GELFAND-TSETLIN THEORY FOR RATIONAL GALOIS ALGEBRAS

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Abstract. In the present paper we study Gelfand-Tsetlin modules defined in terms of BGG differential operators. The structure of these modules is described with the aid of the Postnikov-Stanley polynomials introduced in [PS09]. These polynomials are used to identify the action of the Gelfand-Tsetlin subalgebra on the BGG operators. We also provide explicit bases of the corresponding Gelfand-Tsetlin modules and prove a simplicity criterion for these modules. The results hold for modules defined over standard Galois orders of type A - a large class of rings that include the universal enveloping algebra of $\mathfrak{gl}(n)$ and the finite W-algebras of type A.

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1. Introduction

The category of Gelfand-Tsetlin modules of the general linear Lie algebra $\mathfrak{gl}(n)$ is an important category of modules that plays a prominent role in many areas of mathematics and theoretical physics. By definition, a Gelfand-Tsetlin module of $\mathfrak{gl}(n)$ is one that has a generalized eigenspace decomposition over a certain maximal commutative subalgebra (*Gelfand-Tsetlin subalgebra*) Γ of the universal enveloping algebra of $\mathfrak{gl}(n)$. This algebraic definition has a nice combinatorial flavor. The concept of a Gelfand-Tsetlin module generalizes the classical realization of the simple finite-dimensional representations of $\mathfrak{gl}(n)$ via the so-called Gelfand-Tsetlin tableaux introduced in [GT50]. The explicit nature of the Gelfand-Tsetlin formulas inevitably raises the question of what infinite-dimensional modules admit tableaux bases - a question that led to the systematic study of the theory of Gelfand-Tsetlin modules. This theory has attracted considerable attention in the last 30 years of the 20th century and have been studied in [DOF91, DFO94, Maz98, Maz01, Mol99, Zhe74], among others. Gelfand-Tsetlin bases and modules are also related to Gelfand-Tsetlin integrable systems that were first introduced for the unitary Lie algebra $\mathfrak{u}(n)$ by Guillemin and Sternberg in [GS83], and later for the general linear Lie algebra $\mathfrak{gl}(n)$ by Kostant and Wallach in [KW06a] and [KW06b].

Recently, the study of Gelfand-Tsetlin modules took a new direction after the theory of singular Gelfand-Tsetlin modules was initiated in [FGR16]. Singular Gelfand-Tsetlin modules are roughly those that have basis of tableaux whose entries may be zeros of the denominators in the Gelfand-Tsetlin formulas. For the last three years remarkable progress has been made towards the study of singular Gelfand-Tsetlin modules of $\mathfrak{gl}(n)$. Important results in this direction were obtained in [FGR15, FGR16, FGR17, Zad17, Vis18, Vis17, RZ17]. In particular, explicit constructions of a Gelfand-Tsetlin module with a fixed singular Gelfand-Tsetlin character were obtained with algebro-combinatorial methods in [RZ17] and with geometric methods in [Vis17]. One notable property of these general constructions is their relations with Schubert calculus and reflection groups. As explained below, this relation is brought to a higher level in the present paper and new connections with Schubert polynomials and generalized Littlewood-Richardson coefficients are established. We hope that these new connections, combined with combinatorial results on skew Schubert polynomials, will help us to bring within a reach the solution of the most important problem in the theory: the classification of all simple Gelfand-Tsetlin modules of $\mathfrak{gl}(n)$.

The study of Gelfand-Tsetlin modules is not limited to the cases of $\mathfrak{gl}(n)$ and $\mathfrak{sl}(n)$. Gelfand-Tsetlin subalgebras are part of a uniform algebraic theory, the theory of Galois orders. Galois orders are special types of rings that were introduced in [FO10] in an attempt to unify the representation theories of generalized Weyl algebras and the universal enveloping algebra of $\mathfrak{gl}(n)$. In addition to the universal enveloping algebra of $\mathfrak{gl}(n)$ examples of Galois orders include the n-th Weyl algebra, the quantum plane, the Witten-Woronowicz algebra, the q-deformed Heisenberg algebra, and finite W-algebras of type A (for details and more examples see for example [Har]).

The representation theory of Galois orders was initiated in [FO14]. In particular, the following finiteness theorem for Gelfand-Tsetlin modules of a Galois order U over an integral domain Γ was proven: given a maximal ideal $\mathfrak{m}$ of Γ there exists only finitely many non-isomorphic simple Gelfand-Tsetlin modules M such that $M[\mathfrak{m}] \neq 0$ (see §4.3 for the definition of $M[\mathfrak{m}]$). This theorem generalizes the finiteness theorem for $\mathfrak{gl}(n)$ obtained in [Ovs02]. Other important results of the Gelfand-Tsetlin theory of $\mathfrak{gl}(n)$ were extended to certain types of Galois orders in [EMV, Har, Maz99]. One such important result is the construction of a Gelfand-Tsetlin module with any fixed Gelfand-Tsetlin character over an orthogonal Gelfand-Tsetlin algebra obtained very recently in [EMV]. Another notable contribution is the new framework of rational Galois orders established in [Har]. Examples of rational Galois orders are the universal enveloping algebra of $\mathfrak{gl}(n)$, restricted Yangians of $\mathfrak{gl}(n)$, orthogonal Gelfand-Tsetlin algebras, finite W-algebras of type A, among others.

The first goal of the present paper is to establish a closer connection of the singular Gelfand-Tstelin theory with the theory of Schubert polynomials and reflection groups. We study a new natural class of Γ-modules that consists of differential operators related to the polynomials introduced in [BGG73]. These BGG differential operators have numerous applications in the cohomology theory of flag varieties. In the present paper, we use a particular aspect of these applications - the Postnikov-Stanley operators. Postnikov-Stanley polynomials were originally defined in [PS09] in order to express degrees of Schubert varieties in the generalized complex flag manifold G/B. The polynomials are given by weighted sums over saturated chains in the Bruhat order and have intimate relations with Schubert polynomials, harmonic polynomials, Demazure characters, and generalized Littlewood-Richardson coefficients. The action of Γ on the module of BGG differential operators is described explicitly in terms of Postnikov-Stanley operators. Using this explicit action, we prove one of our main results - an upper bound for the size of Jordan blocks of the generators of Γ, see Theorem 6.6. The explicitness of the action helps us also to understand better the structure of the U-module consisting of BGG operators and, in particular, is used as a technical tool in the proof of the simplicity criterion for this U-module

The other goal of the paper is to deepen the study of Gelfand-Tsetlin modules of rational Galois orders. We define a special class of rational Galois orders that we call standard Galois orders of type A that includes most of the examples of rational Galois orders listed above. Then we construct Gelfand-Tsetlin modules of arbitrary character over these Galois orders and provide explicit bases of these modules, see Theorem 8.3. Our last main result, Theorem 8.5, is a sufficient condition for these modules to be simple. This simplicity criterion generalizes the criterion for orthogonal Gelfand-Tsetlin algebras obtained in [EMV]. It is worth noting, that as an immediate corollary, our result provides new examples of simple modules of any finite W-algebra of type A.

The organization of the paper is as follows. Preliminary results on reflection groups are collected in Section 2. In Section 3 we include the needed background on BGG differential operators and Postnikov-Stanley differential operators. Definitions and properties of Galois orders and Gelfand-Tsetlin modules are included in Section 4. In Section 5 we discuss generalities on rational Galois orders. The Γ-module of BGG operators is defined in Section 6, where we study its structure with the aid of Postnikov-Stanley operators. In this section we also give an upper bound for the size of a Jordan block of any γ of Γ considered as an endomorphism of the Γ-module of BGG differential operators. The U-module structure of a (larger) space of BGG differential operators is studied in Section 7. In Section 8 we provide a basis of the U-module defined in Section 7, prove that this module is a Gelfand-Tsetlin module, and establish a simplicity criterion for this module.

We finish the introduction with a few notational conventions, which will be used throughout the paper. Unless otherwise stated, the ground field will be $\mathbb{C}$. By $\mathbb{N}$ we denote the set of positive integer numbers. A *reflection group* will always be a finite group isomorphic to a subgroup of $\mathrm{O}(n,\mathbb{R})$ for some $n \in \mathbb{N}$ and generated by reflections. Given a ring R and a monoid $\mathcal{M}$ acting on R by ring morphisms, by $R\#\mathcal{M}$ we denote the *smash product* of R and $\mathcal{M}$, i.e. the free R-module with basis $\mathcal{M}$ and product given by $r_1m_1 \cdot r_2m_2 = r_1m_1(r_2)m_1m_2$ for any $r_1, r_2 \in R$ and any $m_1, m_2 \in \mathcal{M}$.
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2. Preliminaries on reflection groups

We recall some basic facts and fix notation on root systems and reflection groups. Our definition of root system is slightly different from the classical one, but is easily seen to be equivalent.

2.1. Root systems and reflection groups. Let V be a complex vector space with a fixed inner product which we denote by $(-,-)$. We use this inner product to identify V with its dual V^* and for each $\alpha \in V^*$ we denote by v_α the unique element of V such that $\alpha(v') = (v', v_\alpha)$ for all $v' \in V$. Given $\alpha \in V^*$ we denote by s_α the orthogonal reflection through the hyperplane $\ker \alpha$, and by s_α^* the corresponding endomorphism of V^*. In this article a *finite root system* over V will be a finite set $\Phi \subset V^*$ such that for each $\alpha \in \Phi$ we have

(R1) $\Phi \cap \mathbb{C}\alpha = \{\pm\alpha\}$ and
(R2) $s_\alpha^*(\Phi) \subset \Phi$.

In classical references such as [Hum90] and [Hil82] root systems are defined as subsets of an Euclidian vector space $V_\mathbb{R}$ with $\mathbb{R}$ instead of $\mathbb{C}$ in $(R1)$. Taking $V = \mathbb{C} \otimes_\mathbb{R} V_\mathbb{R}$ for an adequate $V_\mathbb{R}$ our definition is equivalent to theirs. We use the definition above since we work with complex vector spaces endowed with the action of a reflection group.

We now review the basic features of the theory of root systems. For more details we refer the reader to the two references above. Fix a root system Φ. The Weyl group associated to Φ is the group $W(\Phi)$ generated by $\{s_\alpha \mid \alpha \in \Phi\}$. Since we do not assume that the root systems are reduced or crystallographic, nor that Φ generates $V_\mathbb{R}^*$, the group $W(\Phi)$ is a finite reflection group which may be decomposable, and its action on V may have a nontrivial stabilizer. Any reflection group $G \subset \mathsf{GL}(V)$ is the Weyl group of some root system $\Phi \subset V^*$ [Hil82, §1.2].

Just as in the case of root systems for Lie algebras, for each root system Φ we can choose a linearly independent subset $\Sigma \subset \Phi$ which is a basis of the $\mathbb{R}$-span of Φ such that the coefficients of each root of Φ in this basis are either all nonnegative or all nonpositive. Such sets are called *bases* or *simple systems*, and its elements are called *simple roots*. Each choice of a base defines a partition $\Phi = \Phi^+ \cup -\Phi^+$, where Φ^+ is the set of all *positive* roots, i.e. those whose coordinates over Σ are nonnegative. If we fix a base Σ then the set S of reflections corresponding to simple roots is a minimal generating set of the reflection group $W = W(\Phi)$, and hence (W, S) is a finite Coxeter system in the sense of [Hum90, 1.9]. Each $s \in W$ of order two is of the form s_α for some $\alpha \in \Phi^+$ [Hum90, Proposition 2.14], and given $s \in W$ of order two we denote by α_s the corresponding positive root.

Fixing a base Σ, or equivalently, a minimal generating set $S \subset W$, we define the *length* $\ell(\sigma)$ of $\sigma \in W$ as the least positive integer ℓ such that σ can be written as a composition of ℓ reflections in S. Any sequence $s_1, \ldots, s_{\ell(\sigma)}$ such that $\sigma = s_1 \cdots s_{\ell(\sigma)}$ is called a *reduced decomposition*; notice that reduced decompositions are not unique. The group W acts faithfully and transitively on Φ. Furthermore, $\ell(\sigma) = |\sigma(\Phi^+) \cap -\Phi^+|$, so W has a unique longest element whose length equals $|\Phi|$. We will denote this element by $\omega_0(W)$, or simply by ω_0 if the group W is clear from the context.

For the rest of this section we fix a root system Φ with base Σ and denote by (W, S) be the corresponding Coxeter system.

2.2. Subsystems, subgroups and stabilizers. In this subsection we follow [Hum90, 1.10], where the reader can find most proofs. Given $\Omega \subset \Sigma$ we denote by $\Phi(\Omega)$ the root subsystem generated by Ω. We will call such subsystems *standard*. If $\Psi \subset \Phi$ is an arbitrary subsystem then we can choose a base $\Omega \subset \Psi$ which can be extended to a base $\overline{\Omega}$ of Φ. By [Hum90, 1.4 Theorem] W acts transitively on the set of all bases of Φ, so for some $\sigma \in W$ we have $\sigma(\overline{\Omega}) = \Sigma$ and hence $\sigma(\Psi)$ is standard.

Let $\theta \subset S$ and denote by W_θ the subgroup of W generated by θ. Then (W_θ, θ) is also a Coxeter system and it determines a standard root system $\Phi_\theta \subset \Phi$ with simple roots $\Sigma_\theta = \{\alpha_s \mid s \in \theta\}$. We will refer to subgroups of the form W_θ as *standard parabolic* subgroups. A parabolic subgroup is any subgroup of W that is conjugate to a standard parabolic subgroup.

If $\sigma \in W_\theta$ then we can compute its length as an element of W with respect to the generating set S or as an element of W_θ with respect to the generating set θ. Both lengths turn out to be equal and will be denoted by $\ell(\sigma)$. Since W_θ is also a Coxeter group it has a unique element of maximal length which we will denote by $\omega_0(\theta)$. The set $W^\theta = \{\sigma \in W \mid \ell(\sigma s) > \ell(\sigma) \text{ for all } s \in \theta\}$ is a set of representatives of the classes in the quotient W/W_θ, and for each $\sigma \in W$ there exist unique elements $\sigma^\theta \in W^\theta$ and $\sigma_\theta \in W_\theta$ such that $\sigma = \sigma^\theta \sigma_\theta$ with $\ell(\sigma) = \ell(\sigma^\theta) + \ell(\sigma_\theta)$. The element σ^θ is the element of minimal length in the coclass σW_θ. It follows that $(\omega_0)_\theta = \omega_0(\theta)$ and therefore $\omega_0^\theta = \omega_0 \omega_0(\theta)^{-1}$.

Given $v \in V$ we denote by $\Phi_0(v)$ the set of all roots in Φ such that $\alpha(v) = 0$, which is clearly a root subsystem of Φ. We also denote by W_v the stabilizer of v in W. We will say that v is Σ-*standard*, or just *standard* when Σ is fixed or clear from the context, if $\Phi_0(v)$ is a Σ-standard subsystem of Φ. It is easy to check that v is standard if and only if W_v is a standard parabolic subgroup, and $W = W(\Phi_0(v))$. Since $W_{\sigma(v)} = \sigma W_v \sigma^{-1}$ and $\Phi_0(\sigma(v)) = \sigma(\Phi_0(v))$ for all $\sigma \in W$, it follows that for every $v \in V$ there exists $\sigma \in W$ such that $\sigma(v)$ is standard and hence $W_{\sigma(v)}$ is a standard parabolic subgroup. If v is standard then we denote by W^v the set of minimal length representatives of the left coclasses W/W_v.

3. Divided differences and Postnikov-Stanley operators

In this section V is a fixed complex vector space, $\Lambda = S(V)$, and L is the fraction field of Λ. Note that following the convention of [PS09], we write $S(V)$ for $Sym(V^*)$. Also, we fix a finite root system Φ with base Σ, and set $W = W(\Phi)$ to be the corresponding reflection group with minimal generating set S. Thus W acts on Λ and L, and we set $\Gamma = \Lambda^W$ and $K = L^W$.

3.1. Divided differences. Since W acts on L we can form the smash product $L\#W$. Recall that the product in this complex algebra is given over generators by $f\sigma \cdot g\tau = f\sigma(g)\sigma\tau$ for all $f, g \in L$ and all $\sigma, \tau \in W$. Dedekind's theorem on linear independence of field homomorphisms implies that the algebra morhpism $L\#W \hookrightarrow \mathrm{End}_{\mathbb{C}}(L)$ defined by mapping $l\sigma \in L\#W$ to the endomorphism $f \mapsto l\sigma(f)$ is an embedding. We identify $L\#W$ with its image, and so must be careful to distinguish the result of applying the endomorphism $l\sigma$ to f, whose result is $l\sigma(f)$, and the product of $l\sigma$ and f in $L\#W$, which is $l\sigma \cdot f = l\sigma(f)\sigma$.

For $s \in W$ we set

$$\nabla_s = \frac{1}{\alpha_s}(1-s) \in L\#W.$$

It is easy to show that for each $f, g \in L$,

$$\nabla_s(fg) = \nabla_s(f)g + s(f)\nabla_s(g)$$

so ∇_s is a twisted derivation of L. Notice that $\ker \nabla_s$ is exactly $L^{\langle s \rangle}$ and so ∇_s is $L^{\langle s \rangle}$-linear. Also it follows from the definition that $\nabla_s(\Lambda) \subset \Lambda$.

Example. *Suppose $V = \mathbb{C}^2$ and let $\{x, y\} \subset (\mathbb{C}^2)^*$ be the dual basis to the canonical basis. Let s be the reflection given by $s(z_1, z_2) = (z_2, z_1)$, so $\alpha_s = x - y$. Then for each $f(x, y) \in \mathbb{C}[x, y]$ we have $\nabla_s(f)(x, y) = \frac{f(x,y) - f(y,x)}{x-y}$. Notice that this quotient is always a polynomial, since $f(x, y) - f(y, x)$ is an antisymmetric polynomial and hence divisible by $x - y$.*

Given $\sigma \in W$ we take a reduced decomposition $\sigma = s_1 \cdots s_\ell$ and set $\partial_\sigma = \nabla_{s_1} \circ \cdots \circ \nabla_{s_\ell}$; this element is called the *divided difference* corresponding to σ and does not depend on the chosen reduced decomposition [Hil82, Chapter IV (1.6)]. Notice though that the definition of ∂_σ does depend on the choice of a base $\Sigma \subset \Phi$.

By definition, an $L\#W$-module Z is an L-vector space endowed with a W-module structure such that the action of L on Z is W-equivariant. A simple induction on the length of σ shows that the divided difference ∂_σ defines a K-linear map over any $L\#W$-module Z. In particular L is such a module, and since $\nabla_s(\Lambda) \subset \Lambda$ for any $s \in S$, it follows that Λ is closed under the action of divided differences.

3.2. Coinvariant spaces and Schubert polynomials. The algebra Λ is $\mathbb{Z}_{\geq 0}$-graded with $\Lambda_1 = V^*$ and Γ is a graded subalgebra of Λ . We denote by I_W the ideal of Λ generated by the elements of Γ of positive degree. By the Chevalley-Shephard-Todd theorem Γ is isomorphic to a polynomial algebra in $\dim V$ variables and Λ is a free Γ-module of rank $|W|$. Also, a set $B \subset \Lambda$ is a basis of the Γ-module Λ if and only if its image in the quotient Λ / I_W is a $\mathbb{C}$-basis. Furthermore, Λ / I_W is naturally a graded W-module isomorphic to the regular representation of W with Hilbert series $\sum_{\sigma \in W} t^{\ell(\sigma)}$. For proofs we refer the reader to [Hil82, Chapter II, Section 3].

We now recall the construction of the basis of Schubert polynomials of Λ / I_W. This construction is due to Bernstein, Gelfand and Gelfand [BGG73] and Demazure [Dem74] in the case when W is a Weyl group, and to Hiller [Hil82, Chapter IV] in the case of arbitrary Coxeter groups. Set $\Delta(\Phi) = \prod_{\alpha \in \Phi^+} \alpha$, and for each $\sigma \in W$ set $\mathfrak{S}^\Sigma_\sigma = \frac{1}{|W|} \partial_{\sigma^{-1}\omega_0} \Delta(\Phi)$. We will often write $\mathfrak{S}_\sigma$ instead of $\mathfrak{S}^\Sigma_\sigma$ when the base Σ is clear from the context. Notice that by definition $\deg \mathfrak{S}_\sigma = \ell(\sigma)$. The polynomials $\{\mathfrak{S}_\sigma \mid \sigma \in W\}$ are known as *Schubert polynomials*, and they form a basis of Λ as a Γ-module, so the projection of this set is a basis of Λ / I_W as a complex vector space. Since $K = L^W$ we know that L is a K-vector space of dimension $|W|$ and so $\{\mathfrak{S}_\sigma \mid \sigma \in W\}$ is also a basis of L over K. Given $f \in L$ we will denote by $f_{(\sigma)}$ the coefficient of $\mathfrak{S}_\sigma$ in the expansion of f relative to this basis, so $f = \sum_{\sigma \in W} f_{(\sigma)} \mathfrak{S}_\sigma$.

Since Schubert polynomials form a basis of Λ/I_W, for all $\sigma, \tau, \rho \in W$ there exists $c^{\rho}_{\sigma,\tau} \in \mathbb{C}$ defined implicitly by the equation

$$\mathfrak{S}_\sigma \mathfrak{S}_\tau = \sum_{\rho \in W} c^{\rho}_{\sigma,\tau} \mathfrak{S}_\rho \mod I_W.$$

The coefficients $c^{\rho}_{\sigma,\tau}$ are the *generalized Littlewood-Richardson coefficients* relative to the base Σ. It follows from the definition that $c^{\rho}_{\sigma,\tau} = 0$ unless $\ell(\sigma) + \ell(\tau) = \ell(\rho)$. If $\theta \subset S$ then the space of W_θ-invariants $(\Lambda/I_W)^{W_\theta}$ is generated by the set $\{\mathfrak{S}_\sigma \mid \sigma \in W^\theta\}$ [Hil82, Chapter IV (4.4)]. In particular, if $\sigma, \tau \in W^\theta$ then $c^{\rho}_{\sigma,\tau} \neq 0$ implies that $\rho \in W^\theta$.

3.3. Postnikov-Stanley operators. Given $\alpha \in V^*$ there is a unique $\mathbb{C}$-linear derivation $\Theta(\alpha) : \Lambda \longrightarrow \Lambda$ such that $\Theta(\alpha)(\beta) = (\beta, \alpha)$ for each $\beta \in V^*$. This map extends uniquely to a morphism $\Theta : \Lambda \longrightarrow \mathrm{Der}_{\mathbb{C}}(\Lambda)$. If we fix an orthonormal basis $x_1, \ldots, x_n$ of V^*, then $S(V) \cong \mathbb{C}[x_1, \ldots, x_n]$ and $\Theta(x_i) = \frac{\partial}{\partial x_i}$.

Let $(-,-)_\Theta : \Lambda \times \Lambda \longrightarrow \mathbb{C}$ be the bilinear form given by $(f,g) = \Theta(f)(g)(0)$. This is a nondegenerate bilinear form which can be used to identify Λ with its graded dual Λ°. For every graded ideal $I \subset \Lambda$ we write $\mathcal{H}_I = \{g \in \Lambda \mid (f,g)_\Theta = 0 \text{ for all } f \in I\}$. Since the pairing $(-,-)_\Theta$ is nodegenerate, the space $\mathcal{H}_I$ is naturally isomorphic to the graded dual $(\Lambda/I)^\circ$. We denote by P^Σ_σ the unique element in $\mathcal{H}_{I_W}$ such that $(P^\Sigma_\sigma, \mathfrak{S}^\Sigma_\tau) = \delta_{\sigma,\tau}$ for all $\sigma, \tau \in W$. Like before, we usually write P_σ instead of P^Σ_σ. It follows that the set $\{P_\sigma \mid \sigma \in W\}$ is a graded basis of $\mathcal{H}_{I_W}$, dual to the Demazure basis of Λ/I_W. Also for each $\theta \subset S$ the set $\{P_\sigma \mid \sigma \in W^\theta\}$ is a graded basis of the dual of $(\Lambda/I_W)^{W_\theta}$. Notice that both these families are bases of the space of W-harmonic polynomials, i.e. those polynomials which are annihilated by W-symmetric differential operators.

Recall that σ *covers* τ, and denote this by $\tau \preceq \sigma$, if $\sigma = \tau s_\alpha$ for some $\alpha \in \Phi$ and $\ell(\sigma) = \ell(\tau) + 1$. The *Bruhat order* of W is the transitive closure of this relation. A *saturated chain* from σ to τ in the Bruhat order is a sequence $\sigma = \sigma_0 \preceq \sigma_1 \preceq \cdots \preceq \sigma_r = \tau$, and we refer to r as the length of the saturated chain. The polynomials P_σ were described by Postnikov and Stanley in terms of saturated chains in the Bruhat order of W in [PS09] when W is a Weyl group.

For each covering relation $\sigma \preceq \sigma s_\alpha$ with $\alpha \in \Phi^+$ we set $m(\sigma, \sigma s_\alpha) = \alpha \in V^* = S(V)_1$, and for a saturated chain $C = (\sigma_1, \sigma_2, \ldots, \sigma_r)$ we denote by m_C the product $\prod_{i=1}^{r-1} m_C(\sigma_i, \sigma_{i+1})$. Set

$$P_{\sigma,\tau} = \frac{1}{(\ell(\tau) - \ell(\sigma))!} \sum_C m_C$$

where the sum is taken over all saturated chains from σ to τ. Now, according to [PS09, Corollary 6.9], if $\sigma \leq \tau$ in the Bruhat order then $P_{\sigma,\tau} = \sum_{\rho \in W} c^{\tau}_{\sigma,\rho} P_\rho$. This identity inspires the following definition.

Definition 3.1. *For each $\sigma, \tau \in W$ with $\tau \leq \sigma$ in the Bruhat order of W we set ${}^\Sigma\mathfrak{D}_\sigma = \Theta(P^\Sigma_\sigma)$ and ${}^\Sigma\mathfrak{D}_{\tau,\sigma} = \sum_{\rho \in W} c^{\sigma}_{\tau,\rho} {}^\Sigma\mathfrak{D}_\rho$. We ommit the superscript Σ whenever it is clear from the context.*

3.4. Notice that although by definition ${}^\Sigma\mathfrak{D}_{\tau,\sigma}$ is a differential operator on Λ, it has a well defined extension to the fraction field L, and we will denote this extenssion by the same symbol. We denote by $\mathfrak{D}^0_\sigma$ and $\mathfrak{D}^0_{\tau,\sigma}$ the linear functional of Λ obtained by applying the corresponding differential operator followed by evaluation at 0. The definition of the polynomials P_σ implies that $\mathfrak{D}^0_\sigma(\gamma f) = \gamma(0)\mathfrak{D}^0_\sigma(f)$ for all $f \in \Lambda$ and $\gamma \in \Gamma$. The following proposition shows that this functional extends to the algebra of rational functions without poles at 0 and gives a generalized Leibniz rule to compute the result of applying this operator to the product of two such functions.

Proposition 3.2. *Let $f \in L$ be regular at zero and let $\sigma \in W$. Then $f_{(\sigma)}$ is also regular at 0 and $f_{(\sigma)}(0) = \mathfrak{D}_\sigma(f)(0) = (\partial_\sigma f)(0)$. Furthermore if $g \in L$ is also regular at 0 then*

$$\mathfrak{D}^0_\sigma(fg) = \sum_{\rho \leq \sigma} \mathfrak{D}^0_{\rho,\sigma}(f)\mathfrak{D}^0_\rho(g) = \sum_{\rho \leq \sigma} \mathfrak{D}^0_\rho(f)\mathfrak{D}^0_{\rho,\sigma}(g).$$

Proof. Let $T \subset \Gamma$ be the set of W-invariant rational functions with nonzero constant term. This is clearly a W-invariant set and hence $T^{-1}\Gamma$ is a subalgebra of $K = \mathrm{Frac}(\Gamma)$. Denoting by A the subalgebra of L consisting of rational functions regular at 0, the product map $T^{-1}\Gamma \otimes \Lambda \longrightarrow A$ is an isomorphism, since any fraction $p/q \in L$ with $p, q \in \Lambda$ can be rewritten so that $q \in \Gamma$. Thus A is a free $T^{-1}\Gamma$-module with basis $\{\mathfrak{S}_\sigma \mid \sigma \in W\}$ and $f_{(\sigma)} \in A$ for all $\sigma \in W$.

As noted in the preamble for each $\gamma \in \Gamma$ we have $\mathfrak{D}^0_\sigma(\gamma f) = \gamma(0)\mathfrak{D}^0_\sigma(f)$, and it follows that the same holds if $\gamma \in A^G$. Thus

$$\mathfrak{D}_\sigma(f)(0) = \sum_\tau f_{(\tau)}(0)\mathfrak{D}_\sigma(\mathfrak{S}_\tau)(0) = f_{(\sigma)}(0)$$

as stated. Analogously ∂_σ is a K-linear operator, and hence

$$(\partial_\sigma f)(0) = \sum_\tau f_{(\tau)}(0)\frac{1}{|W|}(\partial_\sigma \partial_{\tau^{-1}\omega_0}\Delta(\Phi))(0).$$

Now $\frac{1}{|W|}\partial_\sigma \partial_{\tau^{-1}\omega_0}\Delta(\Phi)$ is zero unless $\ell(\tau\sigma) = \ell(\tau) + \ell(\sigma)$, in which case it equals $\mathfrak{S}_{\tau\sigma^{-1}}$. The evaluation of this polynomial at 0 is zero except when $\tau = \sigma$, and in this case the polynomial is just the constant $1 \in \mathbb{C}$. Therefore, $(\partial_\sigma f)(0) = f_{(\sigma)}(0) = \mathfrak{D}^0_\sigma(f)$.

Finally,

$$\mathfrak{D}^0_\sigma(fg) = \sum_{\tau,\rho} f_{(\tau)}(0)g_{(\rho)}(0)\mathfrak{D}_\sigma(\mathfrak{S}_\tau\mathfrak{S}_\rho)(0) = \sum_{\tau,\rho} c^\sigma_{\tau,\rho}\mathfrak{D}^0_\tau(f)\mathfrak{D}^0_\rho(g),$$

which proves the first identity in the proposition. To prove the second identity, we use that $c^\sigma_{\tau,\rho} = c^\sigma_{\rho,\tau}$. □

4. Galois orders and Gelfand-Tsetlin modules

Throughout this section Γ is a noetherian integral domain, K is its field of fractions, and L is a finite Galois extension of K with Galois group G. Hence $K = L^G$.

4.1. Galois orders. We first recall the notion of a Galois ring (order), that was introduced in [FO10]. Let $\mathcal{M}$ be a monoid acting on L by ring automorphisms, such that for all $t \in \mathcal{M}$ and all $\sigma \in G$ we have $\sigma \circ t \circ \sigma^{-1} \in \mathcal{M}$. Then the action of G extends naturally to an action on the smash product $L\#\mathcal{M}$. We assume that the monoid $\mathcal{M}$ is *K-separating*, that is given $m, m' \in \mathcal{M}$, if $m|_K = m'|_K$ then $m = m'$.

Definition 4.1. *Set $\mathcal{K} = L\#\mathcal{M}$.*

(i) A Galois ring over Γ *is a finitely generated* Γ*-subring* $U \subset (L\#\mathcal{M})^G$ *such that* $UK = KU = \mathcal{K}$.

(ii) Set $S = \Gamma \setminus \{0\}$. *A Galois ring* U *over* Γ *is a* right (respectively, left) Galois order, *if for any finite-dimensional right (respectively left)* K*-subspace* $W \subset U[S^{-1}]$ *(respectively,* $W \subset [S^{-1}]U$*), the set* $W \cap U$ *is a finitely generated right (respectively, left)* Γ*-module. A Galois ring is* Galois order *if it is both a right and a left Galois order.*

We will always assume that Galois rings are complex algebras. In this case we say that a Galois ring is a Galois algebra over Γ.

4.2. Principal and co-principal Galois orders. Notice that $L\#\mathcal{M}$ acts on L, where for each $X = \sum_{m\in\mathcal{M}} l_m m \in L\#M$ we define its action on $f \in L$ by $X(f) = \sum_m l_m m(f)$.

As an example of a Galois order, Hartwig introduced *the standard Galois* Γ*-order in* $\mathcal{K}$ defined as $\mathcal{K}_\Gamma = \{X \in \mathcal{K} \mid X(\Gamma) \subset \Gamma\}$, see [Har, Theorem 2.21]. In this article the term "standard Galois order" has a different meaning, and for sake of clarity will refer to the algebra above as *the left Hartwig order of* $\mathcal{K}$. A *principal Galois order* is any Galois order $U \subset \mathcal{K}_\Gamma$. By restriction Γ is a left U-module for any principal Galois order, and hence its complex dual Γ^* is a right U-module.

Denote by $\mathcal{M}^{-1}$ the monoid formed by the inverses of the elements in $\mathcal{M}$. Following [Har], we define an anti-isomorphism $-^\dagger : L\#\mathcal{M} \longrightarrow L\#\mathcal{M}^{-1}$ by $(lm)^\dagger = m^{-1} \cdot l = m^{-1}(l)m^{-1}$ for any $l \in L, m \in \mathcal{M}$. The *right Hartwig order* is thus defined as ${}_\Gamma\mathcal{K} = \{X \in \mathcal{K} \mid X^\dagger(\Gamma) \subset \Gamma\}$, and a *co-principal Galois order* is any Galois order contained in ${}_\Gamma\mathcal{K}$. Thus Γ^* is a left U-module for any co-principal Galois order, with action given by $X \cdot \chi = \chi \circ X^\dagger$ for any $X \in U$ and $\chi \in \Gamma^*$.

4.3. Gelfand-Tsetlin modules. Let U be a Galois order over Γ and let M be any U-module. Given $\mathfrak{m} \in \operatorname{Specm}\Gamma$ we set $M[\mathfrak{m}] = \{x \in M \mid \mathfrak{m}^k x = 0 \text{ for } k \gg 0\}$. Since ideals in $\operatorname{Specm}\Gamma$ are in one-to-one correspondence with characters $\chi : \Gamma \longrightarrow \mathbb{C}$ we also set $M[\chi] = \{x \in M \mid (\gamma - \chi(\gamma))^k x = 0 \text{ for all } \gamma \in \Gamma \text{ and } k \gg 0\}$. If χ is given by the natural projection $\Gamma \longrightarrow \Gamma/\mathfrak{m} \cong \mathbb{C}$ then $M[\mathfrak{m}] = M[\chi]$.

Definition 4.2. *A Gelfand-Tsetlin module is a finitely generated* U*-module* M *such that its restriction* $M|_\Gamma$ *to* Γ *can be decomposed as a direct sum* $M|_\Gamma = \bigoplus_{\mathfrak{m}\in\operatorname{Specm}\Gamma} M[\mathfrak{m}]$.

A U-module M is a Gelfand-Tsetlin module if and only if for each $x \in M$ the cyclic Γ-module $\Gamma \cdot x$ is finite dimensional over $\mathbb{C}$ [DFO94, §1.4], which easily implies the following result.

Lemma 4.3. *A* U*-submodule of a Gelfand-Tsetlin module is again a Gelfand-Tsetlin module.*

For every maximal ideal $\mathfrak{m}$ of Γ we denote by $\varphi(\mathfrak{m})$ the number of non-isomorphic simple Gelfand-Tsetlin modules M for which $M[\mathfrak{m}] \neq 0$. Sufficient conditions for the number $\varphi(\mathfrak{m})$ to be nonzero and finite were established in [FO14].

Consider the integral closure $\overline{\Gamma}$ of Γ in L. It is a standard fact that if Γ is finitely generated as a complex algebra then any character of Γ has finitely many extensions to characters of $\overline{\Gamma}$. Let $\overline{\mathfrak{m}}$ be any lifting of $\mathfrak{m}$ to $\overline{\Gamma}$, and $\mathcal{M}_\mathfrak{m}$ be the stabilizer of $\overline{\mathfrak{m}}$ in $\mathcal{M}$. Note that the monoid $\mathcal{M}_\mathbf{m}$ is defined uniquely up to G-conjugation. Thus the cardinality of $\mathcal{M}_\mathfrak{m}$ does not depend on the choice of the lifting. We denote this cardinality by $|\mathfrak{m}|$.

Theorem 4.4. [FO14, Main Theorem and Theorem 4.12] *Let Γ be a commutative domain which is finitely generated as a complex algebra and let $U \subset (L\#\mathcal{M})^G$ be a right Galois order over Γ. Let also $\mathfrak{m} \in \operatorname{Specm}\Gamma$ be such that $|\mathfrak{m}|$ is finite. Then the following hold.*

(i) The number $\varphi(\mathfrak{m})$ is nonzero.

(ii) If U is a Galois order over Γ then the number $\varphi(\mathfrak{m})$ is finite and uniformly bounded.

(iii) If U is a Galois order over Γ and Γ is a normal Noetherian algebra, then for every simple Gelfand-Tsetlin module M the space $M[\mathfrak{m}]$ is finite dimensional and bounded.

5. Rational Galois orders

Recall that V is a complex vector space with an inner product. We set $\Lambda = S(V)$, and $L = \operatorname{Frac}(\Lambda)$. Recall that an element $g \in \mathsf{GL}(V)$ is called a *pseudo-reflection* if it has finite order and fixes a hyperplane of codimension 1. By definition every reflection is a pseudo-reflection, and the converse holds over $\mathbb{R}$ but not over $\mathbb{C}$, which is why finite groups generated by pseudo-reflections are called *pseudo-reflection groups* or *complex reflection groups*. We fix $G \subset \mathsf{GL}(V)$ a pseudo-reflection group. As usual the action of G on V induces actions on Λ and L, and we denote by Γ the algebra of G-invariant elements of Λ and set $K = L^G$.

5.1. Let $L \hookrightarrow \operatorname{End}_{\mathbb{C}}(L)$ be the $\mathbb{C}$-algebra morphism that sends any rational function $f \in L$ to the $\mathbb{C}$-linear map $m_f : f' \in L \mapsto ff' \in L$. Although $\operatorname{End}_{\mathbb{C}}(L)$ is not a L-algebra, it is an L-vector space with $f \cdot \varphi = m_f \circ \varphi$ for all $\varphi \in \operatorname{End}_{\mathbb{C}}(L)$. Also G acts on $\operatorname{End}_{\mathbb{C}}(L)$ by conjugation and $\sigma \cdot m_f = \sigma \circ m_f \circ \sigma^{-1} = m_{\sigma(f)}$ for each $\sigma \in G$, so the map $f \mapsto m_f$ is G-equivariant. For simplicity we will write f for the operator m_f.

Given $v \in V$ we define a map $a_v : V \longrightarrow V$ given by $a_v(v') = v' + v$. This in turn induces an endomorphism of Λ, which we denote by t_v, given bt $t_v(f) = f \circ a_v$; we sometimes write $f(x+v)$ for $t_v(f)$. Each map t_v can be extended to a $\mathbb{C}$-linear operator on L and $t_v \circ t_{v'} = t_{v+v'}$, so V acts on L by automorphisms and we can form the smash product $L\#V$. Once again there is an algebra morphism $L\#V \to \operatorname{End}_{\mathbb{C}}(L)$, and the definitions imply that this map is G-equivariant.

Lemma 5.1. *Let G, V, and L be as above, and let $Z \subset V$ be an arbitrary subset. Then the set $\{t_z \mid z \in Z\} \subset \operatorname{End}_{\mathbb{C}}(L)$ is linearly independent over L, and the map $L\#V \longrightarrow \operatorname{End}_{\mathbb{C}}(L)$ is injective.*

Proof. Put $T = \sum_{i=1}^N f_i t_{z_i}$ where $f_i \in L^\times$ and each $z_i \in Z$, and assume $T = 0$. Given $p \in \Lambda$ we obtain that $0 = T(p) = \sum_i f_i p(x+z_i)$, or, equivalently, $p(x)\sum_i f_i = \sum_i [p(x+z_i) - p(x)]f_i$. Let $v \in V$ be arbitrary and choose a polynomial p of positive degree such that $p(v) = p(v+z_j)$ for all $j \neq i$ but $p(v) + 1 = p(v+z_i)$. Then $0 = p(v+z_i)f_i(v)$ so $f_i(v) = 0$. Since v is arbitrary this implies that $f_i = 0$ so the set $\{t_z \mid z \in Z\}$ is L-linearly independent. Since the morphism $L\#V \longrightarrow \operatorname{End}_{\mathbb{C}}(L)$ is L-linear and sends an L-basis of $L\#V$ to a linearly independent subset, it must be injective. $\square$

5.2. Rational Galois orders. Given a character $\chi : G \longrightarrow \mathbb{C}^\times$, the space of relative invariants $\Lambda_\chi^G = \{f \in \Lambda \mid \sigma \cdot p = \chi(\sigma)p \text{ for all } \sigma \in G\} \subset \Lambda$ is a Λ^G-submodule of Λ. By a theorem of Stanley [Hil82, 4.4 Proposition] Λ_χ^G is a free Λ^G-module of rank 1. The generator of Λ_χ^G is $d_\chi = \prod_{H \in \mathcal{A}(G)} (\alpha_H)^{a_H}$, where $\mathcal{A}(G)$ is the set of hyperplanes that are fixed pointwise by some element of G, each α_H is a linear form such that $\ker \alpha_H = H$, and $a_H \in \mathbb{Z}_{\geq 0}$ is minimal with the property $\det[s_H^*]^{a_H} = \chi(s_H)$ for an arbitrary generator s_H of the stabilizer of H in G. Note that a_H is independent on the choice of s_H, and that if G is a Coxeter group then a_H is either 1 or 0.

Definition 5.2 ([Har, Definition 4.3])**.** *A* co-rational Galois order *is a subalgebra $U \subset \mathrm{End}_{\mathbb{C}}(L)$ generated by Γ and a finite set of operators $\mathcal{X} \subset (L\#V)^G$ such that for each $X \in \mathcal{X}$ there exists $\chi \in \hat{G}$ with $Xd_\chi \in \Lambda\#V$.*

Given $X \in L\#V$ we define its *support* as the set of all $v \in V$ such that t_v appears with nonzero coefficient in X. Note that the support is well-defined since the set $\{t_v \mid v \in V\}$ is free over L. We denote the support of X by $\operatorname{supp} X$. Given a co-rational Galois order $U \subset (L\#V)^G$ we denote by $\mathsf{Z}(U)$ the additive monoid generated by $\{\operatorname{supp} X \mid X \in U\}$ in V. By [Har, Theorem 4.2] U is a co-principal Galois order in $(L\#\mathsf{Z}(U))^G$. In particular Γ^* is a left U-module.

Let $v \in V$, let $\mathsf{ev}_v : \Gamma \longrightarrow \mathbb{C}$ be the character given by evaluation at v, and let $\mathfrak{m} = \ker \mathsf{ev}_v$. Then the cyclic U-module $U \cdot \mathsf{ev}_v \subset \Gamma$ is a Gelfand-Tsetlin module [Har, Theorem 3.3], and since $\mathsf{ev}_v \in U \cdot \mathsf{ev}_v[\mathfrak{m}]$ we have a new proof that $\varphi(\mathfrak{m}) \neq 0$ for rational Galois orders. The following sections are devoted to study a different module associated to v, which always contains ev_v and turns out to be equal to $\mathsf{ev}_v \cdot U$ for generic v (see Theorem 8.5).

6. Structure of Γ-modules associated to Postnikov-Stanley operators

Throughout this section we fix a complex vector space V, and a root system Φ. We also fix a root subsytem $\Psi \subset \Phi$ with base $\Omega \subset \Psi$. We denote by G the Weyl group associated to Φ and by W the one associated to Ψ. Like before, $\Lambda = S(V)$, $L = \mathrm{Frac}(\Lambda)$, $\Gamma = \Lambda^G$, and $K = L^G$. Since $W \subset G$, the group W also acts on the vector spaces Λ, Γ, etc. All Schubert polynomials, Postnikov-Stanley operators, standard elements, etc. are defined with respect to the subsystem Ψ and the base Ω unless otherwise stated.

Lemma 6.1. *Let $v \in V$ and let $\pi^W : \Lambda \longrightarrow \Lambda/I_W$ be the natural projection. Then $\pi^W(t_v(\Gamma)) = (\Lambda/I_W)^{W_v}$.*

Proof. Recall that K is the fixed field of G in L, and hence the fraction field of Γ. Since the extension $L^W \subset L$ is a Galois extension with Galois group W, the field $L^W t_v(K) \subset L$ must be the fixed field of a subgroup $\widetilde{W} \subset W$. If $\sigma \in W_v$ and $f \in K$ then $\sigma \cdot t_v(f) = t_{\sigma(v)}(\sigma \cdot f) = t_v(f)$, so $W_v \subset \widetilde{W}$. On the other hand, if $\sigma \in \widetilde{W}$, then $t_v(f) = t_{\sigma(v)}(f)$. So, in this case, $t_{\sigma(v)-v}(f) = f$ for all $f \in K$ and this implies that $\sigma(v) = v$ so $\sigma \in W_v$. Thus $L^W t_v(K) = L^{W_v}$ which implies that $\Lambda^W t_v(\Gamma) = \Lambda^{W_v}$. Since all non-constant polynomials in Λ^W are in the kernel of π^W we see that $\pi^W(\Lambda^W t_v(\Gamma)) = \pi^W(t_v(\Gamma))$, so this last space equals $\pi^W(\Lambda^{W_v}) = (\Lambda/I_W)^{W_v}$. $\square$

6.1. Let $\mathfrak{D}(\Omega, v)$ be the complex subspace of L^* (the complex dual of L) spanned by $\{{}^{\Omega}\mathfrak{D}_\sigma^v \mid \sigma \in W\}$. From now on we omit the superscript Ω. The generalized Leibniz rule from Proposition 3.2 implies that $\mathfrak{D}(\Omega, v)$ is a Λ-submodule of $\mathrm{Hom}_{\mathbb{C}}(L, \mathbb{C})$, since for each $f \in \Lambda$ and $g \in L$ we have

$$\begin{aligned}(f \cdot \mathfrak{D}^v_\sigma)(g) &= \mathfrak{D}^0_\sigma(t_v(f)t_v(g)) = \sum_{\tau \leq \sigma} \mathfrak{D}^0_{\tau,\sigma}(t_v(f))\mathfrak{D}^0_\tau(t_v(g)) \\ &= \sum_{\tau \leq \sigma} \mathfrak{D}^v_{\tau,\sigma}(f)\mathfrak{D}^v_\tau(g).\end{aligned}$$

Now let $\mathcal{D}^v_\sigma$ denote the restriction of $\mathfrak{D}^v_\sigma$ to Γ and let $\mathcal{D}(\Omega, v)$ be the subspace of Γ^* spanned by $\{\mathcal{D}^v_\sigma \mid \sigma \in W\}$. We call $\mathcal{D}(\Omega, v)$ *the space of BGG differential operators associated to* Ω *and* v. The same computation as above shows that $\mathcal{D}(\Omega, v)$ is a Γ-submodule of Γ^*. We record this result in the following theorem.

Theorem 6.2. *Let $v \in V$. The space $\mathcal{D}(\Omega, v)$ is a Γ-submodule of Γ^* and for each $\gamma \in \Gamma$*

$$\gamma \cdot \mathcal{D}^v_\sigma = \gamma(v)\mathcal{D}^v_\sigma + \sum_{\tau < \sigma} \mathfrak{D}^v_{\sigma,\tau}(\gamma)\mathcal{D}^v_\tau.$$

6.2. The structure of $\mathcal{D}(\Omega, v)$ as Γ-module. The modules $\mathcal{D}(\Omega, v)$ will play an important role in our study of Gelfand-Tsetlin modules over a co-rational Galois order. We thank David Speyer for pointing out the following technical result in [Spe17], which greatly simplified our presentation.

Recall that W^v is the set of minimal length representatives of the left W_v-cosets.

Lemma 6.3. *Let $v \in V$ be Ω-standard, let $A = (\Lambda/I_W)^{W_v}$, and let ω_0^v be the longest element in W^v. Then the bilinear form $(a, b) \in A \times A \mapsto \mathfrak{D}^v_{\omega_0^v}(ab) \in \mathbb{C}$ is non-degenerate.*

Proof. By the Chevalley-Shephard-Todd theorem Λ^{W_v} and Λ^W are polynomial algebras, generated by algebraically independent sets $p_1, \dots, p_r$ and $q_1, \dots, q_s$ respectively. Clearly $p_i \in \Lambda^{W_v}$ and $A = \mathbb{C}[q_1, \dots, q_s]/J$, where J is the ideal generated by the p_i's. This implies that A is a finite-dimensional complete intersection, and hence a graded Artinian self-injective ring.

Set $r = \ell(\omega_0^v)$. Then $A_n = 0$ for $n > r$, while A_r is spanned over $\mathbb{C}$ by $\mathfrak{S}_{\omega_0^v}$ and the bilinear form in the statement is given by taking the coefficient of $\mathfrak{S}_{\omega_0^v}$ in the product fg. By [Lam99, (16.22) and (16.55)], A is a symmetric algebra and there exists a nonsingular associative bilinear form $B : A \times A \longrightarrow \mathbb{C}$; where by associative we mean that $B(a, bc) = B(ab, c)$ for every $a, b, c \in A$. To finish the proof we show that we can choose B so that $B(a, b) = \mathfrak{D}^{ab}_{\omega_0^v}$. Since B is non-degenerate there exists $a' \in A$ such that $B(a', \mathfrak{S}_{\omega_0^v}) = 1$, and if a' were of positive degree then $B(a', \mathfrak{S}_{\omega_0^v}) = B(1, a'\mathfrak{S}_{\omega_0^v}) = 0$ so $a' \in \mathbb{C}$. Without loss of generality we may assume that $a' = 1$, which implies that $B(f, g) = B(1, fg) = \mathfrak{D}_{\omega_0^v}(fg)$. ☐

Proposition 6.4. *Suppose that $v \in V$ is Ω-standard.*

(a) The set $\{\mathcal{D}^v_\sigma \mid \sigma \in W^v\}$ is a basis of $\mathcal{D}(\Omega, v)$, and $\mathcal{D}^v_\sigma = 0$ for all $\sigma \notin W^v$.
(b) Let $x = \sum_{\sigma \in W^v} a_\sigma \mathcal{D}^v_\sigma$. Then $\mathcal{D}(\Omega, v) = \Gamma \cdot x$ if and only if $a_{\omega_0^v} \neq 0$.
(c) With the same notation as as in part b, $(\gamma - \gamma(v))x = 0$ for all $\gamma \in \Gamma$ if and only if $a_\sigma = 0$ for all $\sigma \neq e$.
(d) Let $v' \in V$. The space $\mathcal{D}(\Omega, v) \cap \mathcal{D}(\Omega, v')$ is non-zero if and only if v' is in the G-orbit of v. Furthermore if v' is in the W-orbit of v then $\mathcal{D}(\Omega, v) = \mathcal{D}(\Omega, v')$.

Proof. Since $\mathcal{D}^v_\sigma$ is a differential operator we have $\mathcal{D}^v_\sigma = \mathfrak{D}^0_\sigma \circ t_v|_\Gamma$, so for every $\gamma \in \Gamma$ $\mathfrak{D}^v_\sigma(\gamma) = \mathfrak{D}^0_\sigma(t_v(\gamma))$. By the definition of $\mathfrak{D}^0_\sigma$, the latter value depends only on the image of $t_v(\gamma)$ modulo the ideal I_W, and, thanks to Lemma 6.1, $\pi^W(t_v(\Gamma))$ is exactly the space of W_v-invariants of Λ/I_W. On the other hand, the set of Schubert polynomials $\{\mathfrak{S}_\sigma \mid \sigma \in W^v\}$ forms a basis of this space. Thus, for each $\sigma \in W^v$ there exists $\gamma_\sigma \in \Gamma$ such that $t_v(\gamma_\sigma) \equiv \mathfrak{S}_\sigma \mod I_W$ and these elements span $\pi^W(t_v(\Gamma))$. Hence $\mathcal{D}^v_\sigma(\gamma_\tau) = \delta_{\sigma,\tau}$ for all $\sigma \in W$ and $\tau \in W^v$, and this implies part (a).

For part (b), note that by Lemma 6.3, for each $\sigma \in W^v$ there exist polynomials γ^*_σ such that $\mathcal{D}^v_{\omega^v_0}(\gamma^*_\sigma\gamma_\tau) = \delta_{\sigma,\tau}$ for all $\tau \in W^v$. This implies that $\gamma^*_\sigma \cdot \mathcal{D}^v_{\omega^v_0} = \mathcal{D}^v_\sigma$ and hence, if x is as in the statement with $a_{\omega^v_0} \neq 0$, then for each $\sigma \in W^v$ there exists $\gamma \in \Gamma$ such that $\gamma \cdot x$ equals the sum of $\mathcal{D}^v_\sigma$ and a linear combination of operators $\mathcal{D}^v_\tau$ with $\tau < \sigma$. This proves part b.

Let $\mathfrak{m} = \ker \mathrm{ev}_v \subset \Gamma$. The adjointness between the Hom and the tensor product functors implies that $\mathrm{Hom}_\Gamma(\Gamma/\mathfrak{m}, \Gamma^*) \cong \mathrm{Hom}_\mathbb{C}(\Gamma/\mathfrak{m}, \mathbb{C}) \cong \mathbb{C}$, so the space of elements in Γ^* annihilated by $\mathfrak{m}$ has complex dimension 1. Since $\gamma\mathcal{D}^v_e = \gamma(v)\mathcal{D}^v_e$ this space is generated by $\mathcal{D}^v_e$ and this implies (c).

It follows from the explicit formulas for the action of $\gamma \in \Gamma$ that each element in $\mathcal{D}(\Omega, v)$ is a generalized eigenvector of γ with eigenvalue $\gamma(v)$. Thus if $\mathcal{D}(\Omega, v) \cap \mathcal{D}(\Omega, v') \neq 0$ we must have $\gamma(v) = \gamma(v')$ for all $\gamma \in \Gamma$ which implies that $v' \in G \cdot v$. Now if $v' = \tau(v)$ for some $\tau \in W$ then $\mathcal{D}^{\tau(v)}_\sigma = \mathfrak{D}^0_\sigma \circ t_{\tau(v)}|\Gamma = \mathfrak{D}^0_\sigma \circ \tau \circ t_v \circ \tau^{-1}|_\Gamma = \mathfrak{D}^0_\sigma \circ \tau \circ t_v$. Since $\mathfrak{D}^0_\sigma \circ \tau$ lies in $\mathcal{H}_W$, for each $\rho \in W$ there exist $c_\rho \in \mathbb{C}$ such that $\mathfrak{D}^0_\sigma \circ \tau = \sum_\rho c_\rho \mathfrak{D}^0_\rho$. Hence, $\mathcal{D}^{\tau(v)}_\sigma = \sum_\rho c_\rho \mathcal{D}^v_\rho$, which proves part (d). ☐

6.3. Jordan blocks of elements in Γ. Let $v \in V$ be Ω-standard. For each $\gamma \in \Gamma$ let us denote by $[\gamma]$ the matrix of the endomorphism of $\mathcal{D}(\Omega, v)$ induced by γ relative to the basis described in Proposition 6.4(a) and ordered by decreasing length. By Theorem 6.2, $[\gamma]$ is a lower triangular matrix all diagonal entries of which equal $\gamma(v)$. Thus, the Jordan form of the matrix consists of Jordan blocks with this eigenvalue. To provide further properties on the Jordan form of $[\gamma]$ for generic elements of Γcwe need the following lemma.

Lemma 6.5. *For each $\sigma \in W$ and each $f \in \Lambda_1$ we have $\mathfrak{D}^0_\sigma(f^{\ell(\sigma)}) = \sum_{C(\sigma)} \prod_{i=1}^{\ell(\sigma)} \mathfrak{D}^0_{s_i}(f)$, where the sum runs over the set $C(\sigma)$ of reduced expressions $\sigma = s_1 s_2 \cdots s_{\ell(\sigma)}$ of σ.*

Proof. We will prove the statement by induction on $r = \ell(\sigma)$. The base case $r = 0$ follows from $f(0) = 0$. Now writing $f^r = ff^{r-1}$ and using Proposition 3.2 and the fact that $\mathfrak{D}^0_\tau(f) = 0$ if $\ell(\tau) \neq 1$, we obtain

$$\begin{aligned}
\mathfrak{D}^0_\sigma(ff^{r-1}) &= \sum_{\ell(\tau)=\ell(\sigma)-1} \mathfrak{D}^0_\tau(f^{r-1})\mathfrak{D}^0_{\tau,\sigma}(f) \\
&= \sum_{\ell(\tau)=\ell(\sigma)-1} \left(\sum_{C(\tau)} \prod_{i=1}^{\ell(\tau)} \mathfrak{D}^0_{s_i}(f^{r-1}) \right) \mathfrak{D}^0_{\tau,\sigma}(f).
\end{aligned}$$

Now note that $\mathfrak{D}^0_{\tau,\sigma} = \mathfrak{D}^0_s$ if $\sigma = \tau s$, and otherwise $\mathfrak{D}^0_{\tau,\sigma} = 0$. This completes the proof. ☐

Theorem 6.6. *Let $v \in V$ be standard and let $\gamma \in \Gamma$. Then the Jordan form of the matrix $[\gamma]$ consists of Jordan blocks of size at most $\ell(\omega_0^v) + 1$ and eigenvalue $\gamma(v)$. Furthermore, there is at most one block of this maximal size, and for a generic element γ of Γ there is exactly one such block.*

Proof. Set $r = \ell(\omega_0^v)$. The formula for the action of Γ given in Theorem 6.2 implies that $(\gamma - \gamma(v))\mathcal{D}_\sigma^v$ is a linear combination of $\mathfrak{D}_\tau^v$ with $\ell(\tau) < \ell(\sigma)$. It follows that

$$(\gamma - \gamma(v))^{\ell(\sigma)+1}\mathcal{D}_\sigma^v = 0$$

so $\gamma(v)$ is the only possible eigenvalue of γ acting on the space $\mathcal{D}(\Omega, v)$, and $\ker(\gamma - \gamma(v))^r$ is contained in the linear span of $\mathcal{D}_{\omega_0^v}^v$. This proves that all Jordan blocks are of size at most $r + 1$, and that there is at most one block of this size. We next show that the Jordan form of $[\gamma]$ has generically one such block.

Denote by N the subset of $\Gamma / \operatorname{ann} \mathcal{D}(\Omega, v)$ consisting of the coclasses of those $\gamma \in \Gamma$ whose Jordan form contains only blocks of size strictly smaller than $r+1$. Equivalently, this is the set of coclasses of γ such that $(\gamma - \gamma(v))^r \mathcal{D}(\Omega, v) = 0$, and this set is a Zariski closed subset of $\Gamma / \operatorname{ann} \mathcal{D}(\Omega, v)$. Now let $S_v \subset W_v$ be the set of all simple transpositions in W_v and let $\mathfrak{S} = \sum_{s \in S_v} \mathfrak{S}_s \in (\Lambda / I_W)^{W_v}$. Furthermore, let $\gamma \in \Gamma$ be such that $\pi \circ t_v(\gamma) = \mathfrak{S}$, which exists by Lemma 6.1. Then $\gamma(v) = 0$ and

$$\gamma^r \cdot \mathcal{D}_{\omega_0^v}(v) = \mathcal{D}_{e, \omega_0^v}^v(\gamma^r)\mathcal{D}_e^v = \mathfrak{D}_{\omega_0^v}^0(\mathfrak{S}^r)\mathcal{D}_e^v.$$

Now, by Lemma 6.5, we have

$$\mathfrak{D}_{\omega_0^v}^0(\mathfrak{S}^r) = \sum_C \prod_{i=1}^r \mathfrak{D}_{s_i}^0(\mathfrak{S}) = \sum_C \prod_{i=1}^r \mathcal{I}_{S_v}(s_i)$$

where the sum is over all reduced decompositions $s_1 \cdots s_r$ of ω_0^v and $\mathcal{I}_{S_v}$ is the indicator function of the set S_v (that is, $\mathcal{I}_{S_v}$ is 1 over S_v and 0 over the complement of S_v). Thus, the product $\prod_{i=1}^r \mathcal{I}_{S_v}(s_i)$ is zero unless each s_i in the reduced decomposition lies in S_v. In view of [Hum90, 1.10 Proposition (b)], there is at least one such reduced decomposition and hence $\mathfrak{D}_{\omega_0^v}^0(\mathfrak{S}^r) \in \mathbb{Z}_{>0}$. This shows that $\gamma \notin N$ and hence N is a Zariski closed proper subset of $\Gamma / \operatorname{ann} \mathcal{D}(\Omega, v)$. Thus the complement of N is dense. □

7. Action of a co-rational Galois order

In this section G is a reflection group acting on V, and hence on $\Lambda = S(V)$ and on its field of rational functions $L = \operatorname{Frac}(\Lambda)$. We fix a co-rational Galois order $U \subset (L\#V)^G$ and denote by $Z \subset V$ the additive monoid generated by $\operatorname{supp} U$.

We assume again that $\Phi \subset V^*$ is a root system with base Σ and $G = W(\Phi)$. We denote by Ψ a standard subsystem with base $\Omega \subset \Sigma$ and set $W = W(\Psi)$. All Schubert polynomials and Postnikov-Stanley differential operators appearing in this section are defined with respect to Ω unless otherwise stated.

7.1. Recall that for each $\sigma \in G$ we introduced a divided difference operator as an element of the smash product $L\#G$. Since $\operatorname{End}_{\mathbb{C}}(L)$ is an $(L\#G)$-module, given $X \in \operatorname{End}_{\mathbb{C}}(L)$ and $\sigma \in G$, we obtain a new operator on L by taking $\partial_\sigma(X)$. Notice that, in general, this operator is *different* from the composition of ∂_σ (regarded as an element of $\mathrm{End}_{\mathbb{C}}(L)$) and X. In the following lemma we collect some properties of these operators.

Lemma 7.1. *Let $X \in \mathrm{End}_{\mathbb{C}}(L)$.*
(a) For each $\sigma \in G$ we have $\partial_\sigma(X)|_K = \partial_\sigma \circ X|_K$.
(b) Let $v \in V$ be Ω-standard. If $\sigma \in W^v$ and $\tau \in W_v$ then

$$\mathfrak{D}^v_\sigma \circ \partial_\tau = \begin{cases} \mathfrak{D}^v_{\sigma\tau} & \text{if } \ell(\sigma\tau) = \ell(\sigma) + \ell(\tau); \\ 0 & \text{otherwise.} \end{cases}$$

(c) Let $\widetilde{\Psi} \subset \Psi$ be a standard subsystem, $W_\theta \subset W$ be the corresponding parabolic subgroup, ω_0^θ be the longest word in W^θ, and $\Delta(\Psi)^\theta := \Delta(\Psi)/\Delta(\widetilde{\Psi})$. If $X \in \mathrm{End}_{\mathbb{C}}(L)^{W_\theta}$, then

$$\sum_{\sigma \in W} \sigma \cdot X = |W_\theta| \partial_{\omega_0^\theta}(X\Delta(\Psi)^\theta).$$

Proof. We prove part (a) by induction on $\ell(\sigma)$. If σ is the identity then the result is obvious. Assume now that $\sigma = s\tau$ with $\ell(\sigma) = 1 + \ell(\tau)$ and $s \in S$, and that the statement holds for τ. Setting $X' = \partial_\tau(X)$, we obtain

$$\begin{aligned} \partial_\sigma(X)(f) = \partial_s(X')(f) &= \frac{1}{\alpha_s}(X'(f) - s \circ X' \circ s(f)) = \frac{1}{\alpha_s}(X'(f) - s(X'(f))) \\ &= \partial_s(X'(f)) = \partial_s(\partial_\tau(X(f))) = \partial_\sigma(X(f)) \end{aligned}$$

which is the desired indentity.

We now prove part (b). The fact that $\tau \in W_v$ implies that $t_v \circ \partial_\tau = \partial_\tau \circ t_v$. Now recall from Proposition 3.2 that $\mathfrak{D}^0_\sigma = \mathsf{ev}_0 \circ \partial_\sigma$, so

$$\begin{aligned} \mathfrak{D}^v_\sigma \circ \partial_\tau &= \mathfrak{D}^0_\sigma \circ \partial_\tau \circ t_v \\ &= \mathsf{ev}_0 \circ \partial_\sigma \circ \partial_\tau \circ t_v = \begin{cases} \mathsf{ev}_0 \circ \partial_{\sigma\tau} \circ t_v = \mathfrak{D}^v_{\sigma\tau} & \text{if } \ell(\sigma\tau) = \ell(\sigma) + \ell(\tau); \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Finally we prove part (c). The statement of [Hil82, Chapter IV (1.6)] implies that $\partial_{\omega_0} = \frac{1}{\Delta(\Phi)} \sum_{\sigma \in G} (-1)^{\ell(\sigma)} \sigma$ as operators on L, and since the map $L\#G \longrightarrow \mathrm{End}_{\mathbb{C}}(L)$ is injective, the identity holds in $L\#G$. Using that and the fact that $\sigma \cdot \Delta(\Phi) = (-1)^{\ell(\sigma)} \Delta(\Phi)$ we deduce that $\sum_{\sigma \in G} \sigma \cdot X = \partial_{\omega_0}(X\Delta(\Phi))$ for any $X \in \mathrm{End}_{\mathbb{C}}(L)$. Certainly, the analogous identity holds if we replace G by any subgroup and Φ by the corresponding root subsystem.

Let ω_0 and ω_1 be the longest elements of W and W_θ, respectively. Then $\omega_0\omega_1^{-1} \in \omega_0 W_\theta$ and its length equals to $\ell(\omega_0) - \ell(\omega_1)$, the smallest possible length of an element in the coset $\omega_0 G_\theta$. Thus $\omega_0^\theta = \omega_0\omega_1^{-1}$ and

$$\sum_{\sigma \in W} \sigma \cdot X = \partial_{\omega_0}(X\Delta(\Psi)) = \partial_{\omega_0^\theta} \partial_{\omega_1}(X\Delta(\widetilde{\Psi})\Delta(\Psi)^\theta).$$

Now both $\Delta(\Psi)^\theta$ and X are W_θ-invariant, so the last expression equals

$$\partial_{\omega_0^\theta}(X\Delta(\Psi)^\theta \partial_{\omega_1}(\Delta(\widetilde{\Psi}))) = |W_\theta| \partial_{\omega_0^\theta}(X\Delta(\Psi)^\theta),$$

which completes the proof. $\square$

Recall that for each $z \in V$ there exists some Ω-standard element in the orbit $W \cdot z$. Thus, given $Z \subset V$ that is stable by the action of W, we can choose a set of Ω-standard representatives of Z/W. The following proposition shows how this fact can be used to express elements of U in different ways.

Proposition 7.2. *Let $X \in (L\#V)^G$ and assume that there exists $\chi \in \hat{G}$ such that $d_\chi X \in \Lambda\#V$.*

(a) For each $z \in \operatorname{supp} X$ there exists $f_z \in \Lambda^{G_z}$ such that

$$X = \sum_{z \in \operatorname{supp} X} \frac{f_z}{d_\chi^z} t_z,$$

where d_χ^z is the product of all $\alpha \in \Phi^+$ dividing d_χ such that $\alpha(z) \neq 0$.

(b) Let Y be a set of Ω-standard representatives of $\operatorname{supp} X/W$, and for each $y \in Y$ denote by ω_0^y the longest element in W^y, and by $\Delta(\Psi)^y$ the product of all roots in Ψ^+ with $\alpha(y) \neq 0$. Then

$$X = \sum_{y \in Y} \frac{1}{|W_y|} \partial_{\omega_0^y} \left(\frac{f_y \Delta(\Psi)^y}{d_\chi^y} t_y \right).$$

Proof. Fix $z \in \operatorname{supp} X$ and let h be the coefficient of t_z in X, which is well defined by Lemma 5.1. Since X is G-invariant we know that $\sigma \cdot X = X$ for any $\sigma \in G_z$, so $\sigma(h) = h$. Writing $h = \frac{g}{d_\chi}$ we have

$$\frac{g}{d_\chi} = \sigma \cdot \frac{g}{d_\chi} = \frac{\sigma \cdot g}{\chi(g) d_\chi}.$$

Therefore, $\sigma \cdot g = \chi(\sigma) g$ for all $\sigma \in G_z$.

Denote by χ' the restriction of χ to G_z. Observe that G_z is the reflection group generated by the reflections fixing z and it acts on Λ by restriction. Thus, by Stanley's theorem, the space of relative invariants $\Lambda_{\chi'}^{G_z}$ is generated over Λ^{G_z} by $d_{\chi'}$, and this polynomial is the product of all roots $\alpha \in \Phi^+$ dividing d_χ such that $\alpha(z) = 0$. Therefore, $g = f_z d_\chi$ for some $f_z \in \Lambda^{G_z}$, which implies that $\frac{g}{d_\chi} = \frac{f_z}{d_\chi / d_{\chi'}} = \frac{f_z}{d_\chi^z}$. This proves part (a).

Since X is G-invariant, it is clear that

$$X = \frac{1}{|W|} \sum_{\sigma \in W} \sigma \cdot X = \sum_{y \in Y} \frac{1}{|W|} \sum_{\sigma \in W} \sigma \cdot \left(\frac{f_y}{d_\chi^y} t_y \right).$$

As we mentioned before, the coefficient of t_y is G_y-invariant, and hence it is W_y-invariant. After applying Lemma 7.1(c) to W, we obtain

$$\sum_{\sigma \in W} \sigma \cdot \left(\frac{f_y}{d_\chi^y} t_y \right) = |W^y| \partial_{\omega_0^y} \left(\frac{f_y \Delta(\Psi)^y}{d_\chi^y} t_y \right)$$

and the result follows. $\square$

7.2. U-submodule of Γ^* associated to v. Recall that to each $v \in V$ we associate the character $\mathsf{ev}_v : \Gamma \longrightarrow \mathbb{C}$ given by evaluation at v. Since Γ consists of G-symmetric polynomials, $\mathsf{ev}_v = \mathsf{ev}_{\sigma(v)}$ for any $\sigma \in G$, so we can assume that v is Ω-standard. Furthermore, note that $\mathsf{ev}_v = \mathcal{D}_e^v \in \mathcal{D}(\Omega, v) \subset \Gamma^*$.

Definition 7.3. *Let $v \in V$ be standard. We denote by $V(\Omega, T(v))$ the space $\sum_{z \in \mathsf{Z}} \mathcal{D}(\Omega, v+z)$.*

Recall that $\Phi_0(v)$ is the set of all roots in Φ such that $\alpha(v) = 0$. The following theorem shows that under certain conditions the space $V(\Omega, T(v))$ is a U-module. This theorem generalizes [EMV, Theorem 10] and [RZ17, 5.6 Theorem] to rational Galois orders.

Theorem 7.4. *Let $v \in V$ be standard and assume that $\Phi_0(v+z) \subset \Psi$ for each $z \in \mathsf{Z}$. Then $V(\Omega, T(v)) \subset \Gamma^*$ is a Gelfand-Tsetlin U-module.*

Proof. By Theorem 6.2, the action of Γ on $V(\Omega, T(v))$ is locally finite, so we only need to show that it is a U-submodule of Γ^*. By definition, U is generated by a finite set $\mathcal{X}$ such that any element $X \in \mathcal{X}^\dagger$ satisfies the hypothesis of Proposition 7.2. Hence it is enough to prove the following: for each $z' \in \mathsf{Z}$, each $\sigma \in G$, and each X satisfying the hypothesis of Proposition 7.2, we have $\mathcal{D}_\sigma^{v+z'} \circ X \in V(\Omega, T(v))$. We will prove this in several steps.

First, let v' be a standard element in the W-orbit of $v+z'$. Since $\mathcal{D}(\Omega, v') = \mathcal{D}(\Omega, v+z')$ by Proposition 6.4(d), the statement in the theorem is equivalent to showing that $\mathcal{D}_\sigma^{v'} \circ X \in V(\Omega, T(v))$. Now let $\widetilde{W} = W_{v'}$ and let $\widetilde{\Psi} = \Psi_0(v')$ be the associated standard root subsystem. By Proposition 7.2(b), X can be written as a sum of operators of the form $\partial_{\widetilde{\omega}_0^z}(F_z t_z)$ for $z \in \mathsf{Z}$, where $\widetilde{\omega}_0^z$ is the longest element of $\widetilde{W}^z$ and $F_z = \frac{f_z \Delta(\widetilde{\Psi})^z}{d_\chi^z}$. Thus

$$\mathcal{D}_\sigma^{v'} \circ X = \sum_{z \in Y} \frac{1}{|\widetilde{W}_z|} \mathfrak{D}_\sigma^{v'} \circ \partial_{\widetilde{\omega}_0^z}(F_z t_z)|_\Gamma,$$

where Y is a set of $\widetilde{\Omega}$-standard representatives of $\operatorname{supp} X/\widetilde{W}$. So, it is enough to show that $\mathfrak{D}_\sigma^{v'} \circ \partial_{\widetilde{\omega}_0^z}(F_z t_z)|_\Gamma \in V(\Omega, T(v))$ for any $z \in Y$.

We claim that F_z is regular at v'. Recall that d_χ^z is the product of all roots α_s such that $\chi(s) = -1$ and $\alpha_s(z) \neq 0$. If one of this factors is such that $\alpha_s(v') = 0$ then $\alpha_s \in \Phi_0(v') = \Phi_0(\tau(v+z')) = \tau(\Phi_0(v+z'))$ for some $\tau \in W$. Now since $\Phi_0(v+z') \subset \Psi$ by hypothesis, and since Ψ is stable by the action of W, it follows that $\Phi_0(v') \subset \Psi$, and hence α_s is also a factor of $\Delta(\widetilde{\Psi})^z$. Thus the term $\Delta(\widetilde{\Psi})^z$ in the numerator cancels out all the linear terms in the denominator which are zero at v'. This proves that F_z is regular at v'.

We make one further simplification. By parts (a) and (b) of Lemma 7.1,

$$\begin{aligned}\mathfrak{D}_\sigma^{v'} \circ \partial_{\widetilde{\omega}_0^z}(F_z t_z)|_\Gamma &= \mathfrak{D}_\sigma^0 \circ t_{v'} \circ \partial_{\widetilde{\omega}_0^z} \circ F_z t_z|_\Gamma = \mathfrak{D}_\sigma^0 \circ \partial_{\widetilde{\omega}_0^z} \circ t_{v'}(F_z) t_{v'+z}|_\Gamma \\ &= \begin{cases} \mathfrak{D}_{\sigma\widetilde{\omega}_0^z}^0 \circ t_{v'}(F_z) t_{v'+z} & \text{if } \ell(\sigma\widetilde{\omega}_0^z) = \ell(\sigma) + \ell(\widetilde{\omega}_0^z); \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Here we have used that t_v and $\partial_{\widetilde{\omega}_0^z}$ commute since $\widetilde{\omega}_0^z \in W_{v'}$. If the result above is 0 then we are done. On the other hand, since F_z is regular at v' then $t_{v'}(F_z)$ is regular at 0. So, writing $t_{v'}(F_z) = \sum_{\rho \in W} (t_{v'}(F_z))_{(\rho)} \mathfrak{S}_\rho$ and recalling from Proposition 3.2 that $(t_{v'}(F_z))_{(\rho)}(0) = \mathfrak{D}^0_\rho(t_{v'}(F_z)) = \mathfrak{D}^{v'}_\rho(F_z)$, we obtain that

$$\mathfrak{D}^{v'}_\sigma \circ \partial_{\widetilde{\omega}_0^z}(F_z t_z)|_\Gamma = \begin{cases} \sum_{\rho \in W} \mathfrak{D}^{v'}_\rho(F_z)(\mathfrak{D}^0_{\sigma\widetilde{\omega}_0^z} \circ \mathfrak{S}_\rho t_{v'+z})|_\Gamma & \text{if } \ell(\sigma\widetilde{\omega}_0^z) = \ell(\sigma) + \ell(\widetilde{\omega}_0^z); \\ 0 & \text{otherwise.} \end{cases}$$

Finally, let $\gamma \in \Gamma$. Then

$$\begin{aligned} (\mathfrak{D}^0_{\sigma\widetilde{\omega}_0^z} \circ \mathfrak{S}_\rho t_{v'+z})(\gamma) &= \mathfrak{D}^0_{\sigma\widetilde{\omega}_0^z}(\mathfrak{S}_\rho t_{v'+z}(\gamma)) \\ &= \sum_{\nu \in W} t_{v'+z}(\gamma)_{(\nu)}(0) \mathfrak{D}^0_{\sigma\widetilde{\omega}_0^z}(\mathfrak{S}_\rho \mathfrak{S}_\nu) \\ &= \sum_{\nu \in W} c^{\sigma\widetilde{\omega}_0^z}_{\rho,\nu} \mathfrak{D}^{v+z'}_\nu(\gamma). \end{aligned}$$

Using the identities above, we obtain

$$\begin{aligned} \mathcal{D}^{v'}_\sigma \circ X &= \sum_{\substack{z \in Y \\ \ell(\sigma\widetilde{\omega}_0^z) = \ell(\sigma) + \ell(\widetilde{\omega}_0^z)}} \frac{1}{|\widetilde{W}_z|} \sum_{\rho,\nu \in W} c^{\sigma\widetilde{\omega}_0^z}_{\rho,\nu} \mathfrak{D}^{v'}_\rho(F_z) \mathcal{D}^{v'+z}_\nu \\ &= \sum_{\substack{z \in Y \\ \ell(\sigma\widetilde{\omega}_0^z) = \ell(\sigma) + \ell(\widetilde{\omega}_0^z)}} \frac{1}{|\widetilde{W}_z|} \sum_{\nu \in W} \mathfrak{D}^{v'}_{\nu,\sigma\widetilde{\omega}_0^z}(F_z) \mathcal{D}^{v'+z}_\nu. \end{aligned}$$

Now $v' + z = \tau(v + z') + z = \tau(v + z' + \tau^{-1}(z))$ and hence $\mathcal{D}^{v'+z}_\nu \in V(\Omega, T(v))$. □

8. Standard Galois orders of type A

In this section we consider a special type of Galois order, for which we find a basis of Postnikov-Stanley operators for the module introduced in Theorem 7.4. We also give a sufficient condition for the simplicity of this module.

8.1. Given $\mu = (\mu_1, \dots, \mu_r) \in \mathbb{N}^r$ we set $\mathbb{C}^\mu = \mathbb{C}^{\mu_1} \times \cdots \times \mathbb{C}^{\mu_r}$ and $\mathbb{I} = \mathbb{I}(\mu) = \{(k,i) \mid 1 \leq k \leq r, 1 \leq i \leq \mu_k\}$. Also, for each $v \in \mathbb{C}^\mu$ and $(k,i) \in \mathbb{I}$, we will denote by v_k the projection of v to the component $\mathbb{C}^{\mu_k}$, and by $v_{k,i}$ the i-th coordinate of v_k. We will denote by $e_{k,i}$ the vector of $\mathbb{C}^\mu$ with $(e_{k,i})_{l,j} = \delta_{k,l}\delta_{i,j}$, and refer to the set $\{e_{k,i} \mid (k,i) \in \mathbb{I}\}$ as the *canonical basis* of $\mathbb{C}^\mu$. We denote by $\{x_{k,i} \mid (k,i) \in \mathbb{I}\}$ the dual basis to the canonical basis, so $\mathbb{C}[X_\mu] = \mathbb{C}[x_{k,i} \mid (k,i) \in \mathbb{I}]$ is the algebra of polynomial functions over $\mathbb{C}^\mu$. We denote the fraction field of this algebra by $\mathbb{C}(X_\mu)$. For each $(k,i) \in \mathbb{I}$ we write $t_{k,i}$ for the automorphism $t_{e_{k,i}} \in \mathrm{End}_\mathbb{C}(\mathbb{C}(X_\mu))$.

For each $1 \leq j \leq r$ the symmetric group S_{μ_j} acts on $\mathbb{C}^{\mu_j}$ by permuting the coordinates of a vector, and hence $S_\mu = S_{\mu_1} \times \cdots \times S_{\mu_r}$ acts on $\mathbb{C}^\mu$. This is a reflection group corresponding to the root system $\Phi = \{x_{k,i} - x_{k,j} \mid (k,i),(k,j) \in \mathbb{I}\}$. We fix $\Sigma = \{x_{k,i} - x_{k,i+1} \mid 1 \leq k \leq r, 1 \leq i < \mu_k\}$ as a base of Φ. Given $\sigma \in S_\mu$ we will denote by $\sigma[k]$ its projection to S_{μ_k}. Also, given $\tau \in S_{\mu_k}$ we will denote by $\tau^{(k)}$ the unique element of S_μ such that $\tau^{(k)}[k] = \tau$ and $\tau^{(k)}[l] = \mathsf{Id}_{S_{\mu_l}}$ for $l \neq k$. We denote by

$\mathrm{sym}_k = \frac{1}{\mu_k!}\sum_{\sigma \in S_{\mu_k}} \sigma^{(k)} \in \mathbb{C}[S_\mu]$, and $\Delta_k = \prod_{1\leq i<j\leq \mu_k}(x_{k,i} - x_{k,j})$. Notice that Δ_k is the generator of the space of relative invariants associated to the character $\mathrm{sg}[k]$ given by $\mathrm{sg}[k](\sigma) = \mathrm{sg}(\sigma[k])$.

The action of S_μ on $\mathbb{C}^\mu$ induces actions on $\mathbb{C}[X_\mu]$ and $\mathbb{C}(X_\mu)$, so we may consider Galois orders in $(\mathbb{C}(X_\mu)\#\mathbb{C}^\mu)^{S_\mu}$. The following definition distinguishes a special class of such rational Galois orders.

Definition 8.1. *Let $\mu = (\mu_1, \dots, \mu_r) \in \mathbb{N}$ and let $U \subset (\mathbb{C}(X_\mu)\#\mathbb{C}^\mu)^{S_\mu}$ be a Galois order. We will say that U is a* standard Galois order of type A *if it is generated by $\mathbb{C}[X_\mu]^{S_\mu}$ and a set $\mathcal{X} = \{X_k^\pm \mid 1 \leq k \leq r'\}$ for some $r' \leq r$ such that*

$$X_k^\pm = \mathrm{sym}_k\left(t_{\pm e_{k,1}} \frac{f_k^\pm}{\prod_{j=2}^{\mu_k}(x_{k,1} - x_{k,j})}\right).$$

Remark. *As indicated earlier, by definition, a standard Galois order of type A is not necessarily a standard Galois order in the sense of Hartwig's definition, in* [Har, Definition 2.30].

Notice that in the definition above $X_k^\pm \Delta_k \in \mathbb{C}[X_\mu]\#\mathbb{Z}^\mu$ so U is a co-rational Galois order. From now on set $\overline{\mu} = (\mu_1, \dots, \mu_{r'}, 0, \dots, 0) \in \mathbb{N}^r$. By definition, $\operatorname{supp} U = \mathbb{Z}^{\overline{\mu}}$ for any U which is a standard Galois order of type A.

Example. *As discussed in* [Har, §4.2], *finite W-algebras of type A are co-rational Galois orders. The explicit formulas given in that paragraph show that they are in fact standard Galois orders of type A. Simmilarly the formulas from* [Har, §4.4] *show that orthogonal Gelfand-Tsetlin algebras, introduced by Mazorchuk in* [Maz99], *are also examples of standard Galois orders of type A.*

8.2. Modules of the form $V(\Omega, T(v))$. Fix $\mu \in \mathbb{N}^r$ and let $U \subset (\mathbb{C}[X_\mu]\#\mathbb{C}^\mu)^{S_\mu}$ be a standard Galois order of type A. We will denote by $\overline{\Phi}$ the root system $\{x_{k,i} - x_{k,j} \mid 1 \leq k \leq r', 1 \leq i < j \leq \mu_k\}$, and by $\overline{\Sigma}$ the base $\Sigma \cap \overline{\Phi}$.

Given $v \in \mathbb{C}^\mu$ we set $\Psi(v) = \{\alpha \in \overline{\Phi} \mid \alpha(v) \in \mathbb{Z}\}$. We will say that v is a *seed* if Ψ is a standard root subsystem of $\overline{\Phi}$ and $\Psi(v) = \overline{\Phi}_0(v)$; notice that this second equality is equivalent to $W_v = W(\Psi(v))$. We claim that for every element $v \in \mathbb{C}^\mu$ there exists a seed $\overline{v}$ of the form $\sigma(v) + z$ for some $z \in \mathbb{Z}^{\overline{\mu}}$ and some $\sigma \in S_{\overline{\mu}}$. Indeed, since $\Psi(v)$ is a root subsystem of $\overline{\Phi}$, there exists $\sigma \in S_{\overline{\mu}}$ such that $\sigma(\Psi(v)) = \Psi(\sigma^{-1}(v))$ is a standard subsystem. In other words, $v' = \sigma^{-1}(v)$ has the property that if $v'_{k,i} - v'_{k,j} \in \mathbb{Z}$ for some $(k,i),(k,j) \in \mathbb{I}(\overline{\mu})$ with $i < j$, then $v'_{k,s} - v'_{k,s+1} \in \mathbb{Z}$ for any $i \leq s < j$. It follows that there exists $z \in \mathbb{Z}^{\overline{\mu}}$ such that $v'' = v' + z$ has an even stronger property: if $v''_{k,i} - v''_{k,j} \in \mathbb{Z}$ for some $(k,i),(k,j) \in \mathbb{I}(\overline{\mu})$ with $i < j$, then $v''_{k,s} = v''_{k,s+1}$ for any $i \leq s < j$, or equivalently v'' is seed.

Fix a seed $\overline{v}$, and set $\Psi = \Psi(\overline{v})$ and $\Omega = \Psi(\overline{v}) \cap \overline{\Sigma}$. We denote by $\mathsf{Z}(\overline{v})$ the set of all $z \in \mathbb{Z}^{\overline{\mu}}$ such that $\alpha(z) \geq 0$ for all $\alpha \in \Omega$. This are the integral points in the fundamental domain of the system Ω seen as a root system over the real vector space $\mathbb{R}^\mu \subset \mathbb{C}^\mu$, see [Hum90, §1.12]. Also, for each $z \in \mathsf{Z}(\overline{v})$, we define an equivalence relation $\sim_z$ on $\mathbb{I}(\overline{\mu})$, by letting $(k,i) \sim_z (l,j)$ if and only if $l = k$ and $(\overline{v} + z)_{k,i} = (\overline{v} + z)_{k,j}$. Denote by $\mathbb{I}(\overline{\mu}, z)$ the set of all equivalence classes of this equivalence relation. Each equivalence class $I \in \mathbb{I}(\overline{\mu}, z)$ is by definition a set of the form $\{(k,i),(k,i+1),\ldots,(k,j)\}$ for some $1 \le i < j \le \mu_k$. We will write $a^+(I)$ for (k,i) and $a^-(I)$ for (k,j), i.e. the first and last elements of I, respectively, with respect to the lexicographic order.

Lemma 8.2. *Let $\overline{v} \in \mathbb{C}^\mu$ be a seed, $\Psi = \Psi(\overline{v}), \Omega = \Psi \cap \overline{\Sigma}$ and $W = W(\Psi)$.*

(i) If $z \in \mathbb{Z}^\mu$ then $\overline{v} + z$ is Ω-standard if and only if $z \in \mathsf{Z}(\overline{v})$.
(ii) If $z, z' \in \mathsf{Z}(\overline{v})$ and $\overline{v} + z = \sigma(\overline{v} + z')$ for some $\sigma \in S_\mu$, then $z = z'$.
(iii) If $z \in \mathsf{Z}(\overline{v})$, then $z \pm \delta^{k,i} \in \mathsf{Z}(\overline{v})$ if and only if $(k,i) = a^\pm(I)$ for some $I \in \mathbb{I}(\overline{\mu}, z)$.

Proof. The definition of a seed implies that $\alpha \in \Psi$ if and only if $\alpha(\overline{v}) = 0$. Now $\overline{v} + z$ is Ω-standard if and only if $\alpha(\overline{v} + z) = \alpha(z) \ge 0$ for all $\alpha \in \Omega$. Hence part (i) follows immediately from these definitions.

Let now $\overline{v} + z = \sigma(\overline{v} + z')$; since $z \in \mathbb{Z}^{\overline{\mu}}$, we can assume that $\sigma \in S_{\overline{\mu}}$. Then $\overline{v} - \sigma(\overline{v}) = \sigma(z') - z$, and so $\overline{v}_{k,i} - \sigma(\overline{v})_{k,i} = \overline{v}_{k,i} - \overline{v}_{k,\sigma[k]^{-1}(i)} \in \mathbb{Z}$ for all $(k,i) \in \mathbb{I}(\overline{\mu})$. By the definition of a seed this is possible if and only if $\sigma(\overline{v}) = \overline{v}$, so $z = \sigma(z')$. As mentioned above, $z, z' \in \mathsf{Z}(\overline{v})$ is equivalent to the property that $\alpha(z), \alpha(z') \in \mathbb{Z}_{\ge 0}$ for all $\alpha \in \Omega$, and by [Hum90, 1.12 Theorem, part (a)] there is exactly one element in $W \cdot z$ with this property, so $z = z'$ and part (ii) is proved.

Finally it is easy to check that $z \in \mathsf{Z}(\overline{v})$ if and only if for each $I' = \{(k,i'),(k,i'+1),\ldots,(k,j')\} \in \mathbb{I}(\overline{\mu}, \overline{v})$ we have $z_{k,i'} \ge z_{k,i'+1} \ge \cdots \ge z_{k,j'}$. Thus if $z + \delta^{k,i} \in \mathsf{Z}(\overline{v})$ then either $i = 1$ or $z_{k,i-1} > z_{k,i} \ge z_{k,i+1}$. In either of the two cases there exists $I \in \mathbb{I}(\overline{\mu}, \overline{v} + z)$ with $a^+(I) = (k,i)$. A similar argument shows that if $z - \delta^{k,i} \in \mathsf{Z}(\overline{v})$ then there must exist an I such that $a^-(I) = (k,i)$ and part (iii) is proved. □

We are now ready to prove the following result that generalizes [RZ17, 5.6 Theorem] and [EMV, Theorem 10] to integral Galois algebras of type A. For the sake of comparison, we note that the sets $\mathsf{Z}(\overline{v})$ and W^z in the following theorem correspond respectively to the sets $\{\xi_j \mid j \in J\}$ and X_j defined in [EMV], and to the sets $\mathcal{N}_\eta$ and $\mathsf{Shuffle}^\eta_{\epsilon(z)}$ defined in [RZ17].

Theorem 8.3. *Let $\overline{v} \in \mathbb{C}^\mu$ be a seed, let $\Psi = \Psi(\overline{v})$, let $\Omega = \Psi \cap \overline{\Sigma}$, and let $W = W(\Psi)$. Then*

$$V(\Omega, T(\overline{v})) = \bigoplus_{z \in \mathsf{Z}(\overline{v})} \mathcal{D}(\Omega, \overline{v} + z).$$

In particular, the set $\{\mathcal{D}^{\overline{v}+z}_\sigma \mid z \in \mathsf{Z}(\overline{v}), \sigma \in W^z\}$ is a basis of $V(\Omega, T(\overline{v}))$ and $V(\Omega, T(\overline{v}))$ is a Gelfand-Tsetlin module over U with respect to Γ.

Proof. By definition $V(\Omega, T(\overline{v})) = \sum_{z \in \mathbb{Z}^{\overline{\mu}}} \mathcal{D}(\Omega, \overline{v} + z)$. Now by [Hum90, 1.12 Theorem] for each $z \in \mathbb{Z}^{\overline{\mu}}$ there exists $\sigma \in W$ such that $\sigma(z) \in \mathsf{Z}(\overline{v})$. Since W is the stabilizer of $\overline{v}$ it follows from part (d) of Proposition 6.4 that $\mathcal{D}(\Omega, \overline{v} + z) = \mathcal{D}(\Omega, \overline{v} + \sigma(z))$. Hence $V(\Omega, T(\overline{v})) = \sum_{z \in \mathsf{Z}(\overline{v})} \mathcal{D}(\Omega, \overline{v} + z)$. We next show that the sum is direct. Notice that the space $\mathcal{D}(\Omega, \overline{v} + z)$ consists of eigenvectors of $\Gamma = \mathbb{C}[X_\mu]^{S_\mu}$ with eigenvalue $\mathsf{ev}_{\overline{v}+z}$. If there exist $z, z' \in \mathsf{Z}(v)$ such that $\gamma(\overline{v} + z) = \gamma(\overline{v} + z')$ for all $\gamma \in \Gamma$ then $\overline{v} + z = \sigma(\overline{v} + z')$ for some $\sigma \in S_\mu$ and, by Lemma 8.2(ii), $z = z'$. Hence the sum is direct. The fact that the set in question is a basis follows from Proposition 6.4(a). □

8.3. Simplicity criterion. In this paragraph U denotes a standard Galois order of type A over $(\mathbb{C}[X_\mu]\#\mathbb{C}^\mu)^{S_\mu}$ with generators $X_k^\pm$ for $1 \leq k \leq r'$. By definition

$$(X_k^\pm)^\dagger = \mathrm{sym}_k \left(\frac{f_k^\pm}{\prod_{j=2}^{\mu_k}(x_{k,1} - x_{k,j})} t_{k,1}^{\mp 1} \right)$$

for some $f_k \in \mathbb{C}[X_\mu]^{H_k}$, where H_k is the stabilizer of $e_{k,1}$ in S_μ. Thus we have

$$(X_k^\pm)^\dagger = \sum_{i=1}^{\mu_k} \left(\frac{f_{k,i}^\pm}{\prod_{j\neq i}^{\mu_k}(x_{k,i} - x_{k,j})} t_{k,i}^{\mp 1} \right)$$

where $f_{k,1}^\pm = (1/\mu_k) f_k^\pm$ and $f_{k,j}^\pm = \sigma \cdot f_{k,1}^\pm$ for any $\sigma \in S_\mu$ such that $\sigma[k](1) = j$. For the rest of this paragraph $f_{k,i}^\pm$ will denote the polynomials appearing in the formulas displayed above.

Fix a seed $\overline{v}$, and let $\Psi = \Psi(\overline{v})$ and $W = W(\Psi)$. For each $I = \{(k,i), \ldots, (k,j)\} \subset \Sigma(\overline{\mu})$ we denote by $S(I)$ the group of permutations of the set I. This is a parabolic subgroup of S_μ (usually called a *Young subgroup*) with minimal generating set $\{s_t^{(k)} \mid i \leq t \leq j-1\} \subset S_\mu$. Using this notation we have $W = \prod_{I\in\mathbb{I}(\overline{\mu},0)} S(I)$ and $W_z = \prod_{I\in\mathbb{I}(\overline{\mu},z)} S(I)$ for each $z \in \mathsf{Z}(\overline{v})$, which are Young subgroups of S_μ.

The following lemma describes the action of U on $V(\Omega, T(\overline{v}))$ in terms of the basis given in Theorem 8.3. In order to state the lemma we need to fix some notation which we will also use in the irreducibility criterion Theorem 8.5. Given $z \in \mathsf{Z}(\overline{v})$ and $1 \leq k \leq r'$ we will denote by $\mathbb{I}_k(\overline{\mu}, z)$ the subset of $\mathbb{I}(\overline{\mu}, z)$ consisting of sets of the form $I = \{(k,i), \ldots, (k,j)\}$. We also write $\sigma^+(I) = (j\ j-1\ \cdots\ i)^{(k)}$ and $\sigma^-(I) = (i\ i+1\ \cdots\ j)^{(k)}$.

Lemma 8.4. *Let $\overline{v} \in \mathbb{C}^\mu$ be a seed and let $z \in \mathsf{Z}(\overline{v})$. For each $1 \leq k \leq r'$ we have*

$$(X_k^\pm)^\dagger = \sum_{I\in\mathbb{I}_k(\overline{\mu},z)} \frac{1}{|\widetilde{W}_{a^\mp(I)}|} \partial_{\sigma^\mp(I)} \left(\frac{f_{a^\mp(I)}^\pm}{\prod_{(k,j)\notin I}(x_{a^\mp(I)} - x_{k,j})} t_{a^\mp(I)}^{\mp 1} \right);,$$

$$\mathcal{D}_\sigma^{\overline{v}+z} \circ (X_k^\pm)^\dagger = \sum_{I\in\mathbb{I}_k(\overline{\mu},z,\sigma^\mp(I))} \sum_{\tau\leq\sigma\sigma^\mp(I)} \mathfrak{D}_{\tau,\sigma\sigma^\mp(I)}^{\overline{v}+z} \left(\frac{f_{a^\mp(I)}^\pm}{\prod\limits_{(k,j)\notin I}(x_{a^\mp(I)} - x_{k,j})} \right) \mathcal{D}_\tau^{\overline{v}+z+\delta(\mp I)},$$

where $\mathbb{I}_k(\overline{\mu}, z, \sigma^\mp(I))$ is the subset of $\mathbb{I}_k(\overline{\mu}, z)$ consisting of all I such that $\ell(\sigma\sigma^\mp(I)) = \ell(\sigma) + \ell(\sigma^\mp(I))$ and $\delta(\mp I) = \delta^{a^\mp(I)}$.

Proof. Set $\widetilde{\Psi} = \Psi_0(\overline{v}+z), \widetilde{\Omega} = \Omega \cap \widetilde{\Psi}$, and $\widetilde{W} = W_z$. It is immediate that $\{\pm e_{a^\pm(I)} \mid I \in \mathbb{I}_k(\overline{\mu}, z)\}$ is a set of $\widetilde{\Omega}$-standard representatives of $\{\pm e_{k,1}, \ldots \pm e_{k,\mu_k}\}/\widetilde{W}$. Let $\widetilde{\omega}_0$ be the longest word in $\widetilde{W}$, and let $\widetilde{W}_{(k,t)}$ be the stabilizer of $e_{k,t}$ in $\widetilde{W}$. Then $\sigma^+(I)$ is the shortest element of the left coclass $\widetilde{\omega}_0 W_{(k,i)}$, while $\sigma^-(I)$ is the shortest element of the left coclass $\widetilde{\omega}_0 W_{(k,j)}$. Thus using part (b) of Proposition 7.2 we can rewrite $(X_k^\pm)^\dagger$ as in the statement, and the formula for $\mathcal{D}_\sigma^{\overline{v}+z} \circ (X_k^\pm)^\dagger$ is identical to the one obtained in Theorem 7.4. $\square$

Note that for each $z \in \mathsf{Z}(\overline{v})$ the longest word in W_z is $\prod_{I\in\mathbb{I}(\overline{\mu},z)} \omega_0(I)$, where $\omega_0(I)$ is the longest word in $S(I)$. We will say that $z' \in \mathsf{Z}(\overline{v})$ *refines* z if the following holds: for each $J \in \mathbb{I}(\overline{\mu}, z')$ there exists $I \in \mathbb{I}(\overline{\mu}, z)$ such that $J \subset I$. For instance this always happens if $z = \overline{v}$. If z' refines z then the longest element in $W_z^{z'}$ is equal to

$$\omega_0(z,z') = \prod_{I\in\mathbb{I}(\overline{\mu},z)} \omega_0(I) \prod_{J\in\mathbb{I}(\overline{\mu},z')} \sigma^+(J).$$

Now $\sigma \in W$ lies in W^z if and only if for each $I = \{(k,i),(k,i+1),\dots,(k,j)\}$ in $\mathbb{I}(\overline{\mu},z)$ we have $\sigma[k](i) < \sigma[k](i+1) < \cdots < \sigma[k](j)$. Hence if z' refines z then the longest word of $W^{z'}$ lies in W^z.

This observation will play a crucial role in the following simplicity criterion which generalizes the simplicity criterion for modules over orthogonal Gelfand-Tsetlin algebras [EMV, Theorem 11] to modules over standard Galois orders. Note that the non-integrality condition in that statement is equivalent to the condition $f^{\pm}_{k,i}(\overline{v}+z) \neq 0$ below when U is an orthogonal Gelfand-Tsetlin algebra.

Theorem 8.5. *Let $\overline{v}$ be a seed. If $f^{\pm}_{k,i}(\overline{v}+z) \neq 0$ for all $z \in \mathsf{Z}(\overline{v})$ and all $(k,i) \in \mathbb{I}(\overline{\mu})$ then $V(\Omega, T(\overline{v}))$ is an irreducible U-module.*

Proof. Set $V = V(\Omega, T(\overline{v}))$. We will show that any nonzero submodule $N \subset V$ is in fact equal to V. For each $z \in \mathsf{Z}(\overline{v})$ denote by $\pi^z : V \longrightarrow \mathcal{D}(\Omega, \overline{v}+z)$ the projection to the direct summand. We proceed in four steps.

Step 1. If $t \in V$ and $z \in \mathsf{Z}(\overline{v})$ are such that $\pi^z(t) \neq 0$, then $\mathcal{D}_e^{\overline{v}+z}$ is in the module Ut generated by t.

Proof of Step 1. First notice that Lemma 4.3 implies $\pi^z(t) \in Ut$. Now let $\mathfrak{m} = \ker \mathcal{D}_e^{\overline{v}+z} \subset \Gamma$. By Theorem 6.2 there exists a minimal $l \in \mathbb{N}$ such that $\mathfrak{m}^l \pi^z(t) = 0$, and part c of Proposition 6.4 implies that $\mathfrak{m}^{l-1}\pi^z(t) = \mathbb{C}\mathcal{D}_e^{\overline{v}+z'} \subset Ut$.

Step 2. $\mathcal{D}_e^{\overline{v}+z} \in N$ for all $z \in \mathsf{Z}(\overline{v})$.

Proof of Step 2. Step 1 implies that there exists $v' = \overline{v} + z'$ with $z' \in \mathsf{Z}(\overline{v})$ such that $\mathcal{D}_e^{v'} \in N$. To prove Step 2, we will show that if $\mathcal{D}_e^{\overline{v}+z} \in N$ then $\mathcal{D}_e^{\overline{v}+z\pm\delta^{k,i}} \in N$ for any $(k,i) \in \mathbb{I}(\overline{\mu})$ such that $v \pm \delta^{k,i} \in \mathsf{Z}(\overline{v})$. Indeed, by Lemma 8.4 and the definition of the action of a co-rational Galois order on Γ^*,

$$\begin{aligned}\pi^{z+\delta(\mp I)}\left(X_k^{\pm} \cdot \mathcal{D}_e^{\overline{v}+z}\right) &= \pi^{z+\delta(\mp I)}\left(\mathcal{D}_e^{\overline{v}+z} \circ (X_k^{\pm})^{\dagger}\right) \\ &= \sum_{\tau \leq \sigma^{\mp}(I)} \mathfrak{D}^{\overline{v}+z}_{\tau,\sigma^{\mp}(I)} \left(\frac{f^{\pm}_{a^{\mp}(I)}}{\prod\limits_{(k,j)\notin I} (x_{a^{\mp}(I)} - x_{k,j})} \right) \mathcal{D}_{\tau}^{\overline{v}+z+\delta(\mp I)}\end{aligned}$$

so the coefficient of $\mathcal{D}^{\overline{v}+\delta(\mp I)}_{\sigma^{\mp}(I)}$ is:

$$\frac{f^{\pm}_{a^{\mp}(I)}(\overline{v}+z)}{\prod\limits_{(k,j)\notin I} (\overline{v}_{a^{\mp}(I)} + z_{a^{\mp}(I)} - \overline{v}_{k,j} - z_{k,j})}.$$

Notice that this coefficient is well defined, since by the definition of $\mathbb{I}(\overline{\mu}, z)$ the denominator is nonzero. Also the hypothesis on $\overline{v}$ implies that the denominator nonzero, so $\pi^{z+\delta(\mp I)}(\mathcal{D}_e^{\overline{v}+z} \circ X_k^{\pm}) \neq 0$, and Step 1 implies that $\mathcal{D}_e^{\overline{v}+z+\delta(\mp I)} \in N$ for all $I \in \mathbb{I}(\overline{\mu}, z)$.

Step 3. $\mathcal{D}(\Omega, \overline{v}+z) \subset N$ for all non-critical z, i.e. for all $z \in \mathsf{Z}(\overline{v})$ such that for each $I = \{(k,i), \ldots, (k,j)\} \in \mathbb{I}(\overline{\mu}, \overline{v})$ we have $z_{k,i} > z_{k,i+1} > \cdots > z_{k,j}$.

Proof of Step 3. Notice that for non-critical z, the stabilizer of z is the trivial subgroup of W so the longest element in W^z is ω_0, the longest element of W. To prove Step 3, we build a sequence $z^{(0)}, z^{(1)}, \ldots$ of elements in $\mathsf{Z}(\overline{v})$ as follows. First set $z^{(0)} \in \mathbb{Z}^{\overline{\mu}}$ such that $z^{(0)}_{k,i} = \min\{z_{l,j} \mid (l,j) \in \mathbb{I}(\overline{\mu})\}$. Now suppose $z^{(s)}$ has been defined, and consider the set $L_s = \{(l,j) \in \mathbb{I}(\overline{\mu}) \mid z^{(s)}_{l,j} < z_{l,j}\}$. If $L_z = \emptyset$ then $z^{(s)} = z$ and we set $z^{(s+1)} = z$, otherwise we take (k_s, i_s) to be the minimal element in L_s with respect to the lexicographic order and set $z^{(s+1)} = z^{(s)} + \delta^{k_s, i_s}$. Clearly $z^{(s)} = z$ for $s \gg 0$.

We prove by induction that $\mathcal{D}^{\overline{v}+z^{(s)}}_{\omega_0^{(s)}} \in N$, where $\omega_0^{(s)}$ is the longest element in $W^{z^{(s)}}$. If $s = 0$ then by definition $z^{(0)}$ is a seed, and hence $\omega_0^{(0)} = e$. Since we already know that $\mathcal{D}_e^{\overline{v}+z^{(0)}} \in N$ the base case of the induction follows. Now take $s \geq 0$ and set $y = z^{(s)}, y' = z^{(s+1)}$ and $(k,i) = (k_s, i_s)$ so $y' = y + \delta^{k,i}$. The definition of (k,i) implies that there exists $j \leq \mu_k$ such that $I = \{(k,i), \ldots, (k,j)\} \in \mathbb{I}(\overline{\mu}, y)$, and also that $\{(k,i)\} \in \mathbb{I}(\overline{\mu}, y')$. It follows from the characterization of the longest word in W_z that $\omega_0^{(s+1)} = \omega_0^{(s)} \sigma^+(I)$. A simple computation shows that $\ell(\omega_0^{(s+1)}) = \ell(\omega_0^{(s)}) + \ell(\sigma^+(I))$ so using Lemma 8.4 as in the previous step and the fact that $\mathcal{D}^v_{\sigma,\sigma} = \mathsf{ev}_v$ for all $\sigma \in W$, we see that the coefficient of $\mathcal{D}^{\overline{v}+y'}_{\omega_0^{(s+1)}}$ in $\mathcal{D}^{\overline{v}+y}_{\omega_0^{(s)}} \circ (X_k^-)^\dagger$ is

$$\frac{f^{\pm}_{a^{\mp}(I)}(\overline{v}+y)}{\prod_{(k,j) \notin I} (\overline{v}_{a^{\mp}(I)} + y_{a^{\mp}(I)} - \overline{v}_{k,j} - y_{k,j})}.$$

and the hypothesis implies that this expression is nonzero. Hence by part (b) of Proposition 6.4 $\mathcal{D}(\Omega, \overline{v} + y') \subset N$, and in particular $\mathcal{D}^{\overline{v}+y'}_{\omega_0^{(s+1)}} \in N$.

Step 4. $\mathcal{D}(\Omega, \overline{v}+z) \subset N$ for arbitrary z. In particular, $N = V(\Omega, T(\overline{v}))$.

Proof of Step 4. Fix $z \in \mathsf{Z}(\overline{v})$. Then there exists non-critical $y^{(0)} \in \mathsf{Z}(\overline{v})$ such that $y^{(0)}_{k,i} \geq z_{k,i}$ for all $(k,i) \in \mathbb{I}(\overline{\mu})$. For each $s \geq 0$ set $y^{(s+1)}$ to be $y^{(s)} - \delta^{(k_s,i_s)}$, where (k_s, i_s) is the maximal element in the set $\{(k,i) \in \mathbb{I}(\overline{\mu}) \mid y^{(s)}_{k,i} > z_{k,i}\}$ with respect to the lexicographic order; and if this set is empty then we set $y^{(s+1)} = y^{(s)} = z$. We claim that $\mathcal{D}(\Omega, \overline{v} + y^{(s)}) \subset N$, and prove this by induction on s. Since $y^{(0)}$ is non-critical the base case of the induction follows from Step 3. Now assume that the inclusion holds for some $s \geq 0$, and set $(k,i) = (k_s, i_s)$. Since $\mathbb{I}(\overline{\mu}, y^{(s+1)})$ is a partition of the set $\mathbb{I}(\overline{\mu})$, there exists $I \in (\overline{\mu}, y^{(s+1)})$ such that $(k,i) \in I$. If $I = \{(k,i)\}$ then by construction $\mathbb{I}(\overline{\mu}, y^{(s)}) = \mathbb{I}(\overline{\mu}, y^{(s+1)})$; otherwise we have $\mathbb{I}(\overline{\mu}, y^{(s)}) = \mathbb{I}(\overline{\mu}, y^{(s+1)}) \cup \{I', (k,i)\} \setminus \{I\}$ for $I' = I \setminus \{(k,i)\}$. From this it follows that for each $I' \in \mathbb{I}(\overline{\mu}, y^{(s)})$ there exists $J \in \mathbb{I}(\overline{\mu}, y^{(s)})$ such that $I' \subset J$. Thus $y^{(s)}$ refines $y^{(s+1)}$, and this implies that the longest element of $W^{y^{(s+1)}}$ lies in $W^{y^{(s)}}$. If we denote this element by $\omega_0^{(s+1)}$ then using Lemma 8.4 and the hypothesis just as in the previous step we see that the coefficient of $\mathcal{D}^{\overline{v}+y^{(s+1)}}_{\omega_0^{(s+1)}}$ in $\pi^{\overline{v}+y^{(s+1)}}\left(\mathcal{D}^{\overline{v}+y^{(s)}}_{\omega_0^{(s+1)}} \circ (X_k^+)^\dagger\right)$ is not zero. Once again part (b) of Proposition 6.4 implies $\mathcal{D}(\Omega, \overline{v} + y^{(s+1)}) \subset N$, and since $y^{(s')} = z$ for $s' \gg 0$, Step 4 is proven. $\square$

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