# TANNAKIAN CATEGORIES IN POSITIVE CHARACTERISTIC

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ABSTRACT. We determine internal characterisations for when a tensor category is (super) tannakian, over fields of positive characteristic. This generalises the corresponding characterisations in characteristic zero by P. Deligne. We also explore notions of Frobenius twists in tensor categories in positive characteristic.

## INTRODUCTION

For a field $\Bbbk$, a tensor category over $\Bbbk$ is a $\Bbbk$-linear abelian rigid symmetric monoidal category where the endomorphism algebra of the tensor unit is $\Bbbk$ (contrary to some authors we do not require objects to have finite length). The standard example is the category $\mathrm{Rep}G$ of finite dimensional algebraic representations of an affine group scheme $G$ over $\Bbbk$. An affine group scheme over an algebraically closed field is determined (up to inner automorphism) by its representation category, so tensor categories can be seen as a generalisation of affine group schemes. This motivates looking for internal characterisations which determine when a tensor category is equivalent to such a representation category. By combining Deligne's results in [De1, De2, De3], we have the following theorem in characteristic zero.

**Theorem A** (Deligne)**.** *Let $\Bbbk$ be an algebraically closed field of characteristic zero and $\mathbf{T}$ a tensor category over $\Bbbk$. The following are equivalent.*

*(i) As a tensor category, $\mathbf{T}$ is equivalent to $\mathrm{Rep}G$ for some affine group scheme $G/\Bbbk$.*
*(ii) For each $X \in \mathbf{T}$ there exists $n \in \mathbb{N}$ such that $\Lambda^n X = 0$.*

It is known that the same statement does not hold true for fields of positive characteristic, see e.g. [BE, GK, GM]. Fix a field $\Bbbk$ of characteristic $p$ and a tensor category $\mathbf{T}$ over $\Bbbk$. Since the group algebra $\Bbbk\mathsf{S}_n$ is not semisimple when $n \geq p$, we need to distinguish between the symmetric power $\mathrm{Sym}^n X = \mathrm{H}_0(\mathsf{S}_n, \otimes^n X)$ of $X \in \mathbf{T}$, which is a quotient of $\otimes^n X$, and the divided power $\Gamma^n X = \mathrm{H}^0(\mathsf{S}_n, \otimes^n X)$, which is a subobject of $\otimes^n X$. For $j \in \mathbb{N}$, we define the object $\mathtt{Fr}_+^{(j)} X$ in $\mathbf{T}$ as the image of the composite morphism

$$\Gamma^{p^j} X \hookrightarrow \otimes^{p^j} X \twoheadrightarrow \mathrm{Sym}^{p^j} X.$$

The choice of notation $\mathtt{Fr}_+^{(j)}$ is motivated by the fact that for a vector space $V$, the space $\mathtt{Fr}_+^{(j)} V$ is canonically identified with the $j$-th Frobenius twist $V^{(j)}$ of $V$. Using this construction, we can now formulate our first main result, proved in Theorems 6.1.1 and 6.4.1. For a definition of the exterior power $\Lambda^n$ we refer to Section 1.3.

**Theorem B.** *Let $\Bbbk$ be an algebraically closed field of positive characteristic and $\mathbf{T}$ a tensor category over $\Bbbk$. The following are equivalent.*

*(i) As a tensor category, $\mathbf{T}$ is equivalent to $\mathrm{Rep}G$ for some affine group scheme $G/\Bbbk$.*
*(ii) For each $X \in \mathbf{T}$*

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(a) *there exists* $n \in \mathbb{N}$ *such that* $\Lambda^n X = 0$;
(b) *we have* $\Lambda^n X = 0$ *when* $\Lambda^n \mathtt{Fr}_+^{(j)}(X) = 0$, *for* $j, n \in \mathbb{N}$.

In Theorem 6.2.1, we prove a similar characterisation of the representation categories of affine supergroup schemes among all tensor categories. In characteristic zero, such an internal characterisation follows from the main result of [De2] which states that any tensor category 'of sub-exponential growth' is equivalent to a representation category of an affine supergroup scheme.

The assignment $X \mapsto \mathtt{Fr}_+^{(1)} X$ actually yields an additive functor $\mathtt{Fr}_+ : \mathbf{T} \to \mathbf{T}$. The following is proved in Theorems 3.2.2 and 3.2.4 and Proposition 4.1.3.

**Theorem C.** *Let* $\Bbbk$ *be a field of characteristic* $p > 0$ *and* $\mathbf{T}$ *a tensor category over* $\Bbbk$. *The following are equivalent.*

(i) *The functor* $\mathtt{Fr}_+ : \mathbf{T} \to \mathbf{T}$ *is exact.*
(ii) *For each filtered object* $X \in \mathbf{T}$, *the canonical epimorphism* $\mathrm{Sym}^\bullet(\mathrm{gr}X) \twoheadrightarrow \mathrm{gr}(\mathrm{Sym}^\bullet X)$ *is an isomorphism.*
(iii) *For each monomorphism* $\mathbb{1} \hookrightarrow X$, *the induced morphism* $\mathbb{1} \to \mathrm{Sym}^p X$ *is non-zero.*
(iv) *There exists an abelian* $\Bbbk$*-linear symmetric monoidal category* $\mathbf{C}$ *and an exact* $\Bbbk$*-linear symmetric monoidal functor* $F : \mathbf{T} \to \mathbf{C}$ *which splits every short exact sequence in* $\mathbf{T}$.

In [EHO, Question 3.5], Etingof, Harman and Ostrik ask whether property C(ii) is always satisfied for $p > 2$. An affirmative answer to that question (and hence every property in Theorem C) is sufficient to ensure that the $p$-adic categorical dimensions $\mathrm{Dim}_\pm : \mathrm{Ob}\mathbf{T} \to \mathbb{Z}_p$ as defined in [EHO] are additive along short exact sequences.

As explained above, in this paper we study when a tensor category is equivalent to the representation category of an affine (super)group scheme. In [Os, Conjecture 1.3], Ostrik proposed a different conjectural extension of the results in [De1, De2]. The conjecture states that tensor categories over algebraically closed fields of characteristic $p$ which are of sub-exponential growth are equivalent to representation categories of affine group schemes in the 'universal Verlinde category' $\mathrm{Ver}_p$. In [Os] this conjecture is proved for symmetric fusion categories. The proof relied in an essential way on a generalisation of the classical Frobenius twist to fusion categories. We prove that our functor $\mathtt{Fr}_+$ is a direct summand of a functor $\mathtt{Fr}$ which, when applied to fusion categories, recovers the functor in [Os]. We hope that our generalisation of Ostrik's Frobenius twist to arbitrary tensor categories might be useful in the exploration of [Os, Conjecture 1.3].

The rest of the paper is organised as follows. In Section 1 we review some properties of tensor categories. In Section 2 we study (modular) representation theory of finite groups in abelian categories. This will be used later on to deal with the representations of the symmetric group, and its subgroups, which originate from the symmetric braiding on tensor categories. In Section 3 we define and study 'locally semisimple' tensor categories, which are the ones in which the equivalent conditions in Theorem C are satisfied. We also derive abstract criteria for existence of tensor functors to semisimple tensor categories. In Section 4, we study the Frobenius twists. In Section 5 we give internal characterisations for objects which are 'locally free', that is objects which become isomorphic to (super) vector spaces after internal extension of scalars. As a consequence of those results we obtain internal characterisations of (super) tannakian categories (as defined in 1.5.6) in Section 6. We also show that each tensor category has a unique maximal (super) tannakian subcategory. Finally we prove the result, announced in [De2], that over algebraically closed fields, super tannakian categories are always representation categories of affine supergroup schemes, by adapting an unpublished argument from Deligne in [De3] with some results in Section 3.

## 1. Preliminaries and notation

Unless further specified, $\Bbbk$ denotes an arbitrary field. We set $\mathbb{N} = \{0, 1, 2, \ldots\}$.

1.1. **Symmetric and cyclic groups.** For a finite group $G$ we denote by $\mathrm{Rep}_{\Bbbk}G$ the category of finite dimensional $\Bbbk G$-modules.

1.1.1. *The symmetric group.* We denote the symmetric group acting on $\{1, 2, \ldots, n\}$ by $\mathsf{S}_n$. For each partition $\lambda \vdash n$ we have the Specht module $S^\lambda$ of $\Bbbk\mathsf{S}_n$, as defined in [Jm, §4]. We will use the dual Specht module

$$S_\lambda \;:=\; S^{\lambda^t} \otimes \mathrm{sgn} \;\simeq\; (S^\lambda)^*,$$

where $\mathrm{sgn} \simeq S^{(1,1,\ldots,1)}$ denotes the sign module. The trivial $\mathsf{S}_n$-module is $S_{(n)} \simeq S^{(n)}$.

1.1.2. *The cyclic group.* We denote by $\mathsf{C}_n$ the cyclic group of order $n$ and we fix the embedding $\mathsf{C}_n < \mathsf{S}_n$ which maps the generator of $\mathsf{C}_n$ to the cycle $(1, 2, \ldots, n) \in \mathsf{S}_n$. Assume that $\mathrm{char}(\Bbbk) = p > 0$. We denote by $M_i$ the indecomposable $\Bbbk\mathsf{C}_p$-module of dimension $i$, for $1 \le i \le p$. In particular $M_1 \simeq \Bbbk$ and $M_p \simeq \Bbbk\mathsf{C}_p$. Every object in $\mathrm{Rep}\mathsf{C}_p$ is a direct sum of these modules.

1.1.3. *Wreath products.* Fix a prime number $p$. For $j \in \mathbb{Z}_{>0}$, we define the subgroups $P_j < \mathsf{S}_{p^j}$ and $Q_j < \mathsf{S}_{p^j}$ iteratively by $P_1 = \mathsf{C}_p$, $Q_1 = \mathsf{S}_p$ and

$$P_{j+1} := P_j \wr \mathsf{C}_p \simeq P_j^{\times p} \rtimes \mathsf{C}_p \quad \text{and} \quad Q_{j+1} := Q_j \wr \mathsf{S}_p \simeq Q_j^{\times p} \rtimes \mathsf{S}_p,$$

**Lemma 1.1.4.** *The group $P_j$ is a Sylow $p$-subgroup of $\mathsf{S}_{p^j}$, and $Q_j$ contains its normaliser $N_{\mathsf{S}_{p^j}}(P_j)$.*

*Proof.* That $P_j$ is a Sylow subgroup is well-known, see [Be], and follows immediately from Legendre's theorem on the prime factorisation of factorials. We set $N_j := N_{\mathsf{S}_{p^j}}(P_j)$. It is also proved *loc. cit.* that $|N_j : P_j| = (p-1)^j$. We start the proof of $N_j < Q_j$ by fixing the embeddings

$$\iota_j : \mathsf{S}_{p^{j-1}} \hookrightarrow (\mathsf{S}_{p^{j-1}})^{\times p} \hookrightarrow \mathsf{S}_{p^j} \quad \text{and} \quad \iota'_j : \mathsf{S}_p \hookrightarrow \mathsf{S}_{p^j},$$

where $\iota_j$ is the composite of the diagonal embedding with the embedding of the Young subgroup and $\iota'_j$ is such that its image is the copy of $\mathsf{S}_p$ in the definition $Q_j = Q_{j-1} \wr \mathsf{S}_p$. We will freely use the fact that the images of $\iota_j$ and $\iota'_j$ are commuting subgroups which generate a copy of $\mathsf{S}_{p^{j-1}} \times \mathsf{S}_p$ in $\mathsf{S}_{p^j}$, and that $\iota_j(Q_{j-1}) < Q_j > \iota'_j(\mathsf{S}_p)$.

We define iteratively subgroups $M_j < Q_j$. We set $M_1 = N_1$ and $M_j = \iota_j(M_{j-1}) \times \iota'_j(N_1)$. By construction, $M_j$ normalises $P_j$ (meaning $M_j < N_j$) and satisfies $|M_j| = (p(p-1))^j$. By Lagrange's theorem, the group generated by $M_j$ and $P_j$ has order divisible by $(p-1)^j|P_j| = |N_j|$. This means the latter group coincides with $N_j$ and concludes the proof. □

1.2. **Monoidal categories.**

1.2.1. *Categories.* When clear in which category we are working, we will denote the morphism sets simply by Hom, End or Aut. For $\Bbbk$-linear categories $\mathbf{A}$ and $\mathbf{B}$, we denote by $\mathbf{A} \otimes_{\Bbbk} \mathbf{B}$ the $\Bbbk$-linear category with objects $(X, Y)$ for $X \in \mathbf{A}$ and $Y \in \mathbf{B}$ and the space of morphisms from $(X, Y)$ to $(Z, W)$ given by $\mathbf{A}(X, Z) \otimes_{\Bbbk} \mathbf{B}(Y, W)$. Then we denote by $\mathbf{A} \boxtimes_{\Bbbk} \mathbf{B}$, or $\mathbf{A} \boxtimes \mathbf{B}$, the Karoubi envelope of $\mathbf{A} \otimes_{\Bbbk} \mathbf{B}$. The object $(X, Y)$ as considered in $\mathbf{A} \boxtimes \mathbf{B}$ will be written as $X \boxtimes Y$.

An abelian $\Bbbk$-linear category in which the endomorphism algebra of each simple object is $\Bbbk$ is called **schurian**. A semisimple schurian category is thus equivalent to a direct sum of copies of the category $\mathbf{vec}_{\Bbbk}$ of finite dimensional vector spaces. If $\mathbf{A}$ and $\mathbf{B}$ are $\Bbbk$-linear abelian with $\mathbf{B}$ semisimple schurian, then $\mathbf{A} \boxtimes \mathbf{B}$ is again abelian.

An object $X$ in an abelian category with subobjects

$$0 = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_d = X$$

will be called an object with filtration of length $d$. Then we write $\mathrm{gr}X$ to denote the associated graded object $\bigoplus_{i=1}^d X_i/X_{i-1}$.

Following, [AGV, §I.8.2], for a locally small category $\mathbf{C}$, we denote by $\mathrm{Ind}\mathbf{C}$ the full subcategory of the category of functors $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ consisting of ind-objects.

1.2.2. We will work with symmetric monoidal categories $(\mathbf{C}, \otimes, \mathbb{1}, \gamma)$ where

(i) $\mathbf{C}$ is $\Bbbk$-linear abelian (with $\mathbb{1} \neq 0$);
(ii) $- \otimes -$ is right exact and $\Bbbk$-linear in both variables.

Here $\gamma$ refers to the binatural family of braiding morphisms $\gamma_{XY} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ which satisfy the constraints of [DM, §1]. For $X \in \mathbf{C}$ and $n \in \mathbb{Z}_{\geq 1}$, we write

$$\otimes^n X = \overbrace{X \otimes X \otimes \cdots \otimes X}^{n} \qquad \text{and} \qquad \otimes^0 X = \mathbb{1},$$

and use similar notation for morphisms. An object $X$ in $\mathbf{C}$ is **flat** if $X \otimes - : \mathbf{C} \to \mathbf{C}$ is exact.

1.2.3. Let $\mathbf{C}$ be as in 1.2.2. For an object $X \in \mathbf{C}$, a dual $X^\vee$ is an object equipped with morphisms $\mathrm{co}_X : \mathbb{1} \to X \otimes X^\vee$ and $\mathrm{ev}_X : X^\vee \otimes X \to \mathbb{1}$ satisfying the two snake relations in [De2, (0.1.4)]. Following [De2, §1.4], we then have bi-adjoint functors $(- \otimes X, - \otimes X^\vee)$. In particular, dualisable object are flat. For dualisable $X, Y \in \mathbf{C}$ we have an isomorphism

$$\mathrm{Hom}(X,Y) \xrightarrow{\sim} \mathrm{Hom}(Y^\vee, X^\vee), \quad f \mapsto f^t := (\mathrm{ev}_Y \otimes \mathrm{Id}_{X^\vee}) \circ (\mathrm{Id}_{Y^\vee} \otimes f \otimes \mathrm{Id}_{X^\vee}) \circ (\mathrm{Id}_{Y^\vee} \otimes \mathrm{co}_X).$$

A direct summand of a dualisable object is also dualisable, see [De2, §1.15].

1.2.4. Following [De1, §2], $\mathbf{C}$ as in 1.2.2 is a **tensor category over** $\Bbbk$ if additionally it is essentially small,

(iii) the canonical morphism $\Bbbk \to \mathrm{End}(\mathbb{1})$ is an isomorphism;
(iv) every object in $\mathbf{C}$ is dualisable.

Now let $\mathbf{T}$ be a tensor category. By 1.2.3, the functor $- \otimes -$ is bi-exact and by [DM, Proposition 1.17], the unit object $\mathbb{1}$ is simple. If every object has finite length, then every morphism space is finite dimensional, see [De2, Proposition 1.1]. If $\Bbbk$ is algebraically closed and every object in $\mathbf{T}$ has finite length, $\mathbf{T}$ is therefore schurian.

1.2.5. A right exact $\Bbbk$-linear functor $F : \mathbf{C} \to \mathbf{C}'$ between two categories $\mathbf{C}$ and $\mathbf{C}'$ as in 1.2.2 is a **tensor functor** if it is equipped with natural isomorphisms $c^F_{XY} : F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y)$ and $F(\mathbb{1}) \xrightarrow{\sim} \mathbb{1}$ satisfying the compatibility conditions of [De1, §2.7], see also [DM, Definition 1.8]. In particular tensor functors are *symmetric* monoidal functors. We will usually just write $F$ for the tensor functor, rather than $(F, c^F)$. The following lemma, see [De1, Corollaire 2.10], is a straightforward consequence of the definitions and the fact that $\mathbb{1}$ is simple in a tensor category.

**Lemma 1.2.6.** *Consider a tensor functor $F : \mathbf{C} \to \mathbf{C}'$. If $X \in \mathrm{Ob}\mathbf{C}$ has a dual $X^\vee$ then $F(X^\vee)$ is a dual of $F(X)$. Any tensor functor out of a tensor category is exact and faithful.*

1.2.7. For categories $\mathbf{C}, \mathbf{C}'$ as in 1.2.2 and tensor functors $F, G : \mathbf{C} \to \mathbf{C}'$, a natural transformation $F \Rightarrow G$ is one of tensor functors if it exchanges the monoidal structures as in [De1, §2.7]. As pointed out *loc. cit.*, for such $\eta : F \Rightarrow G$ and dualisable $X \in \mathbf{C}$, the morphisms $\eta_X$ and $(\eta_{X^\vee})^t$ are mutually inverse. In particular, a natural transformation of tensor functors out of a tensor category is *automatically an isomorphism,* see also [DM, Proposition 1.13].

1.2.8. For $\mathbf{C}$ as in 1.2.2. We denote the category of commutative algebras in $\mathbf{C}$ by alg$\mathbf{C}$. Such an algebra is a triple $(\mathcal{A}, m, \eta)$, with $\mathcal{A}$ an object in $\mathbf{C}$ and morphisms $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and $\eta : \mathbb{1} \to \mathcal{A}$ satisfying the traditional commutative (with respect to $\gamma$) algebra relations.

For non-zero $\mathcal{A} \in \mathrm{alg}\mathbf{C}$, we denote the category of $\mathcal{A}$-modules in $\mathbf{C}$ by $\mathbf{C}_{\mathcal{A}}$. Then $(\mathbf{C}_{\mathcal{A}}, \otimes_{\mathcal{A}}, \mathcal{A})$ is a monoidal category as in 1.2.2 with right exact tensor product $- \otimes_{\mathcal{A}} -$, as the coequaliser of $- \otimes \mathcal{A} \otimes - \rightrightarrows - \otimes -$, introduced in [De1, §7.5]. For an algebra morphism $\mathcal{A} \to \mathcal{B}$ we have the corresponding tensor functor $\mathcal{B} \otimes_{\mathcal{A}} - : \mathbf{C}_{\mathcal{A}} \to \mathbf{C}_{\mathcal{B}}$.

1.2.9. For the remainder of this subsection, fix a tensor category $\mathbf{T}$. By [De1, §7.5] the category Ind$\mathbf{T}$ is again naturally a symmetric monoidal category satisfying (i)-(iii) above. Furthermore, the functor $- \otimes -$ is bi-exact, even though only objects in the subcategory $\mathbf{T}$ are dualisable, see [De2, §2.2]. It also follows from the definitions that the functor $- \otimes X : \mathrm{Ind}\mathbf{T} \to \mathrm{Ind}\mathbf{T}$ is faithful, for any $X \in \mathrm{Ind}\mathbf{T}$. We abbreviate the notation of 1.2.8 as $\mathrm{Alg}\mathbf{T} = \mathrm{algInd}\mathbf{T}$ and $\mathbf{Mod}_{\mathcal{A}} = (\mathrm{Ind}\mathbf{T})_{\mathcal{A}}$, or $\mathbf{Mod}^{\mathbf{T}}_{\mathcal{A}}$ when there is risk of ambiguity. We have a tensor functor

$$F_{\mathcal{A}} = \mathcal{A} \otimes - : \ \mathbf{T} \to \mathbf{Mod}_{\mathcal{A}}. \tag{1}$$

For the tensor category $\mathbf{vec} = \mathbf{vec}_{\Bbbk}$, we have that $\mathbf{Vec} = \mathrm{Ind}\mathbf{vec}$ is the category of all vector spaces and Alg$\mathbf{vec}$ is the category $\mathbf{Alg}_{\Bbbk}$ of commutative $\Bbbk$-algebras.

An algebra $\mathcal{A}$ is **simple** if its only ideal subobjects are 0 and $\mathcal{A}$ itself. This is equivalent to $\mathcal{A}$ being simple as an object in $\mathbf{Mod}_{\mathcal{A}}$.

**Lemma 1.2.10.** *Every non-zero algebra $\mathcal{A}$ in* Alg$\mathbf{T}$ *has a simple quotient.*

*Proof.* Since $\mathbf{T}$ is essentially small, Ind$\mathbf{T}$ has a generator and the class of subobjects of any object in Ind$\mathbf{T}$ forms a set. By Zorn's lemma, $\mathcal{A}$ contains a maximal ideal subobject $J$. The algebra $\mathcal{B} = \mathcal{A}/J$ is simple. □

1.2.11. *Tensor subcategories.* A full subcategory $\mathbf{T}'$ of $\mathbf{T}$ is a tensor subcategory if it is closed under the operations of taking subquotients, tensor products, duals and direct sums. In particular $\mathbf{T}'$ is replete in $\mathbf{T}$ and a tensor category itself. For $E$ a collection of objects or full subcategories of $\mathbf{T}$, we denote by $\langle E \rangle$ the minimal tensor subcategory of $\mathbf{T}$ which contains all objects in $E$. We say that $\mathbf{T}$ is **finitely generated** if $\mathbf{T} = \langle X \rangle$, for some object $X \in \mathbf{T}$.

We denote by $\Gamma^{\mathbf{T}}_{\mathbf{T}'}$, or simply $\Gamma_{\mathbf{T}'}$, the right adjoint to the inclusion functor $\mathrm{Ind}\mathbf{T}' \to \mathrm{Ind}\mathbf{T}$. In other words, $\Gamma_{\mathbf{T}'}$ is the left exact lax-monoidal functor which sends an object in Ind$\mathbf{T}$ to its maximal subobject in Ind$\mathbf{T}'$.

If $\mathbf{T}$ and $\mathbf{V}$ are tensor categories where $\mathbf{V}$ is semisimple schurian, then $\mathbf{T} \boxtimes \mathbf{V}$ is again a tensor category. We can and will identify $\mathbf{T}$ and $\mathbf{V}$ with tensor subcategories of $\mathbf{T} \boxtimes \mathbf{V}$.

The following observations are standard, see e.g. [De2, §2.11].

**Lemma 1.2.12.** *Consider a tensor subcategory $\mathbf{T}' \subset \mathbf{T}$.*

*(i) If $\mathcal{A}$ is in* Alg$\mathbf{T}$, *then $\Gamma_{\mathbf{T}'}\mathcal{A}$ is a subalgebra, and hence an object in* Alg$\mathbf{T}'$.
*(ii) For $X \in \mathrm{Ind}\mathbf{T}$ and $Y \in \mathbf{T}'$, the natural morphism $\Gamma_{\mathbf{T}'}(X) \otimes Y \to \Gamma_{\mathbf{T}'}(X \otimes Y)$ is an isomorphism.*
*(iii) For $\mathcal{A} \in \mathrm{Alg}\mathbf{T}$, set $\mathcal{R} := \Gamma_{\mathbf{T}'}\mathcal{A}$. The lax monoidal structure of $\Gamma_{\mathbf{T}'} : \mathrm{Ind}\mathbf{T} \to \mathrm{Ind}\mathbf{T}'$ induces one as a functor $\Gamma_{\mathbf{T}'} : \mathbf{Mod}^{\mathbf{T}}_{\mathcal{A}} \to \mathbf{Mod}^{\mathbf{T}'}_{\mathcal{R}}$ via the universality of the coequaliser $- \otimes_{\mathcal{R}} -$. Furthermore,*

$$\Gamma_{\mathbf{T}'}(M) \otimes_{\mathcal{R}} \Gamma_{\mathbf{T}'}(N) \ \to \ \Gamma_{\mathbf{T}'}(M \otimes_{\mathcal{A}} N)$$

*is an isomorphism if $M \simeq \mathcal{A} \otimes M_0$ and $N \simeq \mathcal{A} \otimes N_0$ for $M_0, N_0 \in \mathbf{T}'$.*

1.3. **Symmetric and divided powers.** Let $\mathbf{C}$ be a symmetric monoidal category as in 1.2.2 and fix $X, Y \in \mathbf{C}$.

1.3.1. We define $\Lambda^2 X$ as the image of the morphism $\gamma_{XX}-1$ in $\mathrm{End}(\otimes^2 X)$. By definition, $\Lambda^2 X$ is a subobject of $\otimes^2 X$. If $2\neq \mathrm{char}(\Bbbk)$, then $\Lambda^2 X$ is a direct summand, equivalently described as the kernel of $\gamma_{XX}+1$. The symmetric algebra

$$\mathrm{Sym}^\bullet X \;=\; \bigoplus_{i\in\mathbb{N}}\mathrm{Sym}^i X \;\in\; \mathrm{alg}\mathbf{C}$$

is the maximal quotient of the tensor algebra of $X$ which is commutative. It is therefore the quotient with respect to the ideal generated by $\Lambda^2 X$. Concretely, for $n\in\mathbb{N}$, we have exact sequences

(2)
$$\bigoplus_{a+b=n-2}\otimes^a X\otimes\Lambda^2X\otimes\otimes^b X\to\otimes^n X\to\mathrm{Sym}^n X\to 0\quad\text{and}\quad\bigoplus_{i=1}^{n-1}\otimes^n X\to\otimes^n X\to\mathrm{Sym}^nX\to 0,$$

where the used endomorphisms of $\otimes^n X$ are $\otimes^{i-1}X\otimes(\gamma_{XX}-1)\otimes\otimes^{n-i-1}X$.

Dually we define the divided power $\Gamma^n X$ by the exact sequence

$$0\;\to\;\Gamma^n X\;\to\;\otimes^n X\;\to\;\bigoplus_{i=1}^{n-1}\otimes^n X$$

using the same endomorphisms of $\otimes^n X$. Equivalently we can set $\mathrm{Sym}^n X=\mathrm{H}_0(\mathsf{S}_n,\otimes^n X)$ and $\Gamma^n X=\mathrm{H}^0(\mathsf{S}_n,\otimes^n X)$, with notation as in Section 2.1.

**Lemma 1.3.2.** *If $X$ is dualisable and* $\mathrm{Sym}^n X$ *is flat,* $\mathrm{Sym}^n X$ *is dualisable with dual* $\Gamma^n(X^\vee)$.

*Proof.* It follows from (2) that the composition $\Gamma^n(X^\vee)\otimes(\otimes^n X)\hookrightarrow(\otimes^n X^\vee)\otimes(\otimes^n X)\xrightarrow{\mathrm{ev}}\mathbb{1}$ factors as $\Gamma^n(X^\vee)\otimes(\otimes^n X)\twoheadrightarrow\Gamma^n(X^\vee)\otimes\mathrm{Sym}^n(X)\xrightarrow{\varepsilon}\mathbb{1}$, for some morphism $\varepsilon$. The flatness of $\mathrm{Sym}^n X$ allows to conclude that the bottom vertical arrow in the diagram

$$\begin{tikzcd}
\mathbb{1} \arrow[r, hook, "\mathrm{co}"] \arrow[d, dashed, "\delta"] & (\otimes^n X)\otimes(\otimes^n X^\vee) \arrow[d] \\
\mathrm{Sym}^n(X)\otimes\Gamma^n(X^\vee) \arrow[r, hook] & \mathrm{Sym}^n(X)\otimes(\otimes^n X^\vee)
\end{tikzcd}$$

is a monomorphism. The existence of a morphism $\delta$ to create a commutative diagram then follows as for $\varepsilon$. That $\varepsilon$ and $\delta$ satisfy the snake relations now follows from construction. For instance, the commutative diagram

$$\begin{tikzcd}
\otimes^n X^\vee \arrow[r, hook, "\otimes^n X^\vee\otimes\mathrm{co}"] & (\otimes^n X^\vee)\otimes(\otimes^n X)\otimes(\otimes^n X^\vee) \arrow[r, "\mathrm{ev}\otimes\otimes^n X^\vee"] & \otimes^n X^\vee \\
& \Gamma^n X^\vee\otimes(\otimes^n X)\otimes(\otimes^n X^\vee) \arrow[u, hook] \arrow[d, two heads] & \\
& \Gamma^n X^\vee\otimes\mathrm{Sym}^n X\otimes(\otimes^n X^\vee) \arrow[uur, "\varepsilon\otimes\otimes^n X^\vee"'] & \\
\Gamma^n X^\vee \arrow[uuu, hook] \arrow[uur, "\Gamma^n X^\vee\otimes\mathrm{co}"] \arrow[r, "\Gamma^n X^\vee\otimes\delta"] & \Gamma^n X^\vee\otimes\mathrm{Sym}^n X\otimes\Gamma^n X^\vee \arrow[u] \arrow[r, "\varepsilon\otimes\Gamma^n X^\vee"] & \Gamma^n X^\vee \arrow[uuu, hook]
\end{tikzcd}$$

ensures that one of the snake relations is inherited from the ones for $(\mathrm{ev},\mathrm{co})$. ∎

1.3.3. *Skew symmetric powers.* There exist several different notions of 'exterior powers' in $\mathbf{C}$. For $n \in \mathbb{N}$, we define $\Lambda^n X$ as the image of the anti-symmetriser $\sum_{\sigma\in\mathsf{S}_n}(-1)^{|\sigma|}\sigma : \otimes^n X \to \otimes^n X$. The essential properties for us are $\Lambda^n\mathbb{1} = 0$ for $n > 1$ and

$$\Lambda^n(X \oplus Y) \simeq \bigoplus_{i=0}^{n} \Lambda^{n-i}X \otimes \Lambda^i Y, \tag{3}$$

which is satisfied whenever $\Lambda^j X$ and $\Lambda^j Y$ are flat for $1 \le j \le n$, or when $X = \mathbb{1}$.

Now assume $\mathrm{char}(\Bbbk) \neq 2$. Then we could alternatively use $\mathrm{H}^0(\mathsf{S}_n, \mathrm{sgn}\otimes\otimes^n X)$ or $\mathrm{H}_0(\mathsf{S}_n, \mathrm{sgn}\otimes \otimes^n X)$, with notation of Section 2, as notion of skew symmetric power and all our results remain valid with these alternative definitions. These two powers satisfy (3) without any flatness condition.

## 1.4. **Semisimplification and the universal Verlinde category.**

1.4.1. *Semisimplification.* For a tensor category $\mathbf{T}$, let $\mathcal{N}$ denote the ideal of negligible morphisms, see [AK, §7.1], which is the unique maximal tensor ideal. We have a canonical symmetric monoidal $\Bbbk$-linear functor $X \mapsto \overline{X}$ from $\mathbf{T}$ to the quotient $\overline{\mathbf{T}} := \mathbf{T}/\mathcal{N}$, which maps an object to itself and a morphism to its equivalence class. As a special case of [AK, Théorème 8.2.2], we find that $\overline{\mathrm{Rep}G}$ is abelian semisimple, for a finite group $G$. We stress that $X \mapsto \overline{X}$ is in general (for instance in the example $\mathrm{Rep}\mathsf{C}_p \to \mathbf{ver}_p$ below) not exact, so not a tensor functor.

1.4.2. *The category of super vector spaces.* Assume $\mathrm{char}(\Bbbk) \neq 2$. The monoidal category **svec** is defined as the category of $\mathbb{Z}/2$-graded vector spaces, or equivalently as $\mathrm{Rep}\mathsf{C}_2$. The braiding is defined via the graded isomorphisms

$$\gamma_{VW} : V \otimes_{\Bbbk} W \to W \otimes_{\Bbbk} V, \quad v \otimes w \mapsto (-1)^{|v||w|} w \otimes v,$$

where $|v| \in \mathbb{Z}/2$ denotes the parity of a homogeneous vector. We denote the one-dimensional super space concentrated in odd degree by $\bar{\mathbb{1}}$.

1.4.3. *Verlinde category.* Assume that $p := \mathrm{char}(\Bbbk) > 0$. In [Os, Definition 3.1], the universal Verlinde category is defined as $\mathbf{ver}_p := \overline{\mathrm{Rep}\mathsf{C}_p}$. With notation as in 1.1.2, the simple objects of $\mathbf{ver}_p$ correspond, up to isomorphism, to $\overline{M}_i$, for $1 \le i < p$. If if $p > 2$, by [EVO, Proposition 2.4], we have

$$\Gamma^{p-j+1}(\overline{M}_j) = \mathrm{Sym}^{p-j+1}(\overline{M}_j) = 0 \quad \text{for all } 1 < j < p. \tag{4}$$

1.5. **Fibre functors.** For the entire subsection we consider tensor categories $\mathbf{T}, \mathbf{V}$, with $\mathbf{V}$ schurian and semisimple. Sometimes we will require the additional assumption, satisfied for instance by **svec** and $\mathbf{ver}_p$, that

$$\text{for every simple object } \mathbf{V} \ni S \not\simeq \mathbb{1} \text{ there exists } N \in \mathbb{N} \text{ for which } \mathrm{Sym}^N S = 0. \tag{5}$$

We recall the following definition from [De2, §3.1].

*Definition* 1.5.1. Assume that $\mathbf{V}$ satisfies (5). A **fibre functor of T over** $\mathcal{R}$, for a non-zero $\mathcal{R}$ in $\mathrm{Alg}\mathbf{V}$, is a tensor functor $\mathbf{T} \to \mathbf{Mod}_{\mathcal{R}}^{\mathbf{V}}$.

**Lemma 1.5.2.** *If* $\mathbf{V}$ *satisfies* (5)*, then every simple algebra* $\mathcal{A}$ *in* $\mathrm{Alg}\mathbf{V}$ *is a field extension of* $\Bbbk$*, interpreted in* $\mathrm{Ind}\mathbf{V} \supset \mathbf{Vec}$*.*

*Proof.* In order to find a contradiction, we assume that there exists a simple subobject $S \not\simeq \mathbb{1}$ of $\mathcal{A}$. Denote by $J$ the ideal generated by $S$, meaning the image of

$$\mathcal{A} \otimes S \hookrightarrow \mathcal{A} \otimes \mathcal{A} \xrightarrow{m} \mathcal{A}.$$

Since the $n$-fold multiplication $\otimes^n\mathcal{A} \to \mathcal{A}$ factors through $\mathrm{Sym}^n\mathcal{A}$ we find by (5) that $J$ is nilpotent in the sense that any subobject in $\mathbf{V}$ of $J$ is sent to zero when multiplied (inside $\mathcal{A}$) with itself enough times. In particular, $J$ does not contain the image of $\eta : \mathbb{1} \to \mathcal{A}$, so $J \neq \mathcal{A}$, which means $J = 0$, a contradiction. In conclusion, $\mathcal{A}$ is contained in $\mathrm{Ind}\mathbf{V}$ and hence a commutative simple $\Bbbk$-algebra. □

*Remark* 1.5.3. Consider tensor categories $\mathbf{T}_1, \mathbf{T}_2$ and non-zero algebras $\mathcal{R}, \mathcal{R}' \in \mathrm{Alg}\mathbf{T}_2$ with an algebra morphism $\mathcal{R} \to \mathcal{R}'$. The composition of any tensor functor $F : \mathbf{T}_1 \to \mathbf{Mod}^{\mathbf{T}_2}_{\mathcal{R}}$ with $\mathcal{R}' \otimes_{\mathcal{R}} -$ yields, by definition, a tensor functor $\mathbf{T}_1 \to \mathbf{Mod}^{\mathbf{T}_2}_{\mathcal{R}'}$.

**Lemma 1.5.4.** *If $\mathbf{T}$ admits a fibre functor as in Definition 1.5.1 then we have the following.*

*(i) Each object in $\mathbf{T}$ has finite length.*

*(ii) There exists a tensor functor $\mathbf{T} \to \mathbf{V}'$ for $\mathbf{V}'$ a semisimple tensor category over $K$, for some field extension $K/\Bbbk$, given by $\mathbf{V}' = \mathbf{vec}_K \boxtimes_{\Bbbk} \mathbf{V}$.*

*Proof.* Part (i) is a direct consequence of part (ii) and the fact that fibre functors are faithful, see Lemma 1.2.6. Now we prove part (ii). Assume we have a fibre functor $\mathbf{T} \to \mathbf{Mod}^{\mathbf{V}}_{\mathcal{R}}$. By Remark 1.5.3 and Lemmata 1.2.10 and 1.5.2, we can replace $\mathcal{R}$ by a field extension $K/\Bbbk$. It follows quickly that $\mathbf{Mod}^{\mathbf{V}}_K \simeq \mathbf{V}'$. □

*Remark* 1.5.5. In [De2, §3.1] the condition that $\mathbf{V}$ be semisimple is not required, but it is assumed that all objects in $\mathbf{T}$ and $\mathbf{V}$ have finite length. By Lemma 1.5.4(i) we thus find that our notion of fibre functor is a special case of the one *loc. cit.*

1.5.6. In case we take $\mathbf{V} = \mathbf{vec}$ in Definition 1.5.1 we recover the classical notion of a fibre functor of [De1, § 1.9]. A tensor category with such a fibre functor is a **tannakian category**, see [De1, §2.8]. When the $\Bbbk$-algebra is simply $\Bbbk$, meaning we have a tensor functor to $\mathbf{vec}$, the category is **neutral tannakian**. A tensor category admitting a fibre functor over an algebra in $\mathbf{V} = \mathbf{svec}$ is a **super tannakian category**, see [De2, §0.9]. Neutral super tannakian categories are defined similarly.

1.5.7. For the reader's convenience we recall some essential facts about the (neutral) tannakian formalism from [De1, §8], in our limited generality. We need a couple of definitions first. An affine group scheme in $\mathbf{T}$ is a functor

$$G : \ \mathrm{Alg}\mathbf{T} \to \mathbf{Grp},$$

which is representable by a commutative Hopf algebra $\Bbbk[G]$ in $\mathrm{Ind}\mathbf{T}$. For a tensor functor $F : \mathbf{T} \to \mathbf{T}'$ to a second tensor category $\mathbf{T}'$, the group functor $\underline{\mathrm{Aut}}^{\otimes}(F)$ sends an algebra $\mathcal{R}$ in $\mathrm{Alg}\mathbf{T}'$ to the group of automorphisms of the tensor functor $(\mathcal{R} \otimes -) \circ F : \mathbf{T} \to \mathbf{Mod}^{\mathbf{T}'}_{\mathcal{R}}$, and is an affine group scheme in $\mathbf{T}'$ represented by the co-end algebra of $F(-)^{\vee} \otimes F(-) : \mathbf{T}^{\mathrm{op}} \times \mathbf{T} \to \mathbf{T}'$. As an example, we have the 'fundamental group' $\pi(\mathbf{T}) = \underline{\mathrm{Aut}}^{\otimes}(\mathrm{Id}_{\mathbf{T}})$.

**Lemma 1.5.8** (Deligne)**.** *Consider a tensor functor $\omega_0 : \mathbf{T} \to \mathbf{V}$.*

*(i) Consider the group scheme $G := \underline{\mathrm{Aut}}^{\otimes}(\omega_0)$ with the canonical homomorphism $p : \pi(\mathbf{V}) \to G$. Let $\mathrm{Rep}(G, p)$ be the category of representations of $G$ in $\mathbf{V}$ which restrict to the evaluation action of $\pi(\mathbf{V})$ under $p$. We have a tensor equivalence $\mathbf{T} \xrightarrow{\sim} \mathrm{Rep}(G, p)$, which yields a commutative diagram with $\omega_0$ and the forgetful functor $\mathrm{Rep}(G, p) \to \mathbf{V}$.*

*(ii) There exists a tensor equivalence $\mathbf{T} \boxtimes \mathbf{V} \xrightarrow{\sim} \mathrm{Rep}G$, which yields a commutative diagram with $\omega : \mathbf{T} \boxtimes \mathbf{V} \to \mathbf{V}$ induced from $(\omega_0, \mathrm{Id}_{\mathbf{V}})$ and the forgetful functor $\mathrm{Rep}G \to \mathbf{V}$.*

*Proof.* Part (i) is [De1, Théorème 8.17], using [De1, (8.15.1)]. Part (ii) is [De1, Proposition 8.22]. □

We end this section with a result which is implicit in Sections 3 and 4 of [De2].

**Proposition 1.5.9** (Deligne)**.** *If $\Bbbk$ is algebraically closed, $\mathbf{V}$ satisfies* (5) *and $\mathbf{T}$ is finitely generated, then any two fibre functors $\mathbf{T} \to \mathbf{V}$ are isomorphic.*

*Proof.* As proved in [De2, §3], for two fibre functors $F, G : \mathbf{T} \to \mathbf{V}$, the co-end

$$\Lambda := \int^{X \in \mathbf{T}} G(X)^\vee \otimes F(X) \ \in \mathrm{Ind}\mathbf{V}$$

inherits the structure of an algebra in $\mathbf{V}$. Furthermore $F, G$ become isomorphic as tensor functors after composition with $\mathcal{R} \otimes -$, for $\mathcal{R} \in \mathrm{Alg}\mathbf{V}$, if and only if there exists an algebra morphism $\Lambda \to \mathcal{R}$. Now if $\mathbf{T}$ is finitely generated, so is the algebra $\Lambda$.

By Lemma 1.2.10, $\Lambda$ has a simple quotient $\mathcal{A}$, which by Lemma 1.5.2 is a field extension $K$ of $\Bbbk$. However, since $\Lambda$ is finitely generated, so is $K$, which means $K = \Bbbk$. In particular, we have an algebra morphism $\Lambda \to \Bbbk = \mathbb{1}$. This means that $F$ and $G$ are indeed isomorphic. □

## 2. Representations in abelian categories

We fix an abelian category $\mathbf{A}$, finite groups $H < G$ and a field $\Bbbk$.

2.1. **Definitions.** We will interpret groups as categories with one object where all morphisms are isomorphisms.

*Definition* 2.1.1. A $G$-object in $\mathbf{A}$ is a functor $G \to \mathbf{A}$. The abelian category of such functors is denoted by $\mathrm{Rep}(G, \mathbf{A})$, the morphism groups by $\mathrm{Hom}_G$ and the forgetful functor by $\mathrm{Res}^G_* : \mathrm{Rep}(G, \mathbf{A}) \to \mathbf{A}$.

Concretely, a $G$-object is of the form $X = (X_0, \phi_X)$, with $X_0 = \mathrm{Res}^G_*(X) \in \mathbf{A}$ and $\phi_X : g \mapsto \phi^g_X$ a group homomorphism $G \to \mathrm{Aut}(X_0)$. A morphism $X \to Y$ in $\mathrm{Rep}(G, \mathbf{A})$ is a morphism $f : X_0 \to Y_0$ in $\mathbf{A}$ such that $f \circ \phi^g_X = \phi^g_Y \circ f$ for all $g \in G$. We obtain a group homomorphism

$$G \to \mathrm{Aut}(\mathrm{Res}^G_*) \ : \ g \mapsto \phi^g, \qquad \text{with } (\phi^g)_X = \phi^g_X \text{ for all } X \in \mathbf{A}. \tag{6}$$

For $X, Y$ in $\mathrm{Rep}(G, \mathbf{A})$, the morphism group $\mathrm{Hom}_G(X, Y)$ can thus be interpreted as the invariants $\mathrm{Hom}(X_0, Y_0)^G$, for the adjoint $G$-action. We have a fully faithful exact functor

$$\iota_{\mathbf{A}} : \mathbf{A} \hookrightarrow \mathrm{Rep}(G, \mathbf{A}), \quad Y \mapsto (Y, \phi_Y) \text{ with } \phi^g_Y := \mathrm{id}_Y \text{ for all } g \in G.$$

We will often omit the functor $\mathrm{Res}^G_*$ and the similarly defined $\mathrm{Res}^G_H$ to simplify notation.

*Example* 2.1.2. We have $\mathrm{Rep}(G, \mathbf{vec}_\Bbbk) = \mathrm{Rep}_\Bbbk G$.

*Definition* 2.1.3. Assume that $\mathbf{A}$ is $\Bbbk$-linear. For $(M, \rho) \in \mathrm{Rep}_\Bbbk G$ with $d = \dim_\Bbbk M$ and $X \in \mathrm{Rep}(G, \mathbf{A})$, we define $Y := M \otimes X$ as an object in $\mathrm{Rep}(G, \mathbf{A})$ with $Y_0 := \bigoplus_{i=1}^d X_0^{(i)}$ for objects $X_0^{(i)}$ in $\mathbf{A}$ with fixed isomorphisms $\alpha_i : X_0 \xrightarrow{\sim} X_0^{(i)}$. Furthermore, we write endomorphisms of $Y_0$ in $\mathbf{A}$ as matrices and set

$$\phi^g_Y \ = \ \left(\rho(g)_{ij}(\alpha_i \circ \phi^g_X \circ \alpha_j^{-1})\right)_{1 \le i,j \le d},$$

where $\rho(g)_{ij} \in \Bbbk$ are the matrix elements of $\rho$ with respect to some fixed basis of $M$.

Alternatively we can define $M \otimes X$ as the object in $\mathrm{Rep}(G, \mathbf{A})$ representing the functor

$$(M^* \otimes \mathrm{Hom}(X_0, -))^G : \ \mathrm{Rep}(G, \mathbf{A}) \to \mathbf{Ab}.$$

We then easily find

$$\mathrm{Hom}_G(M \otimes X, Y) \ \simeq \ \mathrm{Hom}_G(X, M^* \otimes Y). \tag{7}$$

If $\mathbf{A}$ is $\Bbbk$-linear, there is a fully faithful $\Bbbk$-linear functor

$$\mathbf{A} \boxtimes \mathrm{Rep}_{\Bbbk} G \to \mathrm{Rep}(G, \mathbf{A}), \quad X \boxtimes M \mapsto M \otimes \iota_{\mathbf{A}}(X).$$

If $\mathbf{A}$ is also semisimple and schurian then this functor is clearly an equivalence.

*Definition* 2.1.4. (i) The right and left adjoint functors of $\iota_{\mathbf{A}}$ are denoted by

$$\mathrm{H}^0(G,-) : \mathrm{Rep}(G,\mathbf{A}) \to \mathbf{A} \quad \text{and} \quad \mathrm{H}_0(G,-) : \mathrm{Rep}(G,\mathbf{A}) \to \mathbf{A}.$$

Concretely, $\mathrm{H}^0(G,-)$ maps $X$ to the maximal subobject of $X_0$ on which each $\phi_X^g$ acts as the identity, for all $g \in G$, and $\mathrm{H}_0(G,-)$ is defined dually. In symbols this gives

$$\mathrm{H}^0(G,X) = \bigcap_{g \in G} \ker(\mathrm{Id}_{X_0} - \phi_X^g).$$

(ii) Applying the unit and counit natural transformations, and using $\mathrm{Res}^G_* \circ \iota_{\mathbf{A}} \simeq \mathrm{Id}$, yields natural transformations of functors $\mathrm{Rep}(G,\mathbf{A}) \to \mathbf{A}$:

$$\mathrm{H}^0(G,-) \Rightarrow \mathrm{Res}^G_* \Rightarrow \mathrm{H}_0(G,-).$$

We denote the image of the composite by $\mathrm{Triv}_G : \mathrm{Rep}(G,\mathbf{A}) \to \mathbf{A}$.

*Example* 2.1.5. In $\mathrm{Rep}_{\Bbbk} G$, the subquotient $\mathrm{Triv}_G(M)$ of $M \in \mathrm{Rep}_{\Bbbk} G$ is isomorphic to the maximal direct summand of $M$ which has trivial $G$-action.

2.1.6. Consider the set $I = G/H$ of left cosets and pick a representative $r_i \in G$ for each $i \in I$. For each $g \in G$ and $i \in I$ we then have some $g(i) \in I$ and $h_i^g \in H$ such that $gr_i = r_{g(i)} h_i^g$. We now also assume that for each $X_0 \in \mathbf{A}$ we have a fixed set of isomorphisms

$$\{\beta_i^{X_0} : X_0 \xrightarrow{\sim} X_0^{(i)} \mid i \in I\} \quad \text{in } \mathbf{A}.$$

*Definition* 2.1.7. The functor

$$\mathrm{Ind}_H^G : \mathrm{Rep}(H,\mathbf{A}) \to \mathrm{Rep}(G,\mathbf{A})$$

maps an object $X$ in $\mathrm{Rep}(H,\mathbf{A})$ to $Y = (Y_0, \phi_Y)$ with $Y_0 = \bigoplus_{i \in I} X_0^{(i)}$ and

$$\phi_Y^g = \left( \delta_{i,g(j)}\, \beta_i^{X_0} \circ \phi_X^{h_j^g} \circ (\beta_j^{X_0})^{-1} \right)_{i,j \in I}.$$

For a morphism $f$ from $X$ to $Z$ in $\mathrm{Rep}(H,\mathbf{A})$ we have $\mathrm{Ind}_H^G(f) = \left( \beta_i^{Z_0} \circ f \circ (\beta_i^{X_0})^{-1} \right)_{i \in I}$.

As in the classical case, the functor $\mathrm{Ind}_H^G$ is left and right adjoint to $\mathrm{Res}_H^G$.

2.2. **Elementary properties.** For $g \in G$ we denote by $H^g$ the subgroup $gHg^{-1} < G$. Since $H \simeq H^g$ we can interpret $H$-representations as $H^g$-representations. Concretely, for $X \in \mathrm{Rep}(H,\mathbf{A})$, we denote by $X^g$ the object in $\mathrm{Rep}(H^g,\mathbf{A})$ which has same underlying object in $\mathbf{A}$, but has action given by $\phi_{X^g}^{ghg^{-1}} = \phi_X^h$.

**Lemma 2.2.1** (Mackey's theorem)**.** *For a subgroup $L < G$, we have natural isomorphisms*

$$\mathrm{Res}_L^G \circ \mathrm{Ind}_H^G X \xrightarrow{\sim} \bigoplus_{s \in L\backslash G/H} \mathrm{Ind}^L_{L\cap H^s} \circ \mathrm{Res}^{H^s}_{L \cap H^s} X^s, \quad \text{for } X \in \mathrm{Rep}(H,\mathbf{A}).$$

*Proof.* The classical proof, see e.g. [Al, Lemma III.8.7], carries over verbatim. □

**Lemma 2.2.2.** *For $X$ in $\mathrm{Rep}(H,\mathbf{A})$, the morphisms in $\mathbf{A}$ given by $(\beta_i^{X_0})_{i \in I} : X \to \mathrm{Ind}_H^G X$ and $((\beta_i^{X_0})^{-1})_{i\in I} : \mathrm{Ind}_H^G X \to X$, induce isomorphisms*

$$\mathrm{H}^0(H,X) \xrightarrow{\sim} \mathrm{H}^0(G, \mathrm{Ind}_H^G X) \quad \text{and} \quad \mathrm{H}_0(G,\mathrm{Ind}_H^G X) \xrightarrow{\sim} \mathrm{H}_0(H,X).$$

*Proof.* We prove the first property, as the second is similar. Take a trivial $G$-representation $Z$ in $\mathrm{Rep}(G,\mathbf{A})$, *i.e.* an object in the image of $\iota_{\mathbf{A}}$. A morphism $f$ from $Z$ to $\mathrm{Ind}_H^G X$ in $\mathbf{A}$ is of the form $(f_i)_{i\in I}$ for some $f_i : Z \to X_0^{(i)}$. Then $f \in \mathrm{Hom}_G(Z, \mathrm{Ind}_H^G X)$ if and only if

$$\beta_{g(j)}^{X_0} \circ \phi_X^{h_j^g} \circ (\beta_j^{X_0})^{-1} \circ f_j \;=\; f_{g(j)}, \quad \text{for all } j \in I \text{ and } g \in G.$$

Fix an arbitrary $i_0 \in I$. The above equation for $j = i_0$ and arbitrary $g \in H^{r_{i_0}}$ implies that $\varphi := (\beta_{i_0}^{X_0})^{-1} \circ f_{i_0}$ is in $\mathrm{Hom}_H(Z, X)$. The equation for $j = i_0$ and $g = r_i r_{i_0}^{-1}$ for all $i \in I$ then shows that $f_i = \beta_i^{X_0} \circ \varphi$ for all $i \in I$. We have thus showed that composing with $(\beta_i^{X_0})_{i\in I} : X \to \mathrm{Ind}_H^G X$ in $\mathbf{A}$ induces an epimorphism

$$\mathrm{Hom}_H(Z, X) \twoheadrightarrow \mathrm{Hom}_G(Z, \mathrm{Ind}_H^G X).$$

Since we compose with an monomorphism in $\mathbf{A}$, the above epimorphism is also a monomorphism. The fact that composition with $(\beta_i^{X_0})_i$ induces an isomorphism for each such $Z$ concludes the proof. $\square$

**Corollary 2.2.3.** *Assume* $\mathbf{A}$ *is* $\Bbbk$*-linear.*

*(i) If the image of* $|G : H|$ *in* $\Bbbk$ *is zero, we have* $\mathrm{Triv}_G \circ \mathrm{Ind}_H^G = 0$*.*
*(ii) If* $|G : H|$ *is zero and* $|G : L|$ *is invertible in* $\Bbbk$*, for* $L < G$*, then* $\mathrm{Triv}_L \circ \mathrm{Res}_L^G \circ \mathrm{Ind}_H^G = 0$*.*
*(iii) If* $|G : H|$ *is invertible in* $\Bbbk$*, then* $\mathrm{Triv}_G \circ \mathrm{Ind}_H^G \simeq \mathrm{Triv}_H$*.*

*Proof.* Lemma 2.2.2 implies that there exists a commutative diagram in $\mathbf{A}$

$$\begin{tikzcd}
\mathrm{H}^0(H,X) \arrow[r, "\sim"] \arrow[d, hook] & \mathrm{H}^0(G,\mathrm{Ind}_H^G X) \arrow[dr, hook] & & \mathrm{H}_0(G,\mathrm{Ind}_H^G X) \arrow[r, "\sim"] & \mathrm{H}_0(H,X) \\
X \arrow[rr, hook] & & \mathrm{Ind}_H^G X \arrow[rr, two heads] \arrow[ur, two heads] & & X \arrow[u, two heads]
\end{tikzcd}$$

such that the composition of the lower horizontal line is $|G : H|$ times $\mathrm{Id}_X$. The morphism from $\mathrm{H}^0(G, \mathrm{Ind}_H^G X)$ to $\mathrm{H}_0(G, \mathrm{Ind}_H^G X)$ defining $\mathrm{Triv}_G \mathrm{Ind}_H^G X$ is therefore, up to composition with isomorphisms, equal to $|G : H|$ times the corresponding morphism from $\mathrm{H}^0(H,X)$ to $\mathrm{H}_0(H,X)$. This proves parts (i) and (iii).

Now we prove part (ii). By Lemma 2.2.1, the functor $\mathrm{Res}_L^G \circ \mathrm{Ind}_H^G$ is a direct sum of inductions to $L$ from subgroups $L' < L$ which are isomorphic to subgroups of $H$. By assumption and Lagrange's theorem we know that $|L : L'|$ is zero in $\Bbbk$, which implies we can apply part (i) for the group $L$. $\square$

**Lemma 2.2.4.** *(i) The object* $\mathrm{Triv}_G X$ *is a subquotient in* $\mathrm{Triv}_H X$*.*
*(ii) If* $H$ *is a normal subgroup of* $G$*, then*

$$\mathrm{H}^0(G, -) \;\simeq\; \mathrm{H}^0(G/H, \mathrm{H}^0(H, -)) \quad \textit{and} \quad \mathrm{H}_0(G, -) \;\simeq\; \mathrm{H}_0(G/H, \mathrm{H}_0(H, -)).$$

*(iii)* $\mathrm{Triv}_H X$ *is canonically a* $G/H$*-object and* $\mathrm{Triv}_G X$ *is a subquotient in* $\mathrm{Triv}_{G/H} \mathrm{Triv}_H X$*.*

*Proof.* Part (i) follows from the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^0(H,X) \arrow[r, two heads] & \mathrm{Triv}_H X \arrow[r, hook] & \mathrm{H}_0(H,X) \arrow[d] \\
\mathrm{H}^0(G,X) \arrow[u, hook] \arrow[rr] & & \mathrm{H}_0(G,X),
\end{tikzcd} \tag{8}$$

where the image of the lower horizontal morphism is $\mathrm{Triv}_G X$.

Now assume that $H < G$ is normal. Part (ii) is obvious. Clearly the map $\mathrm{H}^0(H,X) \to \mathrm{H}_0(H,X)$ is $G/H$-equivariant. Part (iii) then follows by definition and extending diagram (8) to include $\mathrm{H}^0(G/H, \mathrm{Triv}_H X)$ and $\mathrm{H}_0(G/H, \mathrm{Triv}_H X)$. $\square$

*Example* 2.2.5. Already for $\mathbf{A} = \mathbf{vec}_{\Bbbk}$, the subquotient $\mathrm{Triv}_G X$ will in general not be isomorphic to $\mathrm{Triv}_{G/H}\mathrm{Triv}_H X$, for $H \lhd G$. An example is given by $\Bbbk$ a field of characteristic 2 and $G = \mathsf{S}_2 \times \mathsf{S}_2$. Indeed, let $X$ be a 3-dimensional indecomposable $G$-representation and $H$ one of the copies of $\mathsf{S}_2$. Then we find $\mathrm{Triv}_G X = 0$ but $\mathrm{Triv}_{G/H}\mathrm{Triv}_H X \simeq \Bbbk$.

Recall the natural automorphisms $\phi^g$ of $\mathrm{Res}^G_*$ in equation (6).

**Lemma 2.2.6.** *Assume that* $\mathbf{A}$ *is* $\Bbbk$*-linear and that* $n := |G : H|$ *is invertible in* $\Bbbk$*.*

*(i) The natural endomorphism* $f := \frac{1}{n}\sum_{i\in I}\phi^{r_i}$ *of* $\mathrm{Res}^G_*$ *restricts to* $h : \mathrm{H}^0(H,-) \Rightarrow \mathrm{H}^0(G,-)$*.*
*(ii) The natural endomorphism* $f' := \frac{1}{n}\sum_{i\in I}\phi^{(r_i^{-1})}$ *of* $\mathrm{Res}^G_*$ *yields* $h' : \mathrm{H}_0(G,-) \Rightarrow \mathrm{H}_0(H,-)$*.*
*(iii) The functor* $\mathrm{Triv}_G$ *is a direct summand of* $\mathrm{Triv}_H$*.*

*Proof.* We fix an arbitrary $X$ in $\mathrm{Rep}(G,\mathbf{A})$. First we prove part (i). We define the morphism $m$ in $\mathbf{A}$ by the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^0(H, \mathrm{Res}^G_H X) \arrow[r, hook] \arrow[dr, "m"] & \mathrm{Res}^G_* X \arrow[d, "f_X"] \\
\mathrm{H}^0(G,X) \arrow[r, hook] & \mathrm{Res}^G_* X.
\end{tikzcd}$$

It then follows by direct computation that $\phi^g_X \circ m = m$ for all $g \in G$, which implies that $m$ factors through $\mathrm{H}^0(G,X)$. Part (ii) is proved similarly.

Now we claim that the morphisms $h_X$ and $h'_X$ as defined in parts (i) and (ii), yield a commutative diagram, natural in $X$,

$$\begin{tikzcd}
\mathrm{H}^0(G,X) \arrow[r, hook] \arrow[d, hook] \arrow[dd, bend right=60, "\mathrm{Id}"'] & X \arrow[r, two heads] \arrow[d, "f'_X"] & \mathrm{H}_0(G,X) \arrow[d, hook, "h'_X"] \arrow[dd, bend left=60, "\mathrm{Id}"] \\
\mathrm{H}^0(H,X) \arrow[r, hook] \arrow[d, "h_X"] & X \arrow[r, two heads] \arrow[d, "f_X"] & \mathrm{H}_0(H,X) \arrow[d] \\
\mathrm{H}^0(G,X) \arrow[r, hook] & X \arrow[r, two heads] & \mathrm{H}_0(G,X),
\end{tikzcd}$$

where the unlabelled morphisms are from diagram (8). That the left upper square is commutative follows from the observation that $f'_X$ restricts to the identity on $\mathrm{H}^0(G,X)$. The lower left square is commutative by part (i). Furthermore, since $f_X \circ f'_X$ restricts to the identity on $\mathrm{H}^0(G,X)$, the composite of the two morphisms in the left column is the identity, which implies in particular that $h_X$ is an epimorphism. The arguments for the right-hand side of the diagram are identical.

By commutativity, the morphisms in the right column restrict to morphisms between the respective subobjects $\mathrm{Triv}_G X$ and $\mathrm{Triv}_H X$. In particular, $\mathrm{Triv}_H X$ is a retract of $\mathrm{Triv}_G X$. By naturality, this proves part (iii). $\square$

**Lemma 2.2.7.** *If* $\mathbf{A}$ *is* $\Bbbk$*-linear and* $|G : H|$ *invertible in* $\Bbbk$*, then the identity functor on* $\mathrm{Rep}(G,\mathbf{A})$ *is a direct summand of* $\mathrm{Ind}^G_H \circ \mathrm{Res}^G_H$*.*

*Proof.* We have a morphism

$$(\beta_i^{X_0} \circ \phi_X^{(r_i^{-1})})_{i\in I} \;:\; X \to \mathrm{Ind}^G_H\mathrm{Res}^G_H(X)$$

and a similarly defined morphism in the other direction which compose to $|G : H|$ times the identity. □

The following proposition can be thought of as a very incomplete categorical generalisation of Green's correspondence, see e.g. [Al, Chapter III].

**Proposition 2.2.8.** *Assume that* $\mathbf{A}$ *is* $\Bbbk$*-linear and that* $p := \mathrm{char}(\Bbbk) > 0$*. Let* $P$ *denote a Sylow* $p$*-subgroup of* $G$ *and* $L = N_G(P)$ *its normaliser. If* $H$ *contains* $L$*, then*

$$\mathrm{Triv}_G \;\simeq\; \mathrm{Triv}_H.$$

*More precisely, the canonical morphism* $\mathrm{H}^0(G,-) \Rightarrow \mathrm{Triv}_H$ *is an epimorphism and* $\mathrm{Triv}_H \Rightarrow \mathrm{H}_0(G,-)$ *is a monomorphism.*

*Proof.* By diagram (8), it suffices to prove the claim for $H = L$. By Sylow's theorems, all Sylow subgroups are conjugate. Since $P \lhd L$, it is the unique Sylow $p$-subgroup of $L$. By Lemma 2.2.1, we have

$$\mathrm{Res}^G_L \circ \mathrm{Ind}^G_L \;\simeq\; \mathrm{Id} \;\oplus\; R,$$

where $R$ corresponds to induction functors from $L^s \cap L$ to $L$, where $s \in G$ is such that $P^s \neq P$. Consequently, $L^s \cap L$ does not contain the Sylow $p$-subgroup of $L$. Corollary 2.2.3(i) thus implies $\mathrm{Triv}_L \circ R = 0$, which yields

$$\mathrm{Triv}_L \circ \mathrm{Res}^G_L \circ \mathrm{Ind}^G_L \;\simeq\; \mathrm{Triv}_L.$$

On the other hand, Lemma 2.2.6(iii) implies

$$\mathrm{Triv}_L \circ \mathrm{Res}^G_L \;\simeq\; \mathrm{Triv}_G \;\oplus\; D,$$

for some functor $D$. It now suffices to prove that $D = 0$. Combining the two equations above with Corollary 2.2.3(iii) shows that

$$\mathrm{Triv}_L \;\oplus\; D \circ \mathrm{Ind}^G_L \;\simeq\; \mathrm{Triv}_L,$$

so $D \circ \mathrm{Ind}^G_L \simeq 0$. By Lemma 2.2.7, we thus find indeed that $D = 0$. □

**Lemma 2.2.9.** *Assume* $\mathbf{A}$ *is* $\Bbbk$*-linear and take* $M \in \mathrm{Rep}_\Bbbk G$ *and* $X \in \mathrm{Rep}(H, \mathbf{A})$*. We have an isomorphism in* $\mathrm{Rep}(G, \mathbf{A})$

$$M \otimes \mathrm{Ind}^G_H X \;\xrightarrow{\sim}\; \mathrm{Ind}^G_H(M \otimes X).$$

*Proof.* This follows from the adjunction between $\mathrm{Ind}^G_H$ and $\mathrm{Res}^G_H$ and equation (7). □

2.3. **Semisimplification of representation categories.** In this subsection we assume that $\Bbbk$ is a *splitting field* for $G$. By this we mean that every indecomposable module of $\Bbbk G$ is absolutely indecomposable. Equivalently, the radical of $\mathrm{End}_G(M)$ is of codimension 1, for every indecomposable $\Bbbk G$-module $M$. Every algebraically closed field is thus a splitting field for any finite group. Recall the semisimplifcation $\mathrm{Rep}G \to \overline{\mathrm{Rep}G},\ X \mapsto \overline{X}$ of 1.4.1.

**Lemma 2.3.1.** *Consider arbitrary indecomposable* $M, N$ *in* $\mathrm{Rep}G$*.*

*(i) The object* $\overline{M}$ *is simple or zero. Set* $n_M = 0$ *when* $\overline{M} = 0$ *and* $n_M = 1$ *otherwise.*
*(ii) If* $\overline{M} \simeq \overline{N}$ *then either* $M \simeq N$ *or* $\overline{M} = 0 = \overline{N}$*.*
*(iii) For* $\delta_{MN}$ *defined by* $\delta_{MN} = 1$ *if* $M \simeq N$ *and* $\delta_{MN} = 0$ *otherwise, we have*

$$\dim_\Bbbk \mathrm{Triv}_G(M^* \otimes N) \;=\; \delta_{MN} n_M.$$

*(iv) The category* $\overline{\mathrm{Rep}G}$ *is schurian.*

*Proof.* For the entire proof, let $M, N \in \mathrm{Rep}G$ be indecomposable $\Bbbk G$-modules. By construction, $\mathrm{End}(\overline{M})$ is a quotient of the local algebra $\mathrm{End}_G(M)$ and thus local or zero. Consequently, $\overline{M}$ is either indecomposable or zero. Since $\overline{\mathrm{Rep}G}$ is semisimple, part (i) follows. Part (iv) follows from part (i) and the assumption that $\Bbbk$ is a splitting field for $G$.

Now assume that $M, N$ are not isomorphic and fix a morphism $f : M \to N$. For any morphism $g : N \to M$ we have that $g \circ f$ is not invertible in $\mathrm{End}_G(M)$. Since $\mathrm{End}_G(M)$ is a local and finite dimensional algebra, $g \circ f$ is thus nilpotent. It follows that the morphism $\overline{g} \circ \overline{f}$ of the simple (or zero) object $\overline{M}$ is nilpotent and hence zero. This proves part (ii).

Now let $M, N$ be arbitrary again. As a special case of part (ii), the only indecomposable module in $\mathrm{Rep}G$ which is mapped to $\mathbb{1}$ in $\overline{\mathrm{Rep}G}$ is the trivial one. By Example 2.1.5, we get isomorphisms of vector spaces

$$\mathrm{Triv}_G(M^* \otimes N) \;\simeq\; \Bbbk^{\oplus[\overline{M}^\vee \otimes \overline{N}:\mathbb{1}]} \;\simeq\; \mathrm{Hom}(\overline{M}, \overline{N}).$$

Part (iii) then follows from parts (ii) and (iv). □

*Remark* 2.3.2. It is easy to see that $\overline{M} = 0$ precisely when $\dim_\Bbbk M$ is divisible by $p$.

2.3.3. For each isomorphism class of indecomposable modules $M$ in $\mathrm{Rep}G$ with $n_M = 1$ (as defined in Lemma 2.3.1(i)) we choose one representative. We denote the corresponding set by $\mathbb{B}\overline{G} \subset \mathrm{Ob}\mathrm{Rep}G$. We can interpret $\mathbb{B}\overline{G}$ as the canonical basis of the Grothendieck group of $\overline{\mathrm{Rep}G}$.

*Definition* 2.3.4. Assume that $\Bbbk G$ is of finite representation type and $\mathbf{A}$ is $\Bbbk$-linear. We define the **semisimplification functor**

$$S_G :\; \mathrm{Rep}(G, \mathbf{A}) \;\to\; \mathbf{A} \boxtimes \overline{\mathrm{Rep}G} \quad \text{by} \quad X \mapsto \bigoplus_{M \in \mathbb{B}\overline{G}} \left(\mathrm{Triv}_G(M^* \otimes X) \boxtimes \overline{M}\right).$$

**Proposition 2.3.5.** *Assume that* $\mathbf{A}$ *is semisimple and schurian. Then the composite of*

$$\mathbf{A} \boxtimes \mathrm{Rep}G \;\xrightarrow{\sim}\; \mathrm{Rep}(G, \mathbf{A}) \;\xrightarrow{S_G}\; \mathbf{A} \boxtimes \overline{\mathrm{Rep}G}$$

*is just the product of the identity functor on* $\mathbf{A}$ *and* $\mathrm{Rep}G \to \overline{\mathrm{Rep}G} :\, M \mapsto \overline{M}$.

*Proof.* For simplicity we consider an indecomposable module $N \in \mathrm{Rep}G$ and some object $X_0 \in \mathbf{A}$. The composite is then

$$X_0 \boxtimes N \;\mapsto\; N \otimes X_0 \;\mapsto\; \bigoplus_{M \in \mathbb{B}\overline{G}} (\mathrm{Triv}_G(M^* \otimes N) \otimes X_0) \boxtimes \overline{M} = X_0 \boxtimes \overline{N},$$

by Lemma 2.3.1 □

2.4. **Examples.** Consider a symmetric monoidal category $\mathbf{C}$ as in 1.2.2.

2.4.1. For every $X \in \mathbf{C}$ and $n \in \mathbb{N}$ the braiding $\gamma$ defines a group homomorphism $\mathsf{S}_n \to \mathrm{Aut}(\otimes^n X)$. The permutation $(1,2)$ is for instance sent to $\gamma_{XX} \otimes (\otimes^{n-2}\mathrm{Id}_X)$. We can thus interpret '$\otimes^n$' as a (non-additive) functor

$$X \mapsto \otimes^n X, \;\; \mathbf{C} \to \mathrm{Rep}(\mathsf{S}_n, \mathbf{C}). \tag{9}$$

Recall the dual Specht modules $S_\lambda$ from 1.1.1.

*Definition* 2.4.2. For $\lambda \vdash n$ and $X \in \mathbf{C}$ we define $\Gamma_\lambda(X) \in \mathbf{C}$ as

$$\Gamma_\lambda(X) \;=\; \mathrm{H}^0(\mathsf{S}_n, S_\lambda \otimes (\otimes^n X)).$$

If $\mathrm{char}(\Bbbk) = 0$, by definition we have

$$\otimes^n X \;\simeq\; \bigoplus_{\lambda \vdash n} S^\lambda \otimes \Gamma_\lambda(X),$$

so in that case, $\Gamma_\lambda$ is the Schur functor '$S_\lambda$' of [De2, §1.4].

**Lemma 2.4.3.** *The object* $\mathrm{Triv}_{\mathsf{S}_{n+1}}(\otimes^{n+1}X)$ *is a subquotient of* $\mathrm{Triv}_{\mathsf{S}_n}(\otimes^n X) \otimes X$*. Consequently,* $\mathrm{Triv}_{\mathsf{S}_n}(\otimes^n X) = 0$ *implies that* $\mathrm{Triv}_{\mathsf{S}_r}(\otimes^r X) = 0$ *for all* $r \geq n$*.*

*Proof.* This is a special case of Lemma 2.2.4(i), together with the fact that $\mathrm{Triv}_{\mathsf{S}_n \times 1}(\otimes^{n+1}X)$ is a quotient of $\mathrm{Triv}_{\mathsf{S}_n}(\otimes^n X) \otimes X$ (where the quotient map is an isomorphism if $X$ is flat). □

## 3. Local semisimplicity and freeness

We fix an arbitrary field $\Bbbk$ for the entire section.

### 3.1. **Definitions.** For this subsection we fix a monoidal category $\mathbf{C}$ as in 1.2.2.

3.1.1. For a monomorphism $\alpha : \mathbb{1} \hookrightarrow X$, with $X$ dualisable, and $n \in \mathbb{N}$, we define

$$\alpha^n \in \mathrm{Hom}(\mathbb{1}, \mathrm{Sym}^n X) \quad \text{as the composition } \mathbb{1} \overset{\otimes^n \alpha}{\hookrightarrow} \otimes^n X \twoheadrightarrow \mathrm{Sym}^n X.$$

In other words, we have $\alpha^n = \mathrm{H}_0(\mathsf{S}_n, \otimes^n \alpha)$. We also define $\overline{\alpha} = \oplus_n \alpha^n$, which is an algebra morphism

$$\overline{\alpha} : \; \mathrm{Sym}^\bullet \mathbb{1} \;\to\; \mathrm{Sym}^\bullet X.$$

3.1.2. Fix a short exact sequence $\Sigma$ of dualisable objects in $\mathbf{C}$

$$\Sigma : \; 0 \to U \to V \to W \to 0.$$

This filtration of length 2 on $V$ induces a filtration of length $n+1$ on $\otimes^n V$ with $\mathrm{gr}(\otimes^n V) \simeq \otimes^n(\mathrm{gr}V)$. The quotient $\mathrm{Sym}^n V$ of $\otimes^n V$ is thus also filtered and we get a canonical graded epimorphism

$$\theta^n_\Sigma : \mathrm{Sym}^n(\mathrm{gr}V) \;\twoheadrightarrow\; \mathrm{gr}(\mathrm{Sym}^n V).$$

A priori this need not be an isomorphism, as $\mathrm{H}_0(\mathsf{S}_n, -)$ is only right exact in general. The morphism $\alpha^n$ of 3.1.1 is the degree 1 component of $\theta^n_\Sigma$ composed with the inclusion $\mathrm{gr}_1 \mathrm{Sym}^n V \hookrightarrow \mathrm{Sym}^n V$, in case $U = \mathbb{1}$.

3.1.3. The epimorphism $\theta^n_\Sigma$ is an isomorphism unless $2 \leq \mathrm{char}(\Bbbk) \leq n$. By [EHO, Example 3.3], there exist tensor categories in which $\theta = \oplus_n \theta^n_\Sigma$ is not always an isomorphism if $\mathrm{char}(\Bbbk) = 2$. In [EHO, Question 3.5], Etingof, Harman and Ostrik pose the question of whether $\theta$ (denoted by $\phi_+$ *loc. cit.*) is always an isomorphism in tensor categories for $\mathrm{char}(\Bbbk) > 2$.

### 3.2. **Locally semisimple categories.**

*Definition* 3.2.1. A tensor category $\mathbf{T}$ is **locally semisimple** if there exists a symmetric monoidal category $\mathbf{C}$ as in 1.2.2 and a tensor functor $F : \mathbf{T} \to \mathbf{C}$ which maps every short exact sequence $\Sigma$ in $\mathbf{T}$ to a split short exact sequence $F(\Sigma)$.

By Lemma 1.5.4(ii), all tensor categories which admit fibre functors in the sense of Definition 1.5.1 are locally semisimple. We can characterise locally semisimple tensor categories internally as follows. We freely use the notation and definitions of Subsection 3.1 and the tensor functor $F_{\mathcal{A}} = \mathcal{A} \otimes -$ from (1). Some related results can be found in [Sc, Proposition 5.3.4].

**Theorem 3.2.2.** *A tensor category* $\mathbf{T}$ *is locally semisimple if and only if one of the following equivalent properties is true.*

*(i) For every short exact sequence* $\Sigma$ *in* $\mathbf{T}$*, the epimorphism* $\theta_\Sigma$ *is an isomorphism.*

*(ii) For every $X \in \mathbf{T}$, $n \in \mathbb{N}$ and non-zero $\alpha \in \mathrm{Hom}(\mathbb{1}, X)$, the morphism $\alpha^n$ is non-zero.*
*(iii) For every short exact sequence $\Sigma$ in $\mathbf{T}$ there exists a non-zero $\mathcal{A} = \mathcal{A}_\Sigma$ in $\mathrm{Alg}\mathbf{T}$ such that $\mathcal{A} \otimes \Sigma$ splits in $\mathbf{Mod}_{\mathcal{A}}$.*
*(iv) There exists non-zero $\mathcal{A} \in \mathrm{Alg}\mathbf{T}$ such that for every short exact sequence $\Sigma$ in $\mathbf{T}$, the sequence $\mathcal{A} \otimes \Sigma$ splits in $\mathbf{Mod}_{\mathcal{A}}$.*

*Remark* 3.2.3. (i) If $\mathrm{char}(\Bbbk) = 0$, Theorem 3.2.2(i) shows that all tensor categories are locally semisimple, see also [De1, Lemme 7.14]. If $\mathrm{char}(\Bbbk) > 0$, we will improve Theorem 3.2.2(ii) to Theorem 3.2.4.
(ii) Theorem 3.2.2(i) implies that [EHO, Question 3.5] is equivalent to the open question of whether all tensor categories are locally semisimple if $\mathrm{char}(\Bbbk) \neq 2$.
(iii) If $\mathrm{char}(\Bbbk) \neq 2$, just as in the proof of Theorem 3.2.2(i), we can show that the canonical monomorphism $\mathrm{gr}(\Gamma_{(1^n)}X) \hookrightarrow \Gamma_{(1^n)}(\mathrm{gr}X)$ is always an isomorphism for a filtered object $X$ in a locally semisimple tensor category. The theorem thus shows that in case $\theta$ is always an isomorphism, so is '$\phi_-$' in [EHO, Question 3.5].

**Theorem 3.2.4.** *A tensor category $\mathbf{T}$ over a field $\Bbbk$ with $p := \mathrm{char}(\Bbbk) > 0$ is locally semisimple if and only if for each non-zero $\alpha : \mathbb{1} \to X$ in $\mathbf{T}$, the morphism $\alpha^p : \mathbb{1} \to \mathrm{Sym}^p X$ is non-zero.*

We fix a tensor category $\mathbf{T}$ and start the proof of the theorems with some preparatory results. The following lemma is essentially a reformulation of [De1, Exemple 7.12].

**Lemma 3.2.5.** *Consider a short exact sequence*

$$\Sigma :\ 0 \to \mathbb{1} \xrightarrow{\alpha} X \to Y \to 0$$

*in $\mathbf{T}$. For $(\mathcal{A}, m, \eta) \in \mathrm{Alg}\mathbf{T}$, the sequence $\mathcal{A} \otimes \Sigma$ splits in $\mathbf{Mod}_{\mathcal{A}}$ if and only if we have an algebra morphism $\mathrm{Sym}^\bullet X \to \mathcal{A}$ yielding a commutative diagram of algebra morphisms*

$$\begin{tikzcd} \mathrm{Sym}^\bullet X \arrow[r] & \mathcal{A} \\ \mathrm{Sym}^\bullet \mathbb{1} \arrow[u, "\overline{\alpha}"] \arrow[r, "\rho"] & \mathbb{1} \arrow[u, "\eta"'] \end{tikzcd}$$

*where $\rho$ restricts to the identity $\mathrm{Sym}^1\mathbb{1} \xrightarrow{=} \mathbb{1}$ in degree 1.*

*Proof.* For any algebra $\mathcal{A}$ we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{A}}(\mathcal{A} \otimes X, \mathcal{A}) \arrow[r, "\sim"] \arrow[d, "-\circ(\mathrm{Id}_{\mathcal{A}}\otimes\alpha)"] & \mathrm{Hom}(X, \mathcal{A}) \arrow[r, "\sim"] \arrow[d, "-\circ\alpha"] & \mathrm{Hom}_{alg}(\mathrm{Sym}^\bullet X, \mathcal{A}) \arrow[d, "-\circ\overline{\alpha}"] \\ \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{A}) \arrow[r, "\sim"] & \mathrm{Hom}(\mathbb{1}, \mathcal{A}) \arrow[r, "\sim"] & \mathrm{Hom}_{alg}(\mathrm{Sym}^\bullet \mathbb{1}, \mathcal{A}), \end{tikzcd}$$

see [De1, Example 7.9]. A morphism $f \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}\otimes X, \mathcal{A})$ splits $\mathcal{A}\otimes\Sigma$ if and only if $(\mathrm{Id}_{\mathcal{A}}\otimes\alpha)\circ f = \mathrm{Id}_{\mathcal{A}}$. With $g \in \mathrm{Hom}_{alg}(\mathrm{Sym}^\bullet X, \mathcal{A})$ the image of $f$ under the isomorphisms, this condition becomes commutativity of the diagram

$$\begin{tikzcd} \mathrm{Sym}^\bullet X \arrow[r, "g"] & \mathcal{A}, \\ \mathrm{Sym}^\bullet \mathbb{1} \arrow[u, "\overline{\alpha}"] \arrow[ur, "\eta\circ\rho"'] & \end{tikzcd}$$

which concludes the proof. □

**Corollary 3.2.6.** *If for a short exact sequence $\Sigma$ as in Lemma 3.2.5 we have $\alpha^n \neq 0$ for all $n \in \mathbb{N}$, there exists non-zero $\mathcal{A} \in \mathrm{Alg}\mathbf{T}$ such that $\mathcal{A} \otimes \Sigma$ splits in $\mathbf{Mod}_{\mathcal{A}}$.*

*Proof.* By Lemma 3.2.5 it suffices to prove that the pushout in Alg$\mathbf{T}$

$$\mathrm{Sym}^\bullet X \sqcup_{\mathrm{Sym}^\bullet \mathbb{1}} \mathbb{1} \;\simeq\; \mathrm{Sym}^\bullet X \otimes_{\mathrm{Sym}^\bullet \mathbb{1}} \mathbb{1} =: \mathcal{B}$$

is non-zero. By construction, in Ind$\mathbf{T}$ we have $\mathcal{B} = \varinjlim \mathrm{Sym}^n X$, where the morphisms are given by the composites

$$\mathrm{Sym}^n X \overset{\mathrm{Id}\otimes\alpha}{\hookrightarrow} (\mathrm{Sym}^n X) \otimes X \twoheadrightarrow \mathrm{Sym}^{n+1} X.$$

Consequently, the collection of monomorphisms $\{\alpha^n : \mathbb{1} \to \mathrm{Sym}^n X\}$ yields a monomorphism $\mathbb{1} \hookrightarrow \mathcal{B}$, which proves that the pushout is non-zero. □

*Proof of Theorem 3.2.2.* Assume we have $F : \mathbf{T} \to \mathbf{C}$ as in Definition 3.2.1. We then have $F(\theta^n_\Sigma) = \theta_{F^n(\Sigma)}$. Since $F(\Sigma)$ splits, clearly $\theta_{F(\Sigma)}$ is an isomorphism. Since $F$ is faithful, see Lemma 1.2.6, it follows that $\theta_\Sigma$ is an isomorphism as well. Hence a locally semisimple tensor category satisfies (i).

Property (i) contains (ii) as a special case. That (ii) implies (iii) follows from Corollary 3.2.6 and the isomorphism between $\mathrm{Ext}^1(X, Y)$ and $\mathrm{Ext}^1(Y^\vee \otimes X, \mathbb{1})$, for $X, Y \in \mathbf{T}$, see e.g. [De1, proof of Lemme 7.14].

If (iii) is true, then for every short exact sequence $\Sigma$ in $\mathbf{T}$ we have an algebra $\mathcal{A}_\Sigma$ in Alg$\mathbf{T}$ which splits $\Sigma$. Since $\mathbf{T}$ is essentially small we can take a set $T$ of short exact sequences such that every short exact sequence in $\mathbf{T}$ is isomorphic to one in $T$. Then

$$\mathcal{A} \;=\; \bigotimes_{\Sigma\in T} \mathcal{A}_\Sigma \;:=\; \varinjlim_{S} \bigotimes_{\Sigma\in S} \mathcal{A}_\Sigma \;\in\; \mathrm{Alg}\mathbf{T},$$

where $S$ ranges over all finite subsets of $T$, satisfies condition (iv).

If (iv) is satisfied, the tensor functor $F_{\mathcal{A}} = \mathcal{A} \otimes -$ from (1) makes $\mathbf{T}$ locally semisimple. □

*Proof of Theorem 3.2.4.* One direction is a special case of Theorem 3.2.2(ii). Now assume that $\alpha^p$ is never zero for non-zero $\alpha$ and pick one such $\alpha : \mathbb{1} \to X$. By iterating $j$ times, we find that the morphism

$$\mathbb{1} \to \mathrm{Sym}^p(\mathrm{Sym}^p(\cdots \mathrm{Sym}^p(X)\cdots))$$

is non-zero. By iteration of Lemma 2.2.4(ii), the above morphism can be written as $\mathrm{H}_0(Q_j, \otimes^{p^j}\alpha)$, for $Q_j < \mathsf{S}_{p^j}$ as in 1.1.3. Since $\mathbb{1} = \otimes^{p^j}\mathbb{1}$ is in particular $Q_j$-invariant, we actually find that $\mathrm{Triv}_{Q_j}(\otimes^{p^j}\alpha) \neq 0$. By Proposition 2.2.8 and Lemma 1.1.4, we thus find that $\mathrm{Triv}_{\mathsf{S}_{p^j}}(\otimes^{p^j}\alpha) \neq 0$, so in particular $\alpha^{p^j} = \mathrm{H}_0(\mathsf{S}_{p^j}, \otimes^{p^j}\alpha) \neq 0$, for all $j \in \mathbb{N}$. Since $\alpha^n = 0$ implies $\alpha^{n+1} = 0$, we thus find that $\alpha^n \neq 0$ for all $n \in \mathbb{N}$. The conclusion now follows from Theorem 3.2.2(ii). □

3.3. **Locally free objects and splitting algebras.** For the entire subsection, we consider tensor categories $\mathbf{T}$ and $\mathbf{V}$, with $\mathbf{V}$ schurian and semisimple.

*Definition* 3.3.1. An object $X \in \mathrm{Ob}\mathbf{T}$ is **locally V-free** if there exist $\mathcal{A} \in \mathrm{Alg}(\mathbf{T} \boxtimes \mathbf{V})$ and $X_0 \in \mathrm{Ob}\mathbf{V}$ such that $\mathcal{A} \otimes X \simeq \mathcal{A} \otimes X_0$ in $\mathbf{Mod}^{\mathbf{T}\boxtimes\mathbf{V}}_{\mathcal{A}}$.

**Lemma 3.3.2.** *(i) If $X, Y \in \mathrm{Ob}\mathbf{T}$ are locally $\mathbf{V}$-free, then so are $X \oplus Y$, $X \otimes Y$ and $X^\vee$.*
*(ii) If $\mathbf{V}$ is pointed, then the locally $\mathbf{V}$-free objects form a tensor subcategory of $\mathbf{T}$.*
*(iii) If $\mathbf{T}$ is locally semisimple, any extension of two locally $\mathbf{V}$-free objects is again locally $\mathbf{V}$-free.*
*(iv) If $\mathbf{V}$ is pointed and $\mathbf{T}$ is locally semisimple, the tensor subcategory of $\mathbf{T}$ in (ii) is a Serre subcategory.*

*Proof.* The first observation is straightforward. To prove (ii) it thus suffices to show that any subquotient of a locally $\mathbf{V}$-free object is again locally $\mathbf{V}$-free. Consider a locally $\mathbf{V}$-free object $X \in \mathrm{Ob}\mathbf{T}$. By Lemma 1.2.10 we may assume that there exists $\mathcal{A} \in \mathrm{Alg}(\mathbf{T} \boxtimes \mathbf{V})$, simple in $\mathbf{Mod}_{\mathcal{A}}$, and $X_0 \in \mathbf{V}$ such that $\mathcal{A} \otimes X \simeq \mathcal{A} \otimes X_0$. Since every simple object $S \in \mathbf{V}$ satisfies

$S \otimes S^\vee \simeq \mathbb{1}$, it follows that $\mathcal{A} \otimes S$ is also simple in $\mathbf{Mod}_{\mathcal{A}}$. Consequently $\mathcal{A} \otimes X$ is semisimple and for any subquotient $Y$ of $X$, the subquotient $\mathcal{A} \otimes Y$ of $\mathcal{A} \otimes X_0$ must be of the form $\mathcal{A} \otimes Y_0$ for some $Y_0 \subset X_0$.

To prove (iii) consider a short exact sequence $\Sigma : 0 \to X \to E \to Y \to 0$ in $\mathbf{T}$ where $X$ and $Y$ are locally $\mathbf{V}$-free. By Theorem 3.2.2(iii) and assumption, there exists an algebra $\mathcal{A}_\Sigma \otimes \mathcal{A}_X \otimes \mathcal{A}_Y$ which ensures $E$ is locally $\mathbf{V}$-free. Claim (iv) is just the combination of (ii) and (iii). □

*Definition* 3.3.3. A **$\mathbf{V}$-splitting algebra $\mathcal{A}$ for $\mathbf{T}$** is a non-zero $\mathcal{A} \in \mathrm{Alg}(\mathbf{T} \boxtimes \mathbf{V})$ such that $\mathcal{A}$ splits every short exact sequence in $\mathbf{T}$ (or equivalently in $\mathbf{T} \boxtimes \mathbf{V}$) and for every $X \in \mathrm{Ob}\mathbf{T}$ (or equivalently, for all $X \in \mathbf{T} \boxtimes \mathbf{V}$) there exists $X_0 \in \mathrm{Ob}\mathbf{V}$ for which $\mathcal{A} \otimes X \simeq \mathcal{A} \otimes X_0$ in $\mathbf{Mod}_{\mathcal{A}}$.

When $\mathbf{V} = \mathbf{vec}$, we say 'splitting algebra', rather than '**vec**-splitting algebra'. Recall from Lemma 1.2.12 that, for $\mathcal{A} \in \mathrm{Alg}(\mathbf{T} \boxtimes \mathbf{V})$, the object $\mathcal{R} := \Gamma_{\mathbf{V}}\mathcal{A}$ is an algebra in $\mathbf{V}$ and that $\Gamma_{\mathbf{V}}$ interpreted as a functor $\mathbf{Mod}_{\mathcal{A}}^{\mathbf{T}\boxtimes\mathbf{V}} \to \mathbf{Mod}_{\mathcal{R}}^{\mathbf{V}}$ is canonically lax monoidal.

**Proposition 3.3.4.** *Fix $\mathcal{R} \in \mathrm{Alg}\mathbf{V}$. There is a fully faithful functor from the category of $\mathbf{V}$-splitting algebras $\mathcal{A}$ for $\mathbf{T}$ with $\Gamma_{\mathbf{V}}\mathcal{A} = \mathcal{R}$, as a full subcategory of the category of $\mathcal{R}$-algebras in $\mathbf{T} \boxtimes \mathbf{V}$, to the category of tensor functors $\mathbf{T} \to \mathbf{Mod}_{\mathcal{R}}^{\mathbf{V}}$, given by*

$$\Phi : \mathcal{A} \mapsto \Phi_{\mathcal{A}} := \Gamma_{\mathbf{V}} \circ (\mathcal{A} \otimes -).$$

*Proof.* For an arbitrary algebra $\mathcal{A}$ in $\mathbf{T} \boxtimes \mathbf{V}$ with $\Gamma_{\mathbf{V}}(\mathcal{A}) = \mathcal{R}$, we can define a left exact and lax-monoidal functor $\Phi_{\mathcal{A}} := \Gamma_{\mathbf{V}} \circ (\mathcal{A} \otimes -)$ from $\mathbf{T}$ to $\mathbf{Mod}_{\mathcal{R}}^{\mathbf{V}}$. In case $\mathcal{A}$ is actually a $\mathbf{V}$-splitting algebra, then $\Phi_{\mathcal{A}}$ is monoidal, as follows from Lemma 1.2.12(iii), and exact by Lemma 1.2.12(ii). Hence in this case $\Phi_{\mathcal{A}}$ is a tensor functor, which means that $\Phi$ as in the proposition is well-defined.

There exists a commutative diagram (up to isomorphism) of functors

$$\begin{tikzcd}
\mathrm{Ind}(\mathbf{T} \boxtimes \mathbf{V}) \arrow[r, "M \mapsto \Gamma_{\mathbf{V}}(M \otimes -)"] \arrow[dr] & \mathrm{Funct}_{\Bbbk}(\mathbf{T}, \mathrm{Ind}\mathbf{V}) \arrow[d, "F \mapsto \{X \boxtimes V \mapsto \mathrm{Hom}(V, FX^\vee)\}"] \\
& \mathrm{Funct}_{\Bbbk}((\mathbf{T} \boxtimes \mathbf{V})^{\mathrm{op}}, \mathbf{Vec}),
\end{tikzcd}$$

where the diagonal arrow is the ordinary (fully faithful) embedding of ind-objects in the category of all $\Bbbk$-linear presheaves. It is easy to see that the downwards functor is fully faitfhul and hence so is the horizontal functor.

By the previous paragraph, for two arbitrary algebras $\mathcal{A}$ and $\mathcal{B}$ in $\mathrm{Alg}(\mathbf{T} \boxtimes \mathbf{V})$ with $\Gamma_{\mathbf{V}}(\mathcal{A}) = \mathcal{R} = \Gamma_{\mathbf{V}}(\mathcal{B})$, we have an isomorphism

$$\mathrm{Hom}(\mathcal{A}, \mathcal{B}) \xrightarrow{\sim} \mathrm{Nat}(\Gamma_{\mathbf{V}}(\mathcal{A} \otimes -), \Gamma_{\mathbf{V}}(\mathcal{B} \otimes -)), \;\; f \mapsto \eta^f = \Gamma_{\mathbf{V}}(f \otimes -).$$

To conclude the proof it remains to show that $f$ is an $\mathcal{R}$-algebra morphism if and only if $\eta^f$ is a natural transformation of functors to $\mathbf{Mod}_{\mathcal{R}}$ and of lax monoidal functors. Indeed, it then suffices to apply all of this to the special cases of $\mathbf{V}$-splitting algebras.

It follows immediately that $\eta^f$ is a natural transformation of functors to $\mathbf{Mod}_{\mathcal{R}}$ if and only if $\Gamma_{\mathbf{V}}(f) = \mathrm{Id}_{\mathcal{R}}$, which is the same as saying that $f$ is a morphism over $\mathcal{R}$. From now on we only consider such $f$.

If $f : \mathcal{A} \to \mathcal{B}$ is also an algebra morphism it follows immediately that $\eta^f$ respects the lax monoidal structures of $\Phi_{\mathcal{A}}$ and $\Phi_{\mathcal{B}}$. Now we prove the reverse implication. If $f$ is not an algebra morphism, then there exist $U, V \in \mathbf{V}$ and $X, Y \in \mathbf{T}$ with morphisms $X^\vee \boxtimes U \to \mathcal{A}$ and $Y^\vee \boxtimes V \to \mathcal{A}$, such that the corresponding compositions

$$(X^\vee \boxtimes U) \otimes (Y^\vee \boxtimes V) \;\to \mathcal{A} \otimes \mathcal{A} \;\rightrightarrows\; \mathcal{B}$$

are not equal. It then follows easily that application of $\eta^f$ leads to a non-commutative diagram

$$\Phi_{\mathcal{A}}(X) \otimes_{\mathcal{R}} \Phi_{\mathcal{A}}(Y) \rightrightarrows \Phi_{\mathcal{B}}(X \otimes Y)$$

and hence $\eta^f$ is not a morphism of lax-monoidal functors. This completes the proof. □

**Proposition 3.3.5.** *Assume that every object in* $\mathbf{T}$ *is locally* $\mathbf{V}$*-free and* $\mathbf{V}$ *satisfies* (5).

*(i) The tensor categories* $\mathbf{T}$ *and* $\mathbf{T} \boxtimes \mathbf{V}$ *are locally semisimple.*
*(ii) There exists a* $\mathbf{V}$*-splitting algebra for* $\mathbf{T}$*.*
*(iii) The tensor category* $\mathbf{T}$ *admits a fibre functor* $\mathbf{T} \to \mathbf{Mod}^{\mathbf{V}}_{\mathcal{R}}$ *over some algebra* $\mathcal{R} \in \mathrm{Alg}\mathbf{V}$*.*

*Proof.* For (i) it suffices to prove that $\mathbf{T}' := \mathbf{T} \boxtimes \mathbf{V}$ is locally semisimple. As every object in $\mathbf{T}'$ is locally free, by taking an (infinite) tensor product over the set of isomorphism classes of objects, we obtain $\mathcal{B} \in \mathrm{Alg}\mathbf{T}'$ such that for every $X \in \mathrm{Ob}\mathbf{T}'$ there exists $X_0 \in \mathrm{Ob}\mathbf{V}$ such that $\mathcal{B} \otimes X \simeq \mathcal{B} \otimes X_0$. Now take a monomorphism $\alpha : \mathbb{1} \hookrightarrow X$ in $\mathbf{T}'$. We will show that $\alpha^n \neq 0$ for all $n \geq 1$, so the conclusion in (i) will follow from Theorem 3.2.2(ii).

Observe that any tensor functor $F : \mathbf{T}' \to ?$ is faithful and satisfies $F(\alpha)^n = F(\alpha^n)$. In particular, we have $\alpha^n \neq 0$ if and only if $(\mathcal{B} \otimes \alpha)^n \neq 0$. We compose $\mathcal{B} \otimes \alpha$ with an isomorphism between $\mathcal{B} \otimes X$ and $\mathcal{B} \otimes X_0$ for some $X_0 \in \mathbf{V}$, which exists by assumption, to get a monomorphism

$$\alpha_0 : \mathcal{B} \hookrightarrow \mathcal{B} \otimes X_0 \quad \text{in } \mathbf{Mod}_{\mathcal{B}}.$$

We must show that $\alpha_0^n \neq 0$. Although this can be shown in general, for convenience we will replace $\mathcal{B}$ by a simple quotient, as we can do by Lemmata 1.2.10 and 1.5.3.

We claim that the simplicity of $\mathcal{B}$ implies that $\mathrm{Hom}_{\mathcal{B}}(\mathcal{B}, \mathcal{B} \otimes S) = 0$ for simple $S \in \mathbf{V}$ when $S \not\simeq \mathbb{1}$. By duality we can equivalently consider a non-zero morphism $\mathcal{B} \otimes S \to \mathcal{B}$ for such $S$, which is automatically an epimorphism. Since $- \otimes_{\mathcal{B}} -$ is right exact this yields an epimorphism between the respective $n$-th tensor powers, for all $n \in \mathbb{N}$. Since $\otimes^n_{\mathcal{B}}\mathcal{B} = \mathcal{B} = \mathrm{Sym}^n_{\mathcal{B}}\mathcal{B}$ and $\mathrm{H}_0(\mathsf{S}_n, -)$ is right exact, we thus find epimorphisms

$$\mathcal{B} \otimes \mathrm{Sym}^n S \simeq \mathrm{Sym}^n_{\mathcal{B}}(\mathcal{B} \otimes S) \twoheadrightarrow \mathcal{B}, \quad \text{for all } n \in \mathbb{N}.$$

This is contradicted by assumption (5). Hence the monomorphism $\alpha_0$ is the embedding of the simple direct summand $\mathcal{B}$ of $\mathcal{B} \otimes X_0$. It follows that $\alpha_0^n \neq 0$.

By part (i) we have an algebra $\mathcal{A} \in \mathrm{Alg}(\mathbf{T} \boxtimes \mathbf{V})$ as in Theorem 3.2.2(iv). The tensor product $\mathcal{A} \otimes \mathcal{B}$ is $\mathbf{V}$-splitting for $\mathbf{T}$, proving (ii). Furthermore, (ii) implies (iii) by Proposition 3.3.4. □

3.4. **Neutrality.** As in the previous subsection we consider tensor categories $\mathbf{T}$ and $\mathbf{V}$, with $\mathbf{V}$ schurian and semisimple.

*Definition* 3.4.1. A **neutral V-splitting algebra** $\mathcal{A}$ **for T** is an algebra $\mathcal{A}$ with $\Gamma_{\mathbf{V}}(\mathcal{A}) = \mathbb{1}$ such that for each $X \in \mathbf{T}$ there exists $X_0 \in \mathbf{V}$ for which $\mathcal{A} \otimes X \simeq \mathcal{A} \otimes X_0$.

By the following lemma, this definition is consistent with Definition 3.3.3.

**Lemma 3.4.2.** *If* $\mathcal{A}$ *is a neutral* $\mathbf{V}$*-splitting algebra, then*

*(i)* $\mathcal{A}$ *is a* $\mathbf{V}$*-splitting algebra;*
*(ii) we have a symmetric monoidal equivalence* $\mathbf{Mod}^{\mathbf{T}\boxtimes\mathbf{V}}_{\mathcal{A}} \simeq \mathrm{Ind}\mathbf{V}$*.*

*Proof.* We start by proving (ii), via the symmetric monoidal functor

$$F : \mathrm{Ind}\mathbf{V} \overset{\mathcal{A}\otimes -}{\to} \mathbf{Mod}^{\mathbf{T}\boxtimes\mathbf{V}}_{\mathcal{A}}.$$

It follows easily from $\Gamma_{\mathbf{V}}(\mathcal{A}) = \mathbb{1}$ that $F$ is fully faithful. As for any algebra in $\mathbf{T} \boxtimes \mathbf{V}$, every $\mathcal{A}$-module is a quotient of a direct sum of free modules $\mathcal{A} \otimes Z$ with $Z \in \mathbf{T} \boxtimes \mathbf{V}$. By the assumptions in 3.4.1, any object in $\mathbf{Mod}_{\mathcal{A}}$ thus has a presentation by objects in the image of $F$. Since $F$ is fully faithful and right exact, it follows it is dense and therefore an equivalence.

By (ii), the target of $\mathcal{A}\otimes - : \mathbf{T}\to \mathbf{Mod}_{\mathcal{A}}$ is semisimple, so clearly the functor $\mathcal{A}\otimes -$ splits every short exact sequence. This proves (i). $\square$

**Theorem 3.4.3.** *We have an equivalence from the category of neutral* $\mathbf{V}$*-splitting algebras* $\mathcal{A}$ *for* $\mathbf{T}$ *with the category of tensor functors* $\mathbf{T}\to\mathbf{V}$*, given by*

$$\mathcal{A} \mapsto \Phi_{\mathcal{A}} := \Gamma_{\mathbf{V}}\circ(\mathcal{A}\otimes -).$$

*Proof.* By Proposition 3.3.4 it suffices to prove that the assignment yields a dense functor. Consider a tensor functor $F:\mathbf{T}\to\mathbf{V}$. By Lemma 1.5.8(ii), we can interpret $\mathbf{T}\boxtimes\mathbf{V}$ as the category of representations of an affine group scheme $G$ in $\mathbf{V}$, represented by some $\mathcal{O}\in\mathrm{Alg}\mathbf{V}$. Furthermore, the functor $\omega:\mathbf{T}\boxtimes\mathbf{V}\to\mathbf{V}$, induced from $(F,\mathrm{Id}_{\mathbf{V}})$ is to be interpreted as the functor forgetting the $G$-action

Denote by $\underline{\mathcal{O}}$ the regular $\mathcal{O}$-comodule, which is an object in $\mathrm{Alg}(\mathbf{T}\boxtimes\mathbf{V})$. It follows from the standard properties of Hopf algebras that $\underline{\mathcal{O}}$ is a neutral $\mathbf{V}$-splitting algebra for $\mathbf{T}$. We sketch a proof below.

For any $M\in\mathbf{T}\boxtimes\mathbf{V}$, composition with the counit of $\mathcal{O}$ yields a natural isomorphism

$$\mathrm{Hom}(M,\underline{\mathcal{O}}) \simeq \mathrm{Hom}(\omega(M),\mathbb{1}). \tag{10}$$

We will freely use that the canonical inclusion $\iota:\mathbf{V}\hookrightarrow\mathbf{T}\boxtimes\mathbf{V}$ satisfies $\omega\circ\iota\simeq\mathrm{Id}_{\mathbf{V}}$. Equation (10) thus implies in particular that $\Gamma_{\mathbf{V}}(\underline{\mathcal{O}})=\mathbb{1}$. Furthermore, since $\mathbf{V}$ is semisimple, (10) shows that $\underline{\mathcal{O}}$ is injective in $\mathrm{Ind}(\mathbf{T}\boxtimes\mathbf{V})$. Therefore $\underline{\mathcal{O}}\otimes -$ splits every short exact sequence in $\mathbf{T}\boxtimes\mathbf{V}$. Finally, (10) shows that $\underline{\mathcal{O}}\otimes X\simeq\underline{\mathcal{O}}\otimes\iota\omega(X)$, for any $X\in\mathbf{T}$.

In conclusion, the tensor functor $F$ is isomorphic to $\Phi_{\underline{\mathcal{O}}}=\Gamma_{\mathbf{V}}\circ(\underline{\mathcal{O}}\otimes -)$. $\square$

The proof of Theorem 3.4.3 and Lemma 1.5.8 yield the following examples.

*Example* 3.4.4. (i) The neutral splitting algebras for $\mathbf{T}$ are the algebras in $\mathrm{Alg}\mathbf{T}$ isomorphic to $\underline{\Bbbk[G]}$, under an equivalence $\mathbf{T}\simeq\mathrm{Rep}G$ with $G/\Bbbk$ an affine group scheme.
(ii) If $p\neq 2$, the neutral **svec**-splitting algebras for $\mathbf{T}$ are the algebras in $\mathrm{Alg}(\mathbf{T}\boxtimes\mathbf{svec})$ isomorphic to $\underline{\Bbbk[G]}$, under an equivalence $\mathbf{T}\boxtimes\mathbf{svec}\simeq\mathrm{Rep}G$ with $G$ an affine supergroup scheme.

We conclude this section with two observations regarding restrictions of tensor functors to $\mathbf{V}$.

**Corollary 3.4.5.** *Let* $\mathcal{A}$ *be a neutral* $\mathbf{V}$*-splitting algebra for* $\mathbf{T}$*, and* $\mathbf{T}^0\subset\mathbf{T}$ *a tensor subcategory.*
*(i) The algebra* $\Gamma_{\mathbf{T}^0\boxtimes\mathbf{V}}\mathcal{A}$ *is a neutral* $\mathbf{V}$*-splitting algebra for* $\mathbf{T}^0$*.*
*(ii) For a neutral* $\mathbf{V}$*-splitting algebra* $\mathcal{A}^0$ *for* $\mathbf{T}^0$*, we have* $\Phi_{\mathcal{A}}|_{\mathbf{T}^0}\simeq\Phi_{\mathcal{A}^0}$ *as tensor functors if and only if there exists an algebra morphism* $\mathcal{A}^0\to\mathcal{A}$*.*

*Proof.* Lemma 1.2.12(ii) shows that (i) is true and that $\Phi_{\mathcal{A}}|_{\mathbf{T}^0}\simeq\Phi_{\Gamma_{\mathbf{T}^0\boxtimes\mathbf{V}}(\mathcal{A})}$.

Theorem 3.4.3 and the above paragraph show that $\Phi_{\mathcal{A}}|_{\mathbf{T}^0}\simeq\Phi_{\mathcal{A}^0}$ if and only if $\mathcal{A}^0\simeq\Gamma_{\mathbf{T}^0\boxtimes\mathbf{V}}\mathcal{A}$. The fact that the category of tensor functors $\mathbf{T}^0\to\mathbf{V}$ is a groupoid, see 1.2.7, together with Theorem 3.4.3, shows that $\mathcal{A}^0\simeq\Gamma_{\mathbf{T}^0\boxtimes\mathbf{V}}\mathcal{A}$ if and only if there exists an algebra morphism $\mathcal{A}^0\to\Gamma_{\mathbf{T}^0\boxtimes\mathbf{V}}\mathcal{A}$, which is the same as an algebra morphism $\mathcal{A}^0\to\mathcal{A}$. $\square$

**Lemma 3.4.6.** [De3] *There exists a tensor subcategory* $\mathbf{T}^0\subseteq\mathbf{T}$ *with a tensor functor* $F:\mathbf{T}^0\to\mathbf{V}$ *for which there is no tensor subcategory* $\mathbf{T}^0\subsetneq\mathbf{T}^1\subseteq\mathbf{T}$ *with a tensor functor* $G:\mathbf{T}^1\to\mathbf{V}$ *which satisfies* $G|_{\mathbf{T}^0}\simeq F$*.*

*Proof.* We start by performing two reductions of the statement. If the claim is proved for small tensor categories, then it follows by applying appropriate equivalences for all (essentially small) tensor categories. Consequently, we henceforth assume $\mathbf{T}$ and $\mathbf{V}$ to be small.

It is an easy but tedious exercise to verify that the existence of a tensor functor $G : \mathbf{T}^1 \to \mathbf{V}$ with $G|_{\mathbf{T}^0} \simeq F$ implies existence of a tensor functor $G' : \mathbf{T}^1 \to \mathbf{V}$ which satisfies $G'|_{\mathbf{T}^0} = F$. Hence, we will prove the claim with the isomorphism replaced by an equality.

Now we consider the set of pairs $(\mathbf{T}_\alpha, \Phi_\alpha)$ of tensor subcategories $\mathbf{T}_\alpha \subset \mathbf{T}$ and tensor functors $\Phi_\alpha : \mathbf{T}_\alpha \to \mathbf{V}$. The set is not empty since we can take $\mathbf{T}_\alpha = \langle \mathbb{1} \rangle$. We define the partial order $\leq$ on this set where $(\mathbf{T}_\alpha, \Phi_\alpha) \leq (\mathbf{T}_\beta, \Phi_\beta)$ means $\mathbf{T}_\alpha \subset \mathbf{T}_\beta$ and $\Phi_\beta|_{\mathbf{T}_\alpha} = \Phi_\alpha$. We can apply Zorn's lemma to this poset which yields the desired maximal pair. □

## 4. Frobenius twists in tensor categories

Consider an arbitrary field with $p := \mathrm{char}(\Bbbk) > 0$ and a monoidal category $\mathbf{C}$ as in 1.2.2.

### 4.1. **The symmetric twist.**

4.1.1. Recall the functor $\otimes^{p^j} : \mathbf{C} \to \mathrm{Rep}(\mathsf{S}_{p^j}, \mathbf{C})$, $X \mapsto \otimes^{p^j} X$ from (9).

*Definition* 4.1.2. The $j$-th **symmetric Frobenius twist** is the functor

$$\mathtt{Fr}_+^{(j)} = \mathrm{Triv}_{\mathsf{S}_{p^j}} \circ \otimes^{p^j} : \mathbf{C} \to \mathbf{C}.$$

We also write $\mathtt{Fr}_+ := \mathtt{Fr}_+^{(1)}$. For an exact (for instance out of a tensor category) tensor functor $F : \mathbf{C} \to \mathbf{D}$, the diagram

$$\begin{tikzcd} \mathbf{C} \arrow[r, "\mathtt{Fr}_+^{(j)}"] \arrow[d, "F"] & \mathbf{C} \arrow[d, "F"] \\ \mathbf{D} \arrow[r, "\mathtt{Fr}_+^{(j)}"] & \mathbf{D} \end{tikzcd} \tag{11}$$

commutes up to natural isomorphism.

**Proposition 4.1.3.** *For a tensor category* $\mathbf{T}$*, the following are equivalent:*
*(i) The tensor category* $\mathbf{T}$ *is locally semisimple.*
*(ii) The functor* $\mathtt{Fr}_+ : \mathbf{T} \to \mathbf{T}$ *is exact.*
*(iii) The functor* $\mathtt{Fr}_+^{(j)} : \mathbf{T} \to \mathbf{T}$ *is exact for every* $j \in \mathbb{N}$.

Before proving the proposition, we return to the more general case of monoidal categories $\mathbf{C}$ as in 1.2.2 and prove that $\mathtt{Fr}_+^{(j)}$ is always additive. It can even be made $\Bbbk$-linear by adjusting the $\Bbbk$-linear structure on the target category, but we will omit this interpretation.

**Lemma 4.1.4.** *The functor* $\mathtt{Fr}_+^{(j)}$ *is additive. So, for all* $X, Y \in \mathbf{C}$*, we have*

$$\mathtt{Fr}_+^{(j)}(X \oplus Y) \simeq \mathtt{Fr}_+^{(j)}(X) \oplus \mathtt{Fr}_+^{(j)}(Y), \quad \textit{for } j \in \mathbb{N}.$$

*Proof.* For $f, g \in \mathrm{Hom}(X, Y)$ with $X, Y \in \mathbf{C}$ and $n \in \mathbb{N}$, we have

$$\otimes^n(f+g) = \sum_{a+b=n} \mathrm{Ind}_{\mathsf{S}_a \times \mathsf{S}_b}^{\mathsf{S}_n}((\otimes^a f) \otimes (\otimes^b g)).$$

For $n = p^j$, the index $|\mathsf{S}_n : \mathsf{S}_a \times \mathsf{S}_b|$ is given by the binomial coefficient $\binom{p^j}{a}$. If $a \notin \{0, p^j\}$, this index $\binom{p^j}{a} = \frac{p^j}{a}\binom{p^j-1}{a-1}$ is divisible by $p$. By Corollary 2.2.3(i), we thus have

$$\mathrm{Triv}_{\mathsf{S}_{p^j}}(\otimes^{p^j}(f+g)) = \mathrm{Triv}_{\mathsf{S}_{p^j}}(\otimes^{p^j} f) + \mathrm{Triv}_{\mathsf{S}_{p^j}}(\otimes^{p^j} g),$$

which demonstrates that the functor is additive. □

*Proof of Proposition 4.1.3.* Assume first that $\mathbf{T}$ is locally semisimple via the tensor functor $F : \mathbf{T} \to \mathbf{C}$. By Lemma 4.1.4 and the assumption that $F$ map every short exact sequence to a split one, the composition $\mathtt{Fr}_+^{(j)} \circ F$ is exact. Hence $F \circ \mathtt{Fr}_+^{(j)}$ is exact by commutativity of (11). Since $F$ is exact and faithful, the functor $\mathtt{Fr}_+^{(j)} : \mathbf{T} \to \mathbf{T}$ is also exact. This proves that (i) implies (iii). Furthermore, property (iii) includes (ii) as a special case.

Now consider a monomorphism $\alpha : \mathbb{1} \to X$ in $\mathbf{T}$. We observe that $\alpha^p$ as defined in 3.1.1 is given by $\mathrm{Triv}_{\mathsf{S}_p}(\otimes^p \alpha) = \mathtt{Fr}_+(\alpha)$ composed with the monomorphism $\mathrm{Triv}_{\mathsf{S}_p}(\otimes^p X) \hookrightarrow \mathrm{Sym}^p X$. Now if $\mathtt{Fr}_+$ is exact, then $\mathtt{Fr}_+(\alpha) : \mathbb{1} \to \mathtt{Fr}_+(X)$ is a monomorphism and thus not zero. Consequently $\alpha^p$ is not zero and we apply Theorem 3.2.4 to show that (ii) implies (i). □

*Example* 4.1.5. Take $V \in \mathbf{vec}$, consider the corresponding algebraic group $\mathsf{GL}(V)$ and the category of algebraic representations $\mathbf{T} := \mathrm{Rep}_{\Bbbk}\mathsf{GL}(V)$. We have that $\Gamma^n V$, respectively $\mathrm{Sym}^n V$, is isomorphic to the Weyl module $V(n\epsilon_1)$, respectively dual Weyl module $\mathrm{H}^0(n\epsilon_1)$, see [Jn, §II.2.16]. It follows from [Jn, Propsition II.4.13] that the image of a nonzero morphism from $V(n\epsilon_1)$ to $\mathrm{H}^0(n\epsilon_1)$ is the simple module of highest weight $n\epsilon_1$. By [Jn, Corollary II.3.17] we thus find $\mathtt{Fr}_+^{(j)} V \simeq V^{(j)}$, where $V^{(j)}$ is the classical $j$-th Frobenius twist of $V$ in $\mathrm{Rep}\mathsf{GL}(V)$.

From Lemma 4.1.4 and equation (4) we find the following examples. The first example demonstrates in particular that $\mathtt{Fr}_+$, which is the image of a natural transformation from a lax monoidal to an oplax monoidal functor, is not a monoidal functor.

*Example* 4.1.6.
(i) If $p \neq 2$, for $\mathbf{T} = \mathbf{svec}$ and $V = V_{\bar{0}} \oplus V_{\bar{1}} \in \mathbf{svec}$, we have $\mathtt{Fr}_+ V \simeq V_{\bar{0}}$.
(ii) More generally, for $X$ in $\mathbf{ver}_p$, we have $\mathtt{Fr}_+ X \simeq \mathbb{1}^{\oplus [X:\mathbb{1}]}$.
(iii) If $p = 2$, let $D$ be the triangular Hopf algebra of [EHO, Example 3.3] and $\mathbf{T}$ the category of finite dimensional $D$-modules. Then $\mathtt{Fr}_+ D = 0$, for $D$ the regular $D$-module.

For $j \in \mathbb{Z}_{>0}$, we denote by $\mathtt{Fr}_+^j : \mathbf{C} \to \mathbf{C}$ the composition $\mathtt{Fr}_+ \circ \mathtt{Fr}_+ \circ \cdots \circ \mathtt{Fr}_+$ of $j$ factors $\mathtt{Fr}_+$.

**Lemma 4.1.7.** *For all $X \in \mathbf{C}$, the object $\mathtt{Fr}_+^{(j)}(X)$ is a subquotient of $\mathtt{Fr}_+^j(X)$.*

*Proof.* By Lemma 1.1.4 and Proposition 2.2.8, we have $\mathtt{Fr}_+^{(j)} X \simeq \mathrm{Triv}_{Q_j}(\otimes^{p^j} X)$, with $Q_j < \mathsf{S}_{p^j}$ introduced in 1.1.3. The lemma thus follows by iteration of Lemma 2.2.4(iii). □

For the rest of the subsection, fix a tensor category $\mathbf{T}$ and $X, Y \in \mathbf{T}$.

**Lemma 4.1.8.** *The object $\mathtt{Fr}_+(X) \otimes \mathtt{Fr}_+(Y)$ is a subquotient of $\mathtt{Fr}_+(X \otimes Y)$ in $\mathbf{T}$.*

*Proof.* We have

$$\mathtt{Fr}_+(X) \otimes \mathtt{Fr}_+(Y) \simeq \mathrm{Triv}_{\mathsf{S}_p \times \mathsf{S}_p}((\otimes^p X) \otimes (\otimes^p Y)) \quad \text{and} \quad \mathtt{Fr}_+(X \otimes Y) \simeq \mathrm{Triv}_{\mathsf{S}_p}((\otimes^p X) \otimes (\otimes^p Y)).$$

The conclusion thus follows from Lemma 2.2.4(i) for the diagonal embedding $\mathsf{S}_p \hookrightarrow \mathsf{S}_p \times \mathsf{S}_p$. □

*Remark* 4.1.9.
(i) If we have $p = 2$ and $X \in \mathbf{T}$, we have a short exact sequence
$$0 \to \Lambda^2 X \to \Gamma^2 X \to \mathtt{Fr}_+ X \to 0$$
and one can check directly that $\mathtt{Fr}_+$ is a symmetric monoidal functor.
(ii) By Example 4.1.5, the commutative diagram (11) and Lemmata 4.1.8 and 1.2.6, we find that in tannakian categories we have $\mathtt{Fr}_+(X) \otimes \mathtt{Fr}_+(Y) \simeq \mathtt{Fr}_+(X \otimes Y)$.
(iii) Example 4.1.6(ii) and Lemma 4.1.7 similarly show that in tensor categories which admit a fibre functor over an algebra in $\mathbf{ver}_p$, we have $\mathtt{Fr}_+^{(j)} \simeq \mathtt{Fr}_+^j$.

**Lemma 4.1.10.** *Let $X$ and $S$ be self-dual objects, where $S$ is simple. If $[\otimes^{p^j} X : S] = 1 = [\mathrm{Sym}^{p^j} X : S]$, then $[\mathtt{Fr}_+^{(j)} X : S] = 1$.*

*Proof.* By self-duality we also have $[\Gamma^{p^j}X : S] = 1$. Since $[\otimes^{p^j}X : S] = 1$, it follows that the unique subquotient isomorphic to $S$ must be a subquotient of the image of $\Gamma^{p^j}X \to \mathrm{Sym}^{p^j}X$. $\square$

4.2. **The skew symmetric and internal twist.**

4.2.1. For $X \in \mathbf{C}$ we can restrict the $\mathsf{S}_p$-action on $\otimes^p X$ to the subgroup $\mathsf{C}_p < \mathsf{S}_p$, yielding

$$\mathbf{C} \to \mathrm{Rep}(\mathsf{C}_p, \mathbf{C}), \quad X \mapsto \otimes^p X.$$

*Definition* 4.2.2. The **internal Frobenius twist** is the functor

$$\mathtt{Fr}_{\mathrm{in}} = \mathrm{Triv}_{\mathsf{C}_p} \circ \otimes^p : \mathbf{C} \to \mathbf{C}.$$

**Lemma 4.2.3.** *The functor* $\mathtt{Fr}_{\mathrm{in}}$ *is additive. Moreover, a tensor category* $\mathbf{T}$ *is locally semisimple if and only if* $\mathtt{Fr}_{\mathrm{in}} : \mathbf{T} \to \mathbf{T}$ *is exact.*

*Proof.* Additivity follows as in Lemma 4.1.4, using Corollary 2.2.3(ii). Now consider a tensor category $\mathbf{T}$. By Lemma 2.2.6(iii), the functor $\mathtt{Fr}_{\mathrm{in}}$ contains $\mathtt{Fr}_+$ as a direct summand. Hence Proposition 4.1.3 shows that if $\mathtt{Fr}_{\mathrm{in}}$ is exact, $\mathbf{T}$ must be locally semisimple. The claim in the other direction follows as in the proof of Proposition 4.1.3. $\square$

For the rest of the subsection we assume that $p > 2$.

*Example* 4.2.4. Set $\mathbf{T} = \mathbf{svec}$ and take $V \in \mathbf{svec}$. We have $\mathtt{Fr}_{\mathrm{in}}(V) = V^{(1)} \simeq V$, the ordinary Frobenius twist of $V$ as a $\Bbbk$-module.

*Definition* 4.2.5. For $j \in \mathbb{N}$, the $j$-th **skew symmetric** Frobenius twist is the functor

$$\mathtt{Fr}_-^{(j)} = \mathrm{Triv}_{\mathsf{S}_{p^j}} \circ (\mathrm{sgn}\otimes) \circ \otimes^{p^j} : \mathbf{C} \to \mathbf{C}, \quad X \mapsto \mathrm{Triv}_{\mathsf{S}_{p^j}}(\mathrm{sgn} \otimes (\otimes^{p^j} X)).$$

The following lemma follows from the definition and as above.

**Lemma 4.2.6.** *Take* $j \in \mathbb{N}$.

*(i)* $\mathtt{Fr}_-^{(j)}$ *is additive.*
*(ii)* $\mathtt{Fr}_-^{(j)}$ *is exact if* $\mathbf{C}$ *is a locally semisimple tensor category.*
*(iii) In* $\mathbf{C} \boxtimes \mathbf{svec}$ *we have*

$$\mathtt{Fr}_-^{(j)}(X) \boxtimes \bar{\mathbb{1}} \simeq \mathtt{Fr}_+^{(j)}(X \boxtimes \bar{\mathbb{1}}).$$

*Question* 4.2.7. Let $\mathbf{T}$ be a tensor category.

(i) If $p = 3$, one finds $\mathtt{Fr}_{\mathrm{in}} = \mathtt{Fr}_+ \oplus \mathtt{Fr}_-$. Is the same equation true for $p > 3$?
(ii) If $p = 3$, is $\mathtt{Fr}_{\mathrm{in}}$ monoidal? Closely related, if $p = 3$, is $\mathtt{Fr}_{\mathrm{in}} \simeq \mathtt{Fr}$, with $\mathtt{Fr}$ as in Definition 4.3.1 below?
(iii) Do we have $\mathtt{Fr}_+ \circ \mathtt{Fr}_- = 0 = \mathtt{Fr}_- \circ \mathtt{Fr}_+$?

4.3. **The external twist.** Recall the semisimplification functor $S$ from Definition 2.3.4 and the Verlinde category $\mathbf{ver}_p = \overline{\mathrm{Rep}\mathsf{C}_p}$ in 1.4.3.

*Definition* 4.3.1. The **external Frobenius twist** is the functor

$$\mathtt{Fr} = S_{\mathsf{C}_p} \circ \otimes^p : \mathbf{C} \to \mathbf{C} \boxtimes \mathbf{ver}_p, \quad X \mapsto \bigoplus_{i=1}^{p-1} \mathrm{Triv}_{\mathsf{C}_p}(M_i \otimes (\otimes^p X)) \boxtimes \overline{M}_i.$$

Note that we have $\mathtt{Fr}_+ \simeq \mathtt{Fr}$ if $\mathrm{char}(\Bbbk) = 2$.

**Lemma 4.3.2.** *The functor* $\mathtt{Fr}$ *is additive. If* $\mathbf{T}$ *is a tensor category, then* $\mathtt{Fr}$ *is exact if and only if* $\mathbf{T}$ *is locally semisimple.*

*Proof.* That $\mathtt{Fr}$ is additive follows as in the proof of Lemma 4.1.4, using additionally Lemma 2.2.9. The statement about locally semisimple tensor categories follows as in Lemma 4.2.3. $\square$

**Proposition 4.3.3.** *If* $\mathbf{T}$ *is semisimple and schurian, then* $\mathtt{Fr}$ *coincides with the similarly denoted functor in* [Os, Definition 3.5]*.*

*Proof.* This follows by comparing the definitions and applying Proposition 2.3.5. $\square$

## 5. Characterising locally free objects

We fix a field $\Bbbk$ with $p := \mathrm{char}(\Bbbk) > 0$ and a tensor category $\mathbf{T}$ over $\Bbbk$. We will provide internal characterisations for locally $\mathbf{V}$-free objects for $\mathbf{V}$ equal to $\mathbf{vec}$ and $\mathbf{svec}$.

5.1. **Locally free objects.** By **locally free objects** in $\mathbf{T}$ we refer to locally $\mathbf{vec}$-free objects. We will also replace $\mathbf{T}\boxtimes\mathbf{vec}$ by the equivalent $\mathbf{T}$ in all definitions. Since $\mathcal{A}\otimes X\simeq\mathcal{A}^{\oplus\alpha}$ for $X\in\mathrm{Ob}\mathbf{T}$ and some cardinality $\alpha$ implies that $\alpha$ is finite, by Lemma 1.2.6, there is no need to specify to 'locally free objects of finite rank'.

*Definition* 5.1.1. For $X\in\mathbf{C}$, with $\mathbf{C}$ a monoidal category as in 1.2.2, we define

$$[X]_{\mathbb{1}} \;=\; \sup\{n\in\mathbb{N}\,|\,\Lambda^n(\mathtt{Fr}_+^{(j)}X)\neq 0,\;\text{ for all } j\in\mathbb{N}\}\quad\in\mathbb{N}\cup\{\infty\}.$$

**Theorem 5.1.2.** *For an object $X\in\mathbf{T}$, the following conditions are equivalent.*

*(i)* $[X]_{\mathbb{1}}\in\mathbb{N}$ *and* $\Lambda^rX=0$ *if* $r>[X]_{\mathbb{1}}$*.*

*(ii)* *(a) there exists $n\in\mathbb{N}$ such that $\Lambda^nX=0$;*

  *(b) if $\Lambda^n\mathtt{Fr}_+^{(j)}(X)=0$ for some $j,n\in\mathbb{N}$, then also $\Lambda^nX=0$.*

*(iii)* *$X$ is locally free.*

*(iv)* *There exists a symmetric monoidal category $\mathbf{C}$ as in 1.2.2, a tensor functor $F:\mathbf{T}\to\mathbf{C}$ and $m\in\mathbb{N}$ such that $F(X)\simeq\mathbb{1}^{\oplus m}$ in $\mathbf{C}$.*

We will show below that, at least in characteristic 2, the characterisation in (ii) of locally free objects is strict in the sense that one cannot restrict to finite a number of $j\in\mathbb{N}$. The equivalence of (iii) and (iv) is also proved, in a slightly different generality, in [Sc, Proposition 4.3.8]. Before proving the theorem, we derive some properties for a monoidal category $\mathbf{C}$ as in 1.2.2.

**Lemma 5.1.3.** *For $Y,Z\in\mathbf{C}$, we have*

*(i)* $[\mathbb{1}\oplus Y]_{\mathbb{1}}=1+[Y]_{\mathbb{1}}$*;*

*(ii)* $[Y\oplus Z]_{\mathbb{1}}=0$ *if* $[Y]_{\mathbb{1}}=0=[Z]_{\mathbb{1}}$*;*

*(iii)* *We have $[Y]_{\mathbb{1}}=0$ if and only if there exists $k\in\mathbb{N}$ such that $\mathtt{Fr}_+^{(j)}Y=0$ for all $j\geq k$.*

*Proof.* Part (i) follows from Lemma 4.1.4 and equation (3). Part (iii) follows from Lemma 2.4.3. Part (ii) follows from part (iii) and Lemma 4.1.4. $\square$

The following result is in the proof of [De2, Lemme 2.8]. This is precisely the part of the proof which does not rely on the assumption of characteristic zero.

**Lemma 5.1.4** (Deligne)**.** *Assume that $\mathbf{C}$ admits arbitrary coproducts and let $M\in\mathbf{C}$ be dualisable. For any $(\mathcal{A},m_{\mathcal{A}},\eta_{\mathcal{A}})\in\mathrm{alg}\mathbf{C}$, we have that $\mathcal{A}$ is a direct summand of $\mathcal{A}\otimes M$ in $\mathbf{C}_{\mathcal{A}}$ if and only if there exists an algebra morphism*

$$\mathrm{Sym}^\bullet(M)\otimes\mathrm{Sym}^\bullet(M^\vee)\;\xrightarrow{f}\;\mathcal{A}\quad\textit{with}\quad f\circ\mathrm{co}_M=\eta_{\mathcal{A}}.$$

**Lemma 5.1.5.** *Consider a dualisable object $V$ in $\mathbf{C}$ with quotient $V\overset{\pi}{\twoheadrightarrow}W$ and dualisable subobject $U\overset{\iota}{\hookrightarrow}V$. The composition $\pi\circ\iota$ is zero if and only if $(\pi\otimes\iota^t)\circ\mathrm{co}_V$ is zero.*

*Proof.* The canonical isomorphism

$$\mathrm{Hom}(U,W) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{1}, W\otimes U^\vee), \quad g\mapsto (g\otimes U^\vee)\circ \mathrm{co}_U,$$

maps $\pi\circ\iota$ to $(\pi\otimes\iota^t)\circ\mathrm{co}_V$. □

**Corollary 5.1.6.** *Assume that* $\mathbf{C}$ *admits arbitrary coproducts and let* $M\in\mathbf{C}$ *be dualisable with* $\mathrm{Sym}^n(M^\vee)$ *flat for all* $n\in\mathbb{N}$*. There exists a non-zero* $\mathcal{A}\in\mathrm{alg}\mathbf{C}$ *for which* $\mathcal{A}$ *is a direct summand of* $\mathcal{A}\otimes M$ *in* $\mathbf{C}_{\mathcal{A}}$ *if and only if* $[M]_{\mathbb{1}}>0$.

*Proof.* We start from Lemma 5.1.4. Like all algebra morphisms, any $f$ as in Lemma 5.1.4 is assumed to satisfy $f\circ\eta=\eta_{\mathcal{A}}$ with $\eta$ the unit of the algebra $\mathrm{Sym}^\bullet(M)\otimes\mathrm{Sym}^\bullet(M^\vee)$. The existence of $\mathcal{A}$ is thus equivalent to the quotient of $\mathrm{Sym}^\bullet(M)\otimes\mathrm{Sym}^\bullet(M^\vee)$ with respect to the ideal generated by $(\eta-\mathrm{co}_M)(\mathbb{1})$ being non-zero. As argued in the proof of [De2, Lemme 2.8] this is equivalent to the composition

$$\mathbb{1} \overset{\otimes^n\mathrm{co}_M}{\hookrightarrow} (\otimes^n M)\otimes(\otimes^n M^\vee) \twoheadrightarrow \mathrm{Sym}^n(M)\otimes\mathrm{Sym}^n(M^\vee)$$

being non-zero for all $n\in\mathbb{N}$.

By Lemmata 1.3.2 and 5.1.5 this is equivalent to the composition

$$\mathrm{H}^0(\mathsf{S}_n,\otimes^n M)=\Gamma^n(M)\hookrightarrow\otimes^n M\twoheadrightarrow\mathrm{Sym}^n M=\mathrm{H}_0(\mathsf{S}_n,\otimes^n M)$$

being non-zero. The latter just means that $\mathrm{Triv}_{\mathsf{S}_n}(\otimes^n M)$ is never zero. By Lemma 2.4.3 the condition is thus equivalent to $\mathtt{Fr}_+^{(j)}M\neq 0$ for all $j\in\mathbb{N}$. □

**Proposition 5.1.7.** *For* $X\in\mathbf{T}$ *and* $d\in\mathbb{N}$*, we have* $[X]_{\mathbb{1}}\geq d$ *if and only if there exists a non-zero* $\mathcal{A}\in\mathrm{Alg}\mathbf{T}$ *and* $N\in\mathbf{Mod}_{\mathcal{A}}$ *such that*

$$\mathcal{A}\otimes X \;\simeq\; \mathcal{A}^{\oplus d}\oplus N, \quad \textit{in } \mathbf{Mod}_{\mathcal{A}}.$$

*Proof.* We start by observing that for any non-zero $\mathcal{R}\in\mathrm{Alg}\mathbf{T}$, we have

$$[\mathcal{R}\otimes X]_{\mathbb{1}} \;=\; [X]_{\mathbb{1}}, \quad \text{for } X\in\mathbf{T}, \tag{12}$$

since $\mathcal{R}\otimes -$ is a (faithful exact) tensor functor. If $\mathcal{A}^{\oplus d}$ is a direct summand of $\mathcal{A}\otimes X$, then Lemma 5.1.3(i) and equation (12) imply that $[X]_{\mathbb{1}}\geq d$.

To prove the other direction, we apply induction on $d$. If $d=0$ there is nothing to prove. Assume that the claim is true for $d-1$. Hence, if $[X]_{\mathbb{1}}\geq d$ we know that there exists $\mathcal{B}$ in $\mathrm{Alg}\mathbf{T}$ and $M$ in $\mathbf{Mod}_{\mathcal{B}}$ such that

$$\mathcal{B}\otimes X \;\simeq\; \mathcal{B}^{\oplus(d-1)}\oplus M.$$

By Lemma 5.1.3(i) and equation (12) we have $[M]_{\mathbb{1}}>0$. By construction, $\mathrm{Sym}^n_{\mathcal{B}}(M^\vee)$ is a direct summand of $\mathcal{B}\otimes\mathrm{Sym}^n(X^\vee)$ and therefore dualisable, so in particular flat. By Corollary 5.1.6 for $\mathbf{C}=\mathbf{Mod}_{\mathcal{B}}$, there exists $\mathcal{A}$ in $\mathrm{alg}\mathbf{Mod}_{\mathcal{B}}$, which we can also interpret in $\mathrm{Alg}\mathbf{T}$, for which

$$\mathcal{A}\otimes X \;\simeq\; \mathcal{A}\otimes_{\mathcal{B}}(\mathcal{B}\otimes X) \;\simeq\; \mathcal{A}^{\oplus d-1}\oplus\mathcal{A}\otimes_{\mathcal{B}}M \;\simeq\; \mathcal{A}^{\oplus d}\oplus N,$$

which concludes the proof. □

*Proof of Theorem 5.1.2.* Take $X$ as in (i) and set $d:=[X]_{\mathbb{1}}$. By Proposition 5.1.7, there exists $\mathcal{A}\in\mathrm{Alg}\mathbf{T}$ such that $\mathcal{A}\otimes X$ is of the form $\mathcal{A}^{\oplus d}\oplus N$. By assumption and (3), we have

$$0 \;\simeq\; \mathcal{A}\otimes\Lambda^{d+1}X \;\simeq\; \Lambda^{d+1}_{\mathcal{A}}(\mathcal{A}\otimes X) \;\simeq\; \bigoplus_{i=0}^{d}\left(\Lambda^{i+1}_{\mathcal{A}}N\right)^{\oplus\binom{d}{i}},$$

which implies $N=0$. Hence (i) implies (iii). Condition (iii) clearly implies (iv). That (iv) implies (ii) follows from the fact that $F$ is a (faithful exact) tensor functor. That (ii) implies (i) is straightforward. □

**Lemma 5.1.8.** *For $X, Y \in \mathbf{T}$ with $[X]_{\mathbb{1}}, [Y]_{\mathbb{1}} \in \mathbb{N}$ we have $[X \oplus Y]_{\mathbb{1}} = [X]_{\mathbb{1}} + [Y]_{\mathbb{1}}$.*

*Proof.* By Proposition 5.1.7, there exists $\mathcal{A} \in \mathrm{Alg}\mathbf{T}$ such that

$$\mathcal{A} \otimes X \;\simeq\; \mathcal{A}^{\oplus [X]_{\mathbb{1}}} \oplus M \quad \text{and} \quad \mathcal{A} \otimes Y \;\simeq\; \mathcal{A}^{\oplus [Y]_{\mathbb{1}}} \oplus N,$$

for $M, N$ in $\mathbf{Mod}_{\mathcal{A}}$. By Lemma 5.1.3(i) and equation (12) we find that $[M]_{\mathbb{1}} = 0 = [N]_{\mathbb{1}}$. By Lemma 5.1.3(i) and (ii) we then find $[\mathcal{A} \otimes (X \oplus Y)]_{\mathbb{1}} = [X]_{\mathbb{1}} + [Y]_{\mathbb{1}}$ and the conclusion follows from equation (12). □

**Lemma 5.1.9.** *Assume $\Bbbk$ is algebraically closed with $\mathrm{char}(\Bbbk) = 2$. For each $n \in \mathbb{Z}_{>0}$ there exists a tensor category $\mathbf{T}_n$ with $X_n \in \mathbf{T}_n$ such that $\Lambda^3 X_n = 0$ and*

$$\Lambda^d \mathtt{Fr}_+^{(j)} X_n = 0 \quad \Rightarrow \quad \Lambda^d X_n = 0, \qquad \textit{for all } d \in \mathbb{N} \textit{ and } j < n,$$

*but $X_n$ is not locally free.*

*Proof.* For $n \in \mathbb{N}$, let $\mathbf{T}_n$ be the tensor category $\mathcal{C}_{2n}$ of [BE, Theorem 2.1], which contains $\mathcal{C}_{2n-2}$ as subcategory if $n > 0$. For $X_n$ we take the similarly named self-dual simple object in [BE, 2.1(iii)], so $X_n = 0$ if and only if $n = 0$. It follows from [BE, 2.1(vi) and (ix)] that

$$[\otimes^{2^j} X_n : X_{n-j}] = 1 \;\text{ and }\; [\otimes^{2^i} X_n : X_{n-j}] = 0, \quad \text{for } i < j < n. \tag{13}$$

By [BE, 2.1(iii) and (vii)] we have $\Lambda^2 X_n = \mathbb{1}$ and $\Lambda^3 X_n = 0$, if $n > 0$. The former of the two equations implies $[\mathrm{Sym}^{2^j} X_n : X_{n-j}] = 1$, by equations (2) and (13). By Lemma 4.1.10, we thus have $[\mathtt{Fr}^{(j)} X_n : X_{n-j}] = 1$ for $j < n$. It follows easily from [BE, 2.1(xi)] that $\mathtt{Fr}^j X_n = X_{n-j}$ for $j \leq n$. By Lemma 4.1.7, we thus have $\mathtt{Fr}^{(j)} X_n = X_{n-j}$ for $j \leq n$. □

5.2. **Locally super free objects.** In this subsection we assume that $p \neq 2$.

*Definition* 5.2.1. For $X \in \mathbf{C}$ with $\mathbf{C}$ as in 1.2.2 we define $[X]_{\bar{\mathbb{1}}} \in \mathbb{N} \cup \{\infty\}$ as

$$[X]_{\bar{\mathbb{1}}} \;=\; \sup\{n \in \mathbb{N} \,|\, \Gamma^n(\mathtt{Fr}_-^{(j)} X) \neq 0, \;\text{ for all } j \in \mathbb{N}\}.$$

Recall the dual Specht modules from 1.1.1 and the functors $\Gamma_\lambda$ from Definition 2.4.2.

**Theorem 5.2.2.** *An object $X \in \mathbf{T}$ is locally* **svec***-free if and only if $([X]_{\mathbb{1}}, [X]_{\bar{\mathbb{1}}}) \in \mathbb{N} \times \mathbb{N}$ and $\Gamma_\lambda X = 0$ for the partition $\lambda = (([X]_{\bar{\mathbb{1}}} + 1)^{[X]_{\mathbb{1}}+1})$.*

**Proposition 5.2.3.** *For $X \in \mathbf{T}$ and $d, d' \in \mathbb{N}$, we have $[X]_{\mathbb{1}} \geq d$ and $[X]_{\bar{\mathbb{1}}} \geq d'$ if and only if there exists non-zero $\mathcal{A} \in \mathrm{Alg}(\mathbf{T} \boxtimes \mathbf{svec})$ and $N \in \mathbf{Mod}_{\mathcal{A}}$ such that*

$$\mathcal{A} \otimes X \;\simeq\; \mathcal{A} \otimes (\mathbb{1}^{\oplus d} \oplus \bar{\mathbb{1}}^{\oplus d'}) \oplus N, \quad \textit{in } \mathbf{Mod}_{\mathcal{A}}.$$

*Proof.* This is proved similarly to Proposition 5.1.7, using Lemma 4.2.6(iii). □

**Lemma 5.2.4.** *Take $r \in \mathbb{N}$ and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l) \vdash r$ with $\mu = (\lambda_1, \lambda_2, \cdots, \lambda_{l-1})$. We have a monomorphism of $\Bbbk(\mathsf{S}_{r-\lambda_l} \times \mathsf{S}_{\lambda_l})$-modules*

$$S_\mu \boxtimes \Bbbk \;=\; S_\mu \boxtimes S_{\lambda_l} \;\hookrightarrow\; \mathrm{Res}^{\mathsf{S}_r}_{\mathsf{S}_{r-\lambda_l} \times \mathsf{S}_{\lambda_l}} S_\lambda.$$

*Proof.* We can label the basis in [Jm, §4] of the Specht module $S^{\lambda^t}$ by all standard Young tableaux of shape $\lambda^t$. It is easy to see that the subset of basis elements for which the last column in the corresponding the tableau has labels in the interval $]r - \lambda_l, r]$ spans a subspace invariant under the action of $\mathsf{S}_{r-\lambda_l} \times \mathsf{S}_{\lambda_l}$, which is isomorphic to $S^{\mu^t} \boxtimes \mathrm{sgn}$. Taking the tensor product with the sign module for $\mathsf{S}_r$ then yields the desired inclusion. □

**Corollary 5.2.5.** *Fix $m, a \in \mathbb{N}$ and $Y \in \mathbf{C}$. If $\Gamma_{(a^{m+1})}(\mathbb{1} \oplus Y) = 0$, then $\Gamma_{(a^m)}(Y) = 0$.*

*Proof.* Set $r = (m+1)a$ and $\lambda = (a^{m+1}) \vdash r$. By Lemmata 2.2.9 and 2.2.2 we have

$$0 = \Gamma_\lambda(\mathbb{1} \oplus Y) \;\simeq\; \bigoplus_{i=0}^{r} \mathrm{H}^0(\mathsf{S}_{r-i} \times \mathsf{S}_i, S_\lambda \otimes (Y^{\otimes r-i} \otimes \mathbb{1}^{\otimes i})).$$

In particular, we have

$$\mathrm{H}^0(\mathsf{S}_{r-a} \times \mathsf{S}_a, (\mathrm{Res}^{\mathsf{S}_r}_{\mathsf{S}_{r-a}\times \mathsf{S}_a} S_\lambda) \otimes Y^{\otimes r-a} \otimes \mathbb{1}^{\otimes a}) = 0.$$

Lemma 5.2.4 and the fact that $\mathrm{H}^0(G,-)$ is left exact show that

$$0 \;=\; \mathrm{H}^0(\mathsf{S}_{r-a}, S_{(a^m)} \otimes Y^{\otimes r-a}) \otimes \mathrm{H}^0(\mathsf{S}_a, \mathbb{1}^{\otimes a}) \;\simeq\; \Gamma_{(a^m)}(Y),$$

which concludes the proof. □

*Proof of Theorem 5.2.2.* One direction of the claim is straightforward. Now assume that $(m,n) := ([X]_{\mathbb{1}}, [X]_{\bar{\mathbb{1}}})$ is as in the theorem. By Proposition 5.2.3, there exists non-zero $\mathcal{A} \in \mathrm{Alg}(\mathbf{T} \boxtimes \mathbf{svec})$ for which

$$\mathcal{A} \otimes X \;\simeq\; \mathcal{A} \otimes (\mathbb{1}^{\oplus m} \oplus \bar{\mathbb{1}}^{\oplus n}) \,\oplus\, N.$$

By assumption, for the partition $\lambda = ((n+1)^{m+1})$, we have

$$0 = \mathcal{A} \otimes \Gamma_\lambda(X) \;\simeq\; \Gamma_\lambda(\mathcal{A} \otimes (\mathbb{1}^{\oplus m} \oplus \bar{\mathbb{1}}^{\oplus n}) \,\oplus\, N).$$

By iterating Corollary 5.2.5 in $\mathbf{C} = \mathbf{Mod}_{\mathcal{A}}$, this implies that

$$0 \;=\; \Gamma^{n+1}_{\mathcal{A}}(\mathcal{A} \otimes \bar{\mathbb{1}}^{\oplus n} \,\oplus\, N) \;\simeq\; \bigoplus_{i=0}^{n} \Gamma^{i+1}_{\mathcal{A}}(N)^{\oplus \binom{n}{i}},$$

from which we can deduce that $N = 0$. □

## 6. Internal characterisations

Fix a field $\Bbbk$ of characteristic $p = \mathrm{char}(\Bbbk) > 0$.

### 6.1. Tannakian categories.

The following generalises [De1, Théorème 7.1] to fields of positive characteristic.

**Theorem 6.1.1.** *For a tensor category* $\mathbf{T}$ *the following conditions are equivalent:*

*(i)* $\mathbf{T}$ *is tannakian.*
*(ii) For every* $X$ *in* $\mathbf{T}$*,*
   *(a) there exists* $n \in \mathbb{N}$ *such that* $\Lambda^n X = 0$*;*
   *(b) if* $\Lambda^n \mathtt{Fr}^{(j)}_+(X) = 0$ *for some* $j, n \in \mathbb{N}$*, then also* $\Lambda^n X = 0$*.*
*(iii) Every* $X$ *in* $\mathbf{T}$ *is locally free.*

*Proof.* First we prove that (i) implies (ii). If $\mathbf{T}$ is Tannakian, it admits a tensor functor to $\mathbf{vec}_K$ for some field extension $K/\Bbbk$, by Lemma 1.5.4. The properties in (ii) are satisfied in $\mathbf{vec}_K$, since the objects $\Lambda^n X$ and $\mathtt{Fr}^{(j)}_+ X$ are the same for $\mathbf{vec}_K$ considered as a $K$-linear or $\Bbbk$-linear category. By Lemma 1.2.6 and diagram (11), they are thus satisfied in $\mathbf{T}$ as well. Theorem 5.1.2 states that (ii) implies (iii). Proposition 3.3.5(iii) shows that (iii) implies (i). □

**Proposition 6.1.2.** *Any tensor category* $\mathbf{T}$ *has a unique maximal tannakian subcategory, the tensor subcategory of locally free objects. If* $\mathbf{T}$ *is locally semisimple, the latter is a Serre subcategory.*

*Proof.* By Theorem 6.1.1 it suffices to prove that the full subcategory of locally free objects is a tensor subcategory, respectively a Serre subcategory. These are special cases of Lemma 3.3.2. □

*Remark* 6.1.3. (i) By Lemma 3.3.2(ii), we can simplify Theorem 6.1.1 as follows. A tensor category $\mathbf{T}$ is tannakian if and only if $\mathbf{T} = \langle E \rangle$ for a set $E$ of locally free objects in $\mathbf{T}$. In particular, if $\mathbf{T}$ is finitely generated, $\mathbf{T} = \langle Y \rangle$ for $Y \in \mathbf{T}$, it suffices to check condition 6.1.1(ii) on $Y$.

(ii) In subsequent work in [CE, Proposition 7.2], it is shown that if $p > 2$ and $\Bbbk$ is algebraically closed, the condition that $\mathbf{T}$ be locally semisimple is redundant for the maximal tannakian subcategory to be a Serre subcategory. Note that for $p = 2$, the condition is necessary, see [EHO, Example 3.3].

6.2. **Super tannakian categories.** In this subsection we assume that $p \neq 2$.

**Theorem 6.2.1.** *For a tensor category* $\mathbf{T}$*, the following conditions are equivalent:*

*(i)* $\mathbf{T}$ *is super tannakian.*

*(ii) For every* $X \in \mathbf{T}$*, we have that* $(m,n) := ([X]_{\mathbb{1}}, [X]_{\bar{\mathbb{1}}}) \in \mathbb{N} \times \mathbb{N}$*, and*

$$\Gamma_\lambda X = 0, \qquad \text{for } \lambda = ((n+1)^{m+1}).$$

*(iii) Every* $X$ *in* $\mathbf{T}$ *is locally* **svec***-free.*

*Proof of Theorem 6.2.1.* That (i) implies (ii) is proved as in the proof of Theorem 6.1.1. That (ii) implies (iii) is in Theorem 5.2.2. That (iii) implies (i) is in Proposition 3.3.5. □

**Proposition 6.2.2.** *Any tensor category* $\mathbf{T}$ *has a unique maximal super tannakian subcategory. If* $\mathbf{T}$ *is locally semisimple, the latter is a Serre subcategory.*

*Proof.* Mutatis mutandis Proposition 6.1.2. □

*Example* 6.2.3. Let $\mathbf{V}$ be a semisimple pointed tensor category. In particular, for simple $S \in \mathbf{V}$, the object $\otimes^n S$ is simple, for all $n \in \mathbb{N}$. We thus either have $\mathrm{Sym}^2 S = 0$ or $\Lambda^2 S = 0$. It then follows that either $\Lambda^n S = \otimes^n S$ or $\Gamma^n S = \otimes^n S$, for all $n \in \mathbb{N}$. In both cases, $S$ is easily seen to be locally **svec**-free. It follows from Theorem 6.2.1 that $\mathbf{V}$ is super tannakian. If $p = 2$ one shows similarly that any pointed semisimple tensor category is tannakian.

6.3. **Affine group schemes.** In order to proceed to the last section on neutrality of tannakian categories we need a short interlude on affine group schemes. We refer to [Wa] for the basic notions and to [Ma] for the corresponding results for supergroup schemes. We prove some facts which are presumably well-known, but for which we did not find references.

Consider an affine supergroup scheme $G$ with homomorphism $p : \mathbb{Z}/2 \to G$ inducing the grading on $\Bbbk[G]$, see [De1, §8.19]. We denote by $\mathrm{Rep}(G,p)$ the category of $G$-representations in **svec** which yield the canonical $\mathbb{Z}/2$-action on super spaces via $p$. As a special case, we can consider an affine group scheme as an affine supergroup scheme and set $p$ to be the trivial homomorphism. Then $\mathrm{Rep}(G,p)$ corresponds to the ordinary category of $G$-representations in **vec**. For a closed sub(super)group $H < G$ we denote by $(\mathrm{Rep}(G,p))^H$ the tensor subcategory of all representations for which $H$ is in the kernel. Recall the notion of 'tensor subcategory' from 1.2.11.

**Theorem 6.3.1.** *Consider an affine supergroup scheme* $G$ *with* $p : \mathbb{Z}/2 \to G$ *as above. There is a bijection between closed normal subgroups* $N \lhd G$ *and tensor subcategories of* $\mathrm{Rep}(G,p)$*:*

$$N \mapsto (\mathrm{Rep}(G,p))^N.$$

*Furthermore, the essential image of the canonical tensor functor* $\mathrm{Rep}(G/N,p) \to \mathrm{Rep}(G,p)$ *is precisely* $(\mathrm{Rep}(G,p))^N$*.*

*Proof.* The statement about the essential image follows immediately from the existence of the quotient $G/N$ in [Wa, Theorem 16.3] and its universality, see [Wa, Theorem 15.4].

For any tensor subcategory $\mathbf{T} \subset \mathrm{Rep}(G,p)$, composition of the inclusion functor and the forgetful functor $\mathrm{Rep}(G,p) \to \mathbf{svec}$ yields a fibre functor $\mathbf{T} \to \mathbf{svec}$. By Lemma 1.5.8(i), there exists a super group scheme $H$ under $\mathbb{Z}/2$ with an equivalence $\mathrm{Rep}(H,p) \simeq \mathbf{T}$ of tensor categories. Since the functor $\mathrm{Rep}(H,p) \simeq \mathbf{T} \hookrightarrow \mathrm{Rep}(G,p)$ admits a commutative diagram (up to isomorphism) with the forgetful functors to **svec**, it follows from Lemma 1.5.8(i) that it induces a homomorphism $G \to H$ under $\mathbb{Z}/2$, which in turns induces the functor $\mathrm{Rep}(H,p) \to \mathrm{Rep}(G,p)$, up to isomorphism. By [DM, Proposition 2.21(a)], the morphism $\Bbbk[H] \to \Bbbk[G]$ is injective, so by definition in [Wa, §15.1], $H$ is a quotient of $G$ and hence of the form $G/N$ for a closed normal subgroup $N$, see [Wa, Corollary 16.3].

By the first paragraph, the above procedure assigning a normal subgroup to a tensor subcategory is a two-sided inverse of the one in the theorem. □

**Corollary 6.3.2.** *Suppose that under the bijection in Theorem 6.3.1, we have $N_1 \mapsto \mathbf{T}_1$ and $N_2 \mapsto \mathbf{T}_2$, for closed normal subgroups $N_1, N_2 \lhd G$. Then we also have $N_1 \cap N_2 \mapsto \langle \mathbf{T}_1, \mathbf{T}_2 \rangle$ and $N_1 N_2 \mapsto \mathbf{T}_1 \cap \mathbf{T}_2$.*

*Proof.* By construction, the bijection is order reversing, for the inclusion orders on both sets. The partially ordered sets are actually lattices, so the join and meet will be interchanged. □

**Corollary 6.3.3.** *For $p : \mathbb{Z}/2 \to G$ as in Theorem 6.3.1 and $\mathbf{T} = \mathrm{Rep}(G,p)$, the following are equivalent:*

*(i) $\mathbf{T}$ is finitely generated as a tensor category;*
*(ii) every tensor subcategory of $\mathbf{T}$ is finitely generated;*
*(iii) $G$ is algebraic, i.e. $\Bbbk[G]$ is finitely generated as an algebra.*

*Proof.* For brevity we leave out reference to 'super'. Condition (iii) implies that the topological space underlying the scheme $G = \mathrm{Spec}\Bbbk[G]$ is noetherian. As the poset of closed normal subgroups is a sub-poset of the poset of all closed subspaces, the implication $(iii) \Rightarrow (ii)$ follows from Theorem 6.3.1. Clearly (ii) implies (i). That (i) implies (iii) is in [DM, Proposition 2.20(ii)]. □

**Proposition 6.3.4.** *Let $G$ be an affine supergroup scheme and $N_1, N_2$ closed normal subgroups with $N_1 \cap N_2 = 1$. The canonical homomorphism*

$$G \;\to\; G/N_1 \times_{G/N_1N_2} G/N_2$$

*is an isomorphism. Equivalently, we have $\Bbbk[G/N_1] \otimes_{\Bbbk[G/N_1N_2]} \Bbbk[G/N_2] \xrightarrow{\sim} \Bbbk[G]$.*

*Proof.* For brevity we write the proof only for groups. For a closed normal subgroup $N \lhd G$, denote by $D^G_N$ the quotient of $G$ and $N$ as functors $\mathbf{Alg}_\Bbbk \to \mathbf{Grp}$. By definition of $G/N$, see [Wa, §16.3], $D^G_N$ is a subfunctor of $G/N$. We clearly have an isomorphism of group functors

$$G \;\xrightarrow{\sim}\; D^G_{N_1} \times_{D^G_{N_1N_2}} D^G_{N_2}.$$

As spelled out in [Wa, Theorem 15.5], $D^G_N$ is a 'fat subfunctor' of $G/N$. It follows as an easy exercise that the right-hand side of the above equation is therefore a fat subfunctor of $G/N_1 \times_{G/N_1N_2} G/N_2$ and consequently that the homomorphism in the proposition is the inclusion of a fat subfunctor. However, since $G$ is itself a sheaf for the fpqc topolgy on $(\mathbf{Alg}_\Bbbk)^{\mathrm{op}}$, see [Wa, §15.6], such an inclusion of $G$ must be an isomorphism. □

**Lemma 6.3.5.** *If $\Bbbk$ is algebraically closed, $G$ is an algebraic supergroup over $\Bbbk$ and $N$ a closed normal subgroup, then the canonical left exact sequence*

$$1 \to N(\Bbbk) \to G(\Bbbk) \to G/N(\Bbbk) \to 1$$

*is exact.*

*Proof.* For groups, this is [Wa, Theorem 15.2]. For supergroups, the result is a consequence of the former and [Ma, Theorem 3.13(3)]. ☐

6.4. **Neutrality over algebraically closed fields.** Deligne announced in [De2] that over algebraically closed fields (of characteristic zero, although that is not essential) all tannakian categories are neutral. However, the proof was deemed 'too painful' to add. In his letter [De3], Deligne sketched the argument, and it was written out in more detail by the author in [Appendix A, arXiv:1812.02452v2]. In this section we present a variation of this argument, perhaps slightly less painful, based on the notion of splitting algebras.

**Theorem 6.4.1.** *If $\Bbbk$ is algebraically closed then any (super) tannakian category is neutral.*

For the rest of the subsection *we assume* $\Bbbk = \overline{\Bbbk}$. For finitely generated tensor categories, Theorem 6.4.1 is proved in [De1, Corollaire 6.20] and [De2, Proposition 4.5].

**Lemma 6.4.2.** *Let $\mathbf{T}$ be a finitely generated (super) tannakian tensor category.*

*(i) $\mathbf{T}$ admits a neutral ($\mathbf{svec}$-)splitting algebra, unique up to isomorphism.*
*(ii) For a a neutral ($\mathbf{svec}$-)splitting algebra $\mathcal{A}$ for $\mathbf{T}$ and a tensor subcategory $\mathbf{T}' \subset \mathbf{T}$ any algebra morphism on $\Gamma_{\mathbf{T}'}\mathcal{A}$ lifts to one on $\mathcal{A}$.*

*Proof.* By Theorem 3.4.3, part (i) is a reformulation of the claim that every finitely generated (super) tannakian category is neutral (as we observed above) and that fibre functors to **vec** or **svec** are unique up to isomorphism, see Proposition 1.5.9.

By Theorem 3.4.3, endomorphisms of a neutral splitting algebra correspond to the automorphisms of the associated fibre functor. By Lemma 1.5.8, the latter constitute the group of rational $\Bbbk$-points of the corresponding affine (super)group scheme. Part (ii) is therefore a reformulation of Lemma 6.3.5, by Theorem 6.3.1 and Corollary 6.3.3. ☐

**Lemma 6.4.3.** *Consider a tensor category $\mathbf{T}$ with tensor subcategories $\mathbf{T}_1$ and $\mathbf{T}_2$ such that $\langle \mathbf{T}_1, \mathbf{T}_2\rangle = \mathbf{T}$ and $\mathbf{T}_2$ is finitely generated. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are neutral ($\mathbf{svec}$-)splitting algebras of respectively $\mathbf{T}_1$ and $\mathbf{T}_2$, then there exists an algebra morphism $\mathcal{A}_{12} := \Gamma_{\mathbf{T}_1\cap\mathbf{T}_2}\mathcal{A}_1 \to \mathcal{A}_2$, and for any such morphism the associated $\mathcal{A} := \mathcal{A}_1 \otimes_{\mathcal{A}_{12}} \mathcal{A}_2$ is a neutral ($\mathbf{svec}$-)splitting algebra for $\mathbf{T}$.*

*Proof.* We leave out 'super' in the proof for brevity. Since $\mathbf{T}_1$ and $\mathbf{T}_2$ are (neutral) tannakian, $\mathbf{T} = \langle \mathbf{T}_1, \mathbf{T}_2\rangle$ is also tannakian, by Lemma 3.3.2(ii) and Theorem 6.1.1. We also observe that $\mathbf{T}_1 \cap \mathbf{T}_2$ is finitely generated by Corollary 6.3.3.

First we consider the special case where $\mathbf{T}_1$ is finitely generated as well. Then the tannakian category $\mathbf{T}$ is also finitely generated, so of the form $\mathrm{Rep}G$, for an algebraic group $G/\Bbbk$. By Theorem 6.3.1, we can associate normal subgroups $N_i \lhd G$ to $\mathbf{T}_i \subset \mathbf{T}$. By Corollary 6.3.2, we have $N_1 \cap N_2 = 1$. By the uniqueness of neutral splitting algebras in Lemma 6.4.2(i) and Example 3.4.4 we find the left and right vertical isomorphisms in the diagram

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[d, "\sim"] & \mathcal{A}_{12} \arrow[l, hook'] \arrow[d, "\sim"] & \mathcal{A}_2 \arrow[d, "\sim"] \\
\Bbbk[G/N_1] & \Bbbk[G/N_1N_2] \arrow[l, hook'] \arrow[r, hook] & \Bbbk[G/N_2].
\end{tikzcd}$$

By Theorem 6.3.1 and [Wa, Lemma 16.3], we obtain the middle vertical isomorphism which makes the left square commutative. We then choose the morphism $\mathcal{A}_{12} \to \mathcal{A}_2$ which creates another commutative square in the above diagram. By Proposition 6.3.4, the algebra $\mathcal{A}_1 \otimes_{\mathcal{A}_{12}} \mathcal{A}_2$ is isomorphic to $\Bbbk[G]$ and hence indeed a splitting algebra. Now assume we take a different algebra morphism $\mathcal{A}_{12} \to \mathcal{A}_2$. It must be a composite $\mathcal{A}_{12} \to \Gamma_{\mathbf{T}_1\cap\mathbf{T}_2}\mathcal{A}_2 \hookrightarrow \mathcal{A}_2$, where the first arrow must be an isomorphism, by Theorem 3.4.3. This means that our new $\mathcal{A}_{12} \to \mathcal{A}_2$ is equal to our first choice, up to composition with an automorphism of $\mathcal{A}_{12}$. This automorphism lifts to $\mathcal{A}_1$ by Lemma 6.4.2(ii). This lift yields a canonical isomorphism between the two algebras of the form $\mathcal{A}_1 \otimes_{\mathcal{A}_{12}} \mathcal{A}_2$, hence the second is also a neutral splitting algebra.

Now we consider the general case, meaning that $\mathbf{T}_1$ need not be finitely generated. Take the set $\{\mathbf{T}_1^\alpha\}$ of all finitely generated tensor subcategories of $\mathbf{T}_1$ with $\mathbf{T}_1^\alpha \cap \mathbf{T}_2 = \mathbf{T}_1 \cap \mathbf{T}_2$ and set $\mathcal{A}_1^\alpha := \Gamma_{\mathbf{T}_1^\alpha}\mathcal{A}_1$. By definition, we have $\mathcal{A}_{12} \subset \mathcal{A}_1^\alpha$. Since $\mathbf{T}_1 = \cup_\alpha \mathbf{T}_1^\alpha$, we have

$$\mathcal{A}_1 \;\simeq\; \varinjlim_\alpha \mathcal{A}_1^\alpha.$$

By Corollary 3.4.5(i), the algebra $\mathcal{A}_1^\alpha$, respectively $\mathcal{A}_{12}$ and $\Gamma_{\mathbf{T}_1\cap\mathbf{T}_2}\mathcal{A}_2$, are neutral splitting algebras for $\mathbf{T}_1^\alpha$ respectively $\mathbf{T}_1 \cap \mathbf{T}_2$. Since $\mathbf{T}_1 \cap \mathbf{T}_2$ is finitely generated, Lemma 6.4.2(i) allows to choose an algebra morphism $\mathcal{A}_{12} \xrightarrow{\sim} \Gamma_{\mathbf{T}_1\cap\mathbf{T}_2}\mathcal{A}_2 \hookrightarrow \mathcal{A}_2$. By the previous paragraph, each $\mathcal{A}_1^\alpha \otimes_{\mathcal{A}_{12}} \mathcal{A}_2$ is a neutral splitting algebra for $\langle \mathbf{T}_1^\alpha, \mathbf{T}_2\rangle$ and hence

$$\mathcal{A} \;=\; \mathcal{A}_1 \otimes_{\mathcal{A}_{12}} \mathcal{A}_2 \;\simeq\; \varinjlim_\alpha (\mathcal{A}_1^\alpha \otimes_{\mathcal{A}_{12}} \mathcal{A}_2)$$

is a splitting algebra for $\mathbf{T} = \cup_\alpha \langle \mathbf{T}_1^\alpha, \mathbf{T}_2\rangle$. Since $\mathbb{1}$ is compact, by definition of Ind$\mathbf{T}$, it follows also that

$$\mathrm{Hom}(\mathbb{1}, \mathcal{A}) \simeq \varinjlim_\alpha \mathrm{Hom}(\mathbb{1}, \mathcal{A}_1^\alpha \otimes_{\mathcal{A}_{12}} \mathcal{A}_2) \simeq \Bbbk,$$

so $\mathcal{A}$ is neutral. □

*Proof of Theorem 6.4.1.* For brevity we leave out the references to 'super' in the proof. Let $\mathbf{T}$ be a tannakian category. We reinterpret Lemma 3.4.6, using Theorem 3.4.3 and Corollary 3.4.5(ii), as follows. There exists a tensor subcategory $\mathbf{T}^0 \subset \mathbf{T}$ with a neutral splitting algebra $\mathcal{A}^0$ such that there exists no tensor subcategory $\mathbf{T}^0 \subsetneq \mathbf{T}^1 \subset \mathbf{T}$ with a neutral splitting algebra $\mathcal{A}^1$ which is an algebra over $\mathcal{A}^0$.

In order to derive a contradiction we assume that $\mathbf{T}^0 \neq \mathbf{T}$. Take $X \in \mathrm{Ob}\mathbf{T}\backslash\mathrm{Ob}\mathbf{T}^0$ and set $\mathbf{S} = \langle X \rangle$ and $\mathbf{S}^0 = \mathbf{T}^0 \cap \mathbf{S}$. By construction $\mathbf{S}$ is tannakian and finitely generated. Let $\mathcal{B}$ denote a neutral splitting algebra for $\mathbf{S}$, guaranteed to exist by Lemma 6.4.2(i). By Lemma 6.4.3, there exists a neutral splitting algebra for $\langle \mathbf{T}^0, \mathbf{S}\rangle \supsetneq \mathbf{T}^0$ of the form

$$\mathcal{A}^1 \;:=\; \mathcal{A}^0 \otimes_{\mathcal{B}^0} \mathcal{B}, \qquad \text{for } \mathcal{B}^0 = \Gamma_{\mathbf{S}^0}\mathcal{A}^0.$$

Clearly we have an algebra morphism $\mathcal{A}^0 \to \mathcal{A}^1$, which contradicts the first paragraph.

In conclusion, $\mathbf{T}^0 = \mathbf{T}$ admits a neutral splitting algebra and the conclusion follows from Theorem 3.4.3. □

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