0^y g_x(x,\eta)\eta d\eta + y(t)\int_0^y f_x(x,\eta)d\eta \leq 0.$$

Also, conditions (i)–(iii) imply that

$$V_1 \leq [\delta_0 - a(\delta + b) + \lambda_1 + \mu_1 r]\,y^2(t) + [-\mu + \lambda_2 + \mu_2 r]z^2(t).$$

Using the inequality $|ab| \leq \frac{1}{2}[a^2 + b^2]$ together with conditions (i)–(iii), it is easy to see that

$$\begin{aligned} V_2 \leq \left(\frac{\beta B}{2} + (1+\beta) + \frac{\beta\delta_0}{2}\right)y^2(t) + \left(\beta^2 + \beta + \frac{\beta\delta_0}{2}\right)y^2(t-r) \\ + \left(\frac{\beta\eta}{2} + (1+\beta)\right)z^2(t) + \left(\frac{\beta\eta}{2} + \beta^2 + \beta + \frac{\beta B}{2}\right)z^2(t-r). \end{aligned}$$

From (2.4) and conditions (ii) and (iii), we have

$$z(t)\Big(\Delta_1 + \Delta_2 + \Delta_3\Big) \leq |z(t)|\,\Big(\,|\Delta_1| + |\Delta_2| + |\Delta_3|\,\Big)$$

$$\leq |z(t)| \Big((K+\delta_0) \int_{t-r}^{t} |y(s)|\, ds + L \int_{t-r}^{t} |z(s)|\, ds \Big)$$
$$= (K+\delta_0) \int_{t-r}^{t} |z(t)|\, |y(s)|\, ds + L \int_{t-r}^{t} |z(t)|\, |z(s)|\, ds.$$

Using again the inequality $|ab| \leq \frac{1}{2}[a^2+b^2]$, we obtain

$$\begin{aligned} z(t)\Big(\Delta_1+\Delta_2+\Delta_3\Big) &\leq \frac{(K+\delta_0)}{2} \int_{t-r}^{t} \big(z^2(t)+y^2(s)\big)\, ds \\ &\quad + \frac{L}{2} \int_{t-r}^{t} \big(z^2(t)+z^2(s)\big)\, ds \\ &\leq \frac{\sigma r}{2} z^2 + \frac{(K+\delta_0)}{2} \int_{t-r}^{t} y^2(s) ds + \frac{L}{2} \int_{t-r}^{t} z^2(s) ds. \end{aligned}$$

In the same way,

$$\begin{aligned} ay(t)\Big(\Delta_1+\Delta_2+\Delta_3\Big) &\leq a\frac{\sigma r}{2} y^2 + \frac{a}{2}\Big(\delta_0+K\Big) \int_{t-r}^{t} y^2(s) ds \\ &\quad + \frac{aL}{2} \int_{t-r}^{t} z^2(s) ds, \\ \beta z(t-r)\Big(\Delta_1+\Delta_2+\Delta_3\Big) &\leq \frac{\beta r\sigma}{2} z^2(t-r) + \Big(\frac{\delta_0}{2}+\frac{K}{2}\Big)\beta \int_{t-r}^{t} y^2(s) ds \\ &\quad + \frac{\beta L}{2} \int_{t-r}^{t} z^2(s) ds. \end{aligned}$$

Hence, V_3 becomes

$$\begin{aligned} V_3 &\leq \frac{\sigma r}{2} z^2 + \frac{a\sigma r}{2} y^2 + \frac{\beta r\sigma}{2} z^2(t-r) + \Big(\frac{\delta_0+K}{2}(1+a+\beta)\Big) \int_{t-r}^{t} y^2(s) ds \\ &\quad + \frac{L}{2}(1+a+\beta) \int_{t-r}^{t} z^2(s) ds. \end{aligned}$$

Using the above estimates, we obtain

$$\begin{aligned} V'_{(2.3)} &\leq \left(\delta_0+\lambda_1+\frac{\beta B}{2}+\frac{\beta\delta_0}{2}+(1+\beta)-a\,(b+\delta)+r\left(\frac{a\sigma}{2}+\mu_1\right)\right) y^2(t) \\ &\quad + \left(-\mu+\lambda_2+1+\beta+\frac{\beta\eta}{2}+\left(\frac{\sigma}{2}\ +\mu_2\right) r\right) z^2(t) \\ &\quad + \left(-\lambda_2+\frac{\beta B}{2}+\beta^2+\beta+\frac{\beta\eta}{2}+\frac{\beta\sigma}{2} r\right) z^2(t-r) \\ &\quad + \left(\frac{\beta\delta_0}{2}+\beta^2+\beta-\lambda_1\right) y^2(t-r) \end{aligned}$$

$$+\left(\frac{1}{2}\left(1+\beta+a\right)\left(\delta_0+K\right)-\mu_1\right)\int_{t-r}^{t}y^2(s)ds$$
$$+\left(\frac{L}{2}\left(1+\beta+a\right)-\mu_2\right)\int_{t-r}^{t}z^2(s)ds.$$

If we now choose

$$\lambda_1=\frac{\beta\delta_0}{2}+\beta^2+\beta \quad \text{and} \quad \lambda_2=\frac{\beta B}{2}+\beta+\beta^2+\frac{\beta\eta}{2}+\varepsilon,$$

then from (2.1) we obtain

$$V'_{(2.3)}\le\left(\delta_0+\lambda_1+\frac{\beta B}{2}+\frac{\beta\delta_0}{2}+(1+\beta)-a\left(b+\delta\right)+r\left(\frac{a\sigma}{2}+\mu_1\right)\right)y^2(t)$$
$$+\left[-\mu+\omega+\beta\eta+\epsilon+\left(\frac{\sigma}{2}+\mu_2\right)r\right]z^2(t)+\left(\frac{\beta\sigma r}{2}-\epsilon\right)z^2(t-r)$$

Thus,

$$V'_{(2.3)}\le-\kappa_1y^2(t)-\kappa_2z^2(t), \tag{2.7}$$

where

$$\kappa_1=a\left(\delta+b\right)-\delta_0-\beta\delta_0-\omega-r\left(\frac{a\sigma}{2}+\mu_1\right)>0$$

and

$$\kappa_2=\mu-\beta\eta-\omega-\left(\frac{\sigma}{2}+\mu_2\right)r-\epsilon>0,$$

provided that

$$r<\min\left\{\frac{a\left(\delta+b\right)-\delta_0-\beta\delta_0-\omega}{\frac{a}{2}\sigma+\mu_1},\ \frac{\mu-\beta\eta-\omega-\epsilon}{\frac{1}{2}\sigma+\mu_2},\ \frac{2\varepsilon}{\beta\sigma}\right\}.$$

From (2.7) we see that $V'_{(2.3)}\le 0$ and $V'_{(2.3)}=0$ on the set $M=\{(x,0,0)\}$. The largest invariant set contained in M is the origin, so by LaSalle's invariance principle, the zero solution of (2.2) is uniformly asymptotically stable (see [5, 10]). □

Remark. The choice of the form of the Lyapunov functional is not totally unlike those used by other authors (see, for example, [1, 2, 20, 24]). What makes the calculations and estimates more complicated here is the presence of the neutral term, i.e., the fact we have $\beta\ne 0$.

3. Boundedness

In this section, we give sufficient conditions for all solutions of equation (1.1) to be bounded. As a consequence of our result, we will be able to obtain the global uniform asymptotic stability of the zero solution of (2.2). We write equation (1.1) as the equivalent system

$$
\begin{cases}
x'(t) = y(t), \\
y'(t) = z(t), \\
(z(t) + \beta z(t-r))' = -g(x(t), y(t))z(t) - f(x(t), y(t)) - h(x(t)) \\
\qquad\qquad + \Delta_1 + \Delta_2 + \Delta_3 + q(t).
\end{cases} \tag{3.1}
$$

Theorem 3.1. *In addition to the conditions of Theorem 2.1, assume that*

(v) $\displaystyle\int_{t_0}^{\infty} |q(s)|ds < \infty.$

Then there exists a positive constant D such that any solution of (2.3) satisfies

$$
|x(t)| \leq D, \quad |y(t)| \leq D, \quad \text{and} \quad |Z(t)| \leq D. \tag{3.2}
$$

Proof. Differentiating (2.5) along the solutions of system (3.1), we obtain

$$
V'_{(3.1)} \leq -\kappa_1 y^2 - \kappa_2 z^2 + Zq(t) + ayq(t),
$$

for all $t \geq t_1$. Applying the inequality $|u| \leq \ u^2 + 1$ leads to

$$
V'_{(3.1)} \leq -\kappa_1 y^2 - \kappa_2 z^2 + \kappa_3 |q(t)| \big(y^2 + Z^2 + 2\big), \tag{3.3}
$$

where $\kappa_3 = \max\{1, a\}$.

From (2.6) we see that

$$
V'_{(3.1)} \leq -\kappa_1 y^2 - \kappa_2 z^2 + \frac{\kappa_3}{K_0} |q(t)| V + 2\kappa_3 |q(t)|. \tag{3.4}
$$

An integration from t_1 to t gives

$$
V(t) \leq V(t_1) + 2\kappa_3 \int_{t_1}^{t} |q(s)|ds + \frac{\kappa_3}{K_0} \int_{t_1}^{t} V(s)|q(s)|ds.
$$

Gronwall's inequality and condition (v) imply that V is bounded, and the conclusion follows. □

Corollary 3.1. *If conditions (i)–(iv) hold, then the zero solution of equation* (2.2) *is globally uniformly asymptotically stable.*

Proof. By Theorem 2.1, the zero solution of equation (2.2) is uniformly asymptotically stable, and by Theorem 3.1 all solutions are bounded. So by LaSalle's invariance principle the uniform asymptotic stability is global. □

4. Square Integrability

We now turn our attention to the question of the square integrability of solutions and their derivatives. Our main result is contained in the following theorem,

Theorem 4.1. *Assume that the conditions of Theorem 3.1 hold; then for any solution x of* (1.1),

$$\int_{t_0}^{\infty}\Big(x^2(s)+x'^2(s)+x''^2(s)\Big)\,ds<\infty.$$

Proof. Define $H(t)$ by

$$H(t)=V(t)+\rho\int_{t_1}^{t}(z^2(s)+y^2(s))ds \quad \text{for all} \quad t\geq t_1\geq\ t_0+r,$$

where $\rho>0$ is a constant to be specified below. Differentiating $H(t)$ and using (3.4), we obtain

$$H'(t)\leq(\rho-\kappa)(z^2(t)+y^2(t))+\big(\frac{\kappa_3}{K_0}V+2\kappa_3\big)|q(t)|,$$

where $\kappa=\min\{\kappa_1,\kappa_2\}$. If we choose $\rho<\kappa$, then the boundedness of V implies

$$H'(t)\leq\kappa_4|q(t)|, \tag{4.1}$$

for some constant $\kappa_4>0$. Integrating (4.1) from t_1 to t and using condition (v), we see that there is a positive constant C such that

$$\int_{t_1}^{\infty}y^2(s)ds\leq C \text{ and } \int_{t_1}^{\infty}z^2(s)ds\leq C.$$

Therefore,

$$\int_{t_1}^{\infty}x'^2(s)ds<\infty \text{ and } \int_{t_1}^{\infty}x''^2(s)ds<\infty. \tag{4.2}$$

To show that $\int_{t_1}^{\infty}x^2(s)ds<\infty$, multiply (1.1) by $x(t-r)$ to obtain

$$X'''(t)x(t-r)+g\big(x(t),x'(t)\big)x''(t)x(t-r)+f\big(x(t-r),x'(t-r)\big)x(t-r)$$
$$+h(x(t-r))x(t-r)=q(t)x(t-r).$$

Integrating from t_1 to t, we have

$$\int_{t_1}^{t}h(x(s-r))x(s-r)ds=L_1(t)+L_2(t)+L_3(t)+L_4(t), \tag{4.3}$$

where

$$L_1(t)=-\int_{t_1}^{t}(x(s)+\beta x(s-r))'''x(s-r)ds,$$

$$L_2(t) = -\int_{t_1}^{t} g\big(x(s), x'(s)\big)x''(s)x(s-r)ds,$$
$$L_3(t) = -\int_{t_1}^{t} f\big(x(s-r), x'(s-r)\big)x(s-r)ds,$$
$$L_4(t) = \int_{t_1}^{t} q(t)x(s-r)ds.$$

An integration of L_1 by parts gives

$$L_1(t) = -\Big[x(t-r)Z(t) - x(t_1-r)Z(t_1))\Big] + \int_{t_1}^{t} (x(s) + \beta x(s-r))''x'(t-s)ds.$$

Hence, from (4.2) and (3.2),

$$\begin{aligned} L_1(t) &\le 2M^2 + \frac{1}{2}\int_{t_1}^{t} \Big((x(s) + \beta x(s-r))''^2 + x'^2(s-r)\Big)ds \\ &= 2M^2 + \frac{1}{2}\int_{t_1}^{t} \left[(1+\beta)x''^2(s) + (\beta+\beta^2)x''^2(s-r) + x'^2(s-r)\right] ds \\ &\le M_1 < \infty \end{aligned}$$

for some $M_1 > 0$.

Applying the Cauchy-Schwarz inequality to $L_2(t)$ and then using (i) and (4.2) gives

$$\begin{aligned} L_2(t) &\le \int_{t_1}^{t} \big|g(x(s), x'(s))x''(s)x(s-r)\big| ds \\ &\le \left\{\int_{t_1}^{t} \left[\big|g(x(s), x'(s))x''(s)\big|\right]^2 ds\right\}^{\frac{1}{2}} \left\{\int_{t_1}^{t} x^2(s-r)ds\right\}^{\frac{1}{2}} \\ &\le M_2 \left\{\int_{t_1}^{t} x^2(s-r)ds\right\}^{\frac{1}{2}} \end{aligned}$$

for some constant $M_2 > 0$.

Similarly, using condition (ii) and (4.2),

$$\begin{aligned} L_3(t) &\le \int_{t_1}^{t} \big|f(x(s-r), x'(s-r))x(s-r)\big| ds \\ &\le \left\{B^2\int_{t_1}^{t} (x'(s-r))^2 ds\right\}^{\frac{1}{2}} \left\{\int_{t_1}^{t} x^2(s-r)ds\right\}^{\frac{1}{2}} \\ &\le M_3 \left\{\int_{t_1}^{t} x^2(s-r)ds\right\}^{\frac{1}{2}} \end{aligned}$$

for some $M_3 > 0$.

Finally, (v) and (3.2) imply

$$L_4(t) \le \int_{t_1}^{t} |q(s)x(s-r)|\, ds \le D \int_{t_1}^{t} |q(s)|\, ds \le M_4 < \infty.$$

On the other hand, from condition (iii), we have

$$\int_{t_1}^{t} x(s-r)h(x(s-r))ds \ge \delta_1 \int_{t_1}^{t} x^2(s-r)ds.$$

Thus, from (4.3), we have

$$\delta_1 \left\{ \int_{t_1}^{t} x^2(s-r)ds \right\} \le M_1 + M_2 \left\{ \int_{t_1}^{t} x^2(s-r)ds \right\}^{\frac{1}{2}} + M_3 \left\{ \int_{t_1}^{t} x^2(s-r)ds \right\}^{\frac{1}{2}} + M_4. \tag{4.4}$$

If

$$\int_{t_1}^{t} x^2(s-r)ds \to \infty \quad \text{as} \quad t \to \infty,$$

then dividing both sides of (4.4) by

$$\left\{ \int_{t_1}^{t} x^2(s-r)ds \right\}^{\frac{1}{2}},$$

we immediately obtain a contradiction. This completes the proof of the theorem. □

We conclude this paper with an example to illustrate our results.

Example 1. Consider third order nonlinear neutral delay differential equation

$$(x + \beta x(t-r))''' + \left(16 - \frac{2}{\pi}\arctan(xy)\right) x'' + 10x'(t-r) - \frac{2}{\pi} x'^{x(t-r)} x'(t-r)) + \left(2x(t-r) + \frac{x(t-r)}{1+ \mid x(t-r) \mid}\right) = \frac{1}{1+t^2}. \tag{4.5}$$

Here we have:

$$a + \mu = 15 \le g(x,y) = 16 - \frac{2}{\pi}\arctan(xy) \le 17 = a + \eta$$

and

$$y g_x(x,y) = -\frac{2}{\pi}\frac{y^2}{1+x^2y^2} \le 0;$$

$$b + \delta = 9 < \frac{f(x,y)}{y} = 10 - \frac{2}{\pi}\arctan(e^x y) < 11 = B;$$

$$-K = -1 \leq f_x(x, y) = -\frac{2}{\pi} \frac{y^2}{1 + (e^x y)^2} \leq 0 \quad \text{and}$$

$$|f_y(x, y)| = \left| 10 - \frac{2}{\pi} \arctan(e^x y) + \frac{2}{\pi} \frac{e^x y}{1 + (e^x y)^2} \right| \leq 12 = L;$$

$$\frac{h(x)}{x} \geq 2 = \delta_1 \quad \text{and} \quad |h'(x)| \leq 3 = \delta_0.$$

Taking $a = 5 > \frac{\delta_0}{2} = \frac{3}{2}$ and $\beta = 0.2$, we see that $\omega = 2.54$, $\mu = 10$, $\eta = 12$, $\sigma = 16$, $\mu_1 = 12.4$, and $\mu_2 = 37.2$. Also,

$$\delta_0 + \beta\delta_0 + \omega - a\,(\delta + b) = -38.86,$$

and

$$-\mu + \frac{\beta\eta}{2} + \omega = -6.26 < -1 = -\varepsilon.$$

Finally, we see the conditions on $q(t)$ are clearly satisfied. Therefore, if $r < \min\{.639, .116, .625\}$, then all solutions of equation (4.5) are bounded, every solution along with its first and second derivatives belong to $L^2[t_0, \infty)$, and the zero solution of the unforced equation is globally uniformly asymptotically stable.

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