Exact Treatment for the Instability of a Class of Inhomogeneous Equilibrium Plasmas

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ı infinite equilibrium plasma homogeneous in the y and z ırections and contained in the x direction by a magnetic field $ = \hat{i}_z B_z(x)$ is studied. We consider the following class of quilibrium distribution functions:

$$1)\quad f_0 = g(v_\perp^2, v_\parallel^2)\exp\left\{-\alpha\left|v_y + \int_0^x \frac{e}{m} B_z(x')dx'\right|\right\},\ \alpha>0$$

$B_z = B_0 - b(x)$ with $\int_{-\infty}^{+\infty} b(x)\,dx < \infty)$ which leads to density profiles verywhere infinitely differentiable. The stability of such a onfiguration against electrostatic perturbations is studied.

et us assume a quantity $F(x,t)$ to have a Laplace transform, e. there exists a finite γ such that $F(x,t)\,e^{\gamma t}$ is integrable. necessary and sufficient condition for $F(x,t)$ to grow ndefinitely in time is that the transform $\hat{F}(x,\omega) = \int^{\infty} dt\, e^{i\omega t} F(x,t)$ have a singularity (not necessarily a pole) in the alf-plane $\mathrm{Im}\,\omega \geqslant 0$. The same condition is true for the ourier transform: $\hat{F}(\underline{K},\omega) = \int d^3r\, e^{i\underline{K}\cdot\underline{r}}\, \hat{F}(x,\omega)$ of $\hat{F}(x,\omega)$ ith K real.

e therefore deduce an equation for the Laplace-Fourier trans- orm of the electric potential and look for the singularities f this potential without explicitly solving the equation. necessary and sufficient condition is obtained for the xistence of singularities in the form:

$$(2)\qquad \mathbb{D}(\omega, \underline{K}) = 0$$

where $\mathbb{D}$ is a functional of the distribution f_0 and is ndependent of the initial conditions.

f the linearized Vlasov equation is solved with the method of haracteristics and the result substituted in the Poisson equation, the following integral equation is obtained for the lectric potential :

$$(3)\qquad K^2\phi(\omega,\underline{K}) = \phi^0(\omega,\underline{K}) + \int_{-\infty}^{+\infty} dK'_x\, \mathbb{K}(\omega,\underline{K}|\underline{K}')\,\phi(\omega,\underline{K}')$$

where $\underline{K} = (K_x, K_y, K_z)$, $\underline{K}' = (K'_x, K_y, K_z)$

$$\phi^0(\omega,\underline{K}) = \sum_{j=e,i} 4\pi e_j \int d^3r \int_0^\infty dt\, e^{i\omega t}\, e^{i\underline{K}\cdot\underline{r}} \int d^3v\, f_j(\underline{r},\underline{v},t'=0)$$

f_j = initial perturbation for species j

$$\mathbb{K}(\omega,\underline{K}|\underline{K}') = -\frac{i}{2\pi}\sum_{j=e,i}\frac{4\pi e_j^2}{m_j}\int_{-\infty}^{+\infty}dx\int d^3v\int_0^\infty du\, e^{i\omega u}\, e^{i(K_x-K'_x)x}\, e^{i\underline{K}'\cdot(\underline{r}-\underline{R})}\, \underline{K}'\cdot\nabla_v f_{0_j}(X,\underline{V})$$

$\underline{R}$, $\underline{V}$ describe the unperturbed orbits (position and velocity espectively) of the particles between initial time $t' = 0$ and ime $t' = t$. Let $\mathbb{K}$ be written as:

$$\mathbb{K} = \mathbb{K}^- - \mathbb{K}^+$$

where $\mathbb{K}^{\mp} = +\frac{i}{2\pi}\sum_{j=e,i}\frac{4\pi e_j^2}{m_j}\int_{x_0}^{\mp\infty}dx\int d^3v\int_0^\infty du\cdots$ with $x_0 = x_0(v_y, B_z)$ and

uch that $v_y + \int_0^{x_0}\frac{e}{m}B_z(x')\,dx' = 0$

f B_z were constant $(B_z = B_0)$ and only f_0 depended on x (which s an approximation valid when $\beta \ll 1$), $(\underline{r} - \underline{R})$ would be ndependent of x. Then

$$\mathbb{K}^{\mp} = \frac{c^{\mp}(\underline{K}',\underline{K},\omega)}{K'_x - (K_x \mp ia)}\ ;\quad a = \alpha\left|\frac{e}{m}\right|$$

where the $c^{\mp}$ are entire functions if, as we assume in the ollowing $\lim_{v_\perp\to\infty} g(v_\perp^2, v_\parallel^2)\exp\{\beta v_\perp\} = 0$, $\beta = \mathrm{const.} > 0$

when B_z depends on x, then: $\mathbb{K}^{\mp} = \dfrac{c^{\mp}}{K'_x - (K_x \mp ia)} + G^{\mp}$

$$G^{\mp} = -\frac{i}{2\pi}\sum_{j=e,i}\frac{4\pi e_j^2}{m_j}\int d^3v\int_{x_0}^{\mp\infty}dx\int_0^{+\infty}du\, e^{i\omega u}\, e^{-i(K'_x-K_x)x}\left[e^{i\underline{K}'\cdot(\underline{r}-\underline{R})}\,\underline{K}'\cdot\nabla_v f_{0_j}(X,\underline{V}) - \lim_{|x|\to\infty} e^{i\underline{K}'\cdot(\underline{r}-\underline{R})}\,\underline{K}'\cdot\nabla_v f_{0_j}\right]$$

Further analysis of $G^{\mp}$ requires knowledge of the form of $b(x)$

We shall consider two classes of $G^{\mp}$ corresponding to the following classes of $b(x)$ (always with the assumption $\int_{-\infty}^{+\infty} b(x)\,dx < \infty$)

1) $b(x)$ goes to zero more rapidly than $e^{-c|x|}$ when $|x| \to \infty$ for all positive c . Here $G^{\mp}$ is an entire function.

2) $b(x)$ goes to zero more slowly than $e^{-\varepsilon|x|}$, when $|x| \to \infty$ for an arbitrary small $\varepsilon > 0$. Then $G^{\mp}$ has a branch point at $K'_x = K_x \mp ia$ with: $\lim_{K'_x = K_x \mp ia}(K'_x - K_x \pm ia)\,G^{\mp} = 0$

The problem would be more complicated with other classes of $b(x)$, but the mean line of the method would remain the same

It can be shown that, when has a singularity at $K_x = K_0(\omega)$ $(\mathrm{Im}\,K_0 = 0)$, it is also singular at $K_x = K_0 \pm ina$, $(n=1,2,.)$

The necessary and sufficient condition for the existence of a singularity at $K_x = K_0(\omega)$ is:

$$\mathbb{D}\left(\frac{K_0}{ia}\right) = 0 \text{ where:}$$

$$\mathbb{D}(\sigma) \equiv \int_{iv_{I_1}-\infty}^{iv_{I_1}+\infty} dv\, \frac{H_+(v)\,e^{-i\sigma v}}{e^{-iv}-1} + \int_{iv_{I_2}-\infty}^{iv_{I_2}+\infty} dv\, \frac{H_-(v)\,e^{-i\sigma v}}{e^{-iv}-1}$$

$$H_+(v) = \frac{1}{ia}\int_0^{+i\infty} h(K_0)\,e^{K_0 v/a}\,dK_0,\quad H_-(v) = \frac{1}{ia}\int_{-i\infty}^{0} h(K_0)\,e^{K_0 v/a}\,dK_0$$

$$h(K_0) = \log\left[\frac{c^-\,e^{K_x^2\rho^2}}{K^2}\right]_{\substack{K'_x = K_0\\ K_x = K_0 + ia}} - \log\left[\frac{c^+\,e^{K_x^2\rho^2}}{K^2}\right]_{\substack{K'_x = K_0\\ K_x = K_0 - ia}}$$

$$\rho^2 = \overline{v_\perp^2}/\Omega^2\ ,\quad \Omega = \left|\frac{e}{m}B_0\right|$$

In the quasi-neutrality approximation, the left hand side in (3) drops out. Consequently, the whole set of singularities $K_0 \pm ina$ appears no more with the singularity $K_0(\omega)$ and the dispersion relation degenerates into the form:

$$c^{\mp}(K'_x = K_x \mp ia\ ,\ K;\omega) = 0$$

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