Journal of Vibration and Acoustics

Technical Briefs

A Relationship Between Defective Systems and Unit-Rank Modification of Classical Damping

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A common assumption within the mathematical modeling of vibrating elastomechanical system is that the damping matrix can be diagonalized by the modal matrix of the undamped model. These damping models are sometimes called ''classical'' or ''proportional.'' Moreover it is well known that in case of a repeated eigenvalue of multiplicity m, there may not exist a full sub-basis of m linearly independent eigenvectors. These systems are generally termed ''defective.'' This technical brief addresses a relation between a unit-rank modification of a classical damping matrix and defective systems. It is demonstrated that if a rank-one modification of the damping matrix leads to a repeated eigenvalue, which is not an eigenvalue of the unmodified system, then the modified system is defective. Therefore defective systems are much more common in mechanical systems with general viscous damping than previously thought, and this conclusion should provide strong motivation for more detailed study of defective systems. [S0739-3717(00)00602-4]

Introduction

The difference between general viscous damping and proportional damping was thoroughly investigated by Caughy [1]. He gave an explicit formula for the family of proportional damping matrices. The basic property of a proportional damping matrix is that it can be diagonalized by the modal matrix of the undamped system. However in many applications a proportional damping is not sufficient to match the model predictions and the data [2]. Hence the proportional damping matrix has to be updated. This issue has been addressed frequently in the last decade (Tong et al. [3], Bellos and Inman [4], Starek and Inman [5], Balmes [6], Garvey et al. [7]). A common updating method is the unit-rank modification [8], which is of particular interest in control problems [9]. As an example consider the proportionally damped 2-DoF model

$$M=I_2,\quad C=\frac{1}{3}\begin{bmatrix}4-\sqrt{5} & 0\\ 0 & 8-\sqrt{5}\end{bmatrix},\quad K=\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix},\tag{1}$$

Contributed by the Technical Committee on Vibration and Sound for publication in the JOURNAL OF VIBRATION AND ACOUSTICS, Manuscript received Oct. 1999. Associate Technical Editor: D. J. Inman.

which has two conjugate pairs of eigenvalues

$$\lambda_1=\lambda_3^*=-\frac{4-\sqrt{5}}{6}+\frac{j}{6}\sqrt{15+8\sqrt{5}}\approx-0.294+0.956j\tag{2}$$

$$\lambda_2=\lambda_4^*=-\frac{8-\sqrt{5}}{6}+\frac{j}{6}\sqrt{75+16\sqrt{5}}\approx-0.961+1.754j.\tag{3}$$

The rank-one modification of the damping matrix

$$\hat{C}=C+\mathbf{x}\mathbf{x}^{\mathsf{T}}\tag{4}$$

where

$$\mathbf{x}=\alpha(1,-1)^{\mathsf{T}}\tag{5}$$

leads for $\alpha=\sqrt{\sqrt{5}/3}$ to a repeated eigenvalue $\lambda=-1+j$. Since the matrix

$$\lambda^2M+\lambda C+\lambda\mathbf{x}\mathbf{x}^{\mathsf{T}}+K=-\frac{1}{3}\begin{bmatrix}1+2j & \sqrt{5}(-1+j)\\ \sqrt{5}(-1+j) & -2(2+j)\end{bmatrix}\tag{6}$$

has rank 1 there exists only one eigenvector for the repeated eigenvalue. Hence the matrices

$$\begin{bmatrix}\hat{C} & M\\ M & 0\end{bmatrix}\quad\text{and}\quad\begin{bmatrix}K & 0\\ 0 & -M\end{bmatrix}\tag{7}$$

of the corresponding first order system cannot be diagonalized simultaneously, i.e., the system is defective. For defective systems only the Jordan decomposition is available [10].

The above example suggests a general relation between unit-rank modification of classical damping and defectiveness. Given a classically damped vibrating elastomechanical system with n degrees of freedom (DoF) which is already transformed into diagonal form, the eigenvalues λ_{oi} of

$$D(\lambda):=\lambda^2I_n+2\lambda\Gamma+\Omega^2\equiv\operatorname{diag}(p_i)_{i=1,\ldots,n},\tag{8}$$

where $\Gamma=\operatorname{diag}(\gamma_i)_{i=1,\ldots,n}$ and $\Omega=\operatorname{diag}(\omega_i)_{i=1,\ldots,n}$, can be calculated from

$$\det(D(\lambda_o))=0=\prod_{i=1}^{n}p_i(\lambda_o)=:p(\lambda_o)\tag{9}$$

with the well known solutions

$$\lambda_{oi}=-\gamma_i\pm j\omega_i\sqrt{1-(\gamma_i/\omega_i)^2}.\tag{10}$$

The scope of this note is to explore whether a rank-one modification of the damping matrix

$$A=D(\lambda)+\lambda\mathbf{x}\mathbf{x}^{\mathsf{T}},\quad \mathbf{x}\in\mathbb{R}^n,\tag{11}$$

can produce a nondefective system with repeated eigenvalues in the case that none of eigenvalues (10) is eigenvalue of (11). Note, that the latter condition implies that $x_i \neq 0$ for all $i=1,\ldots,n$. Of course the modes corresponding to $x_i=0$ are not affected by the modification. Thus the modes are decoupled into those affected by the modification $(x_i \neq 0)$ and those not affected $(x_i=0)$. It will be assumed that this has been done and that only the subsystem of modes affected by the modification is considered in the following analysis.

If the eigenvalue λ_i is repeated of multiplicity m then the characteristic polynomial contains the factor $(\lambda-\lambda_i)^m$. Hence the derivative of the characteristic polynomial vanishes at λ_i, that is (Lancaster and Tismenetsky [11], p. 346)

$$\left.\frac{d\det(A)}{d\lambda}\right|_{\lambda=\lambda_i}=\left.\operatorname{trace}\left[A^{\mathrm{ad}}\frac{dA}{d\lambda}\right]\right|_{\lambda=\lambda_i}=0. \tag{12}$$

Here the superscript ad denotes the adjungate of A. Its element in row i and column k is defined by

$$(A^{\mathrm{ad}})_{ik}=(-1)^{i+k}\det(A^{(k|i)}), \tag{13}$$

where the superscript $(i|k)$ denotes the deletion of row i and column k. Only in the case $A^{\mathrm{ad}}(\lambda_i)=0$, which implies rank $(A(\lambda_i))<n-1$, there exist at least two linearly independent eigenvectors. In general, for a repeated eigenvalue λ of multiplicity r, the necessary and sufficient condition for nondefectiveness is

$$\operatorname{rank}(A(\lambda))=n-r. \tag{14}$$

Only if (14) holds true there exist r linearly independent eigenvectors for that eigenvalue. The main result of this note is that, under the assumptions made, a rank-one modification of the damping cannot produce a nondefective system with a repeated eigenvalue.

The next section contains the proof of the main result of this paper. If desired the preliminary material may be omitted taking the reader directly to the theorem at the end of the next section. Examples are given in the last section.

1 Theorem on Defective Systems

As it has been explained above a necessary condition for the system described by A to be nondefective for a repeated eigenvalue λ is

$$A^{\mathrm{ad}}(\lambda)=0 \tag{15}$$

which means that all n^2 minors

$$m_{ik}:=\det[A^{(i|k)}(\lambda)],\quad i,k\in\{1,\ldots,n\} \tag{16}$$

of size $n-1$ of $A(\lambda)$ have to be zero. A basic tool of the following investigation is the well known formula (see for instance Lancaster and Tismenetsky [11], p. 65)

$$\det(Z+\mathbf{x}\mathbf{y}^{\mathsf{T}})=\det(Z)+\mathbf{y}^{\mathsf{T}}Z^{\mathrm{ad}}\mathbf{x}, \tag{17}$$

where Z is an arbitrary square matrix and $\mathbf{x},\mathbf{y}$ are arbitrary n-dimensional vectors. As an immediate consequence one finds (see also Veselić [12])

$$\det(A(\lambda))=\det(D(\lambda)+\lambda\mathbf{x}\mathbf{x}^{\mathsf{T}})=p+\lambda\sum_{\ell=1}^{n}x_\ell^2p^{(\ell)}, \tag{18}$$

where the superscript (ℓ) denotes the deletion of the ℓth factor (see Eq. (9)) in p. Since the principal minors m_{ii} of A have the same structure as A we find

$$\begin{aligned}m_{ii}=\det(A^{(i|i)})&=\det(D^{(i|i)})+\lambda\sum_{\ell=1}^{n-1}(x_\ell^{(i)})^2p^{(i\ell)}\\&=p^{(i)}+\lambda\sum_{\ell\neq i}x_\ell^2p^{(i\ell)},\end{aligned} \tag{19}$$

where $\mathbf{x}^{(i)}$ is an $(n-1)$-dimensional vector resulting from deletion of the ith element of $\mathbf{x}$. To calculate the off-diagonal minors one needs some definitions. Suppressing the argument λ we define for $i<k$ the diagonal matrix

$$D_{(i,k)}:=\begin{bmatrix}p_i & & 0\\ & \ddots & \\ 0 & & p_k\end{bmatrix}\in\mathbb{C}^{(k-i+1)\times(k-i+1)}. \tag{20}$$

Moreover it is convenient to define the strictly upper triangular matrix

$$N_{(i,k)}:=\begin{bmatrix}0 & D_{(i,k)}\\ 0 & 0\end{bmatrix}\in\mathbb{C}^{(k-i+2)\times(k-i+2)}, \tag{21}$$

then one finds for $i<k$

$$A^{(i|k)}=\underbrace{\begin{bmatrix}D_{(1,i-1)} & 0 & 0\\ 0 & N_{(i+1,k-1)} & 0\\ 0 & 0 & D_{(k+1,n)}\end{bmatrix}}_{=D^{(i|k)}}+\lambda\mathbf{x}^{(i)}\mathbf{x}^{(k)\mathsf{T}}\in\mathbb{C}^{(n-1)\times(n-1)}, \tag{22}$$

Note that $\det(D^{(i|k)})=\delta_{ik}p^{(i)}$, which means that $D^{(i|k)}$ on the right side of (22) is singular because the $k-1$th row and the i column are zero. Hence all minors of that matrix are zero except the minor

$$\det([D^{(i|k)}]^{(k-1|i)})=p^{(ik)}. \tag{23}$$

This result holds true also for the cases $i,k\in\{1,n\}$. This can be verified from the above formulas with the convention to delete all block rows or columns that contain expressions which are not defined because they are outside the index range $\{1,\ldots,n\}$, as for instance $P_{(n+1,n)}$, $N_{(i+1,0)}$ or I_0. As a summarizing result we find

$$[D^{(i|k)}]^{\mathrm{ad}}=\begin{Bmatrix}(-1)^{i+k-1}p^{(ik)}E_{ik-1}, & i<k\\ \operatorname{diag}(p^{(ij)})_{j\neq i}, & i=k\\ (-1)^{i+k-1}p^{(ik)}E_{i-1k}, & i>k\end{Bmatrix}\in\mathbb{C}^{(n-1)\times(n-1)}, \tag{24}$$

where in general $E_{ik}:=\mathbf{e}_i\mathbf{e}_k^{\mathsf{T}}$ and $\mathbf{e}_i$ is the ith column vector of the identity matrix. Applying Eq. (17) to Eq. (22) one finds for $i\neq k$

$$m_{ik}=(-1)^{k+i-1}\lambda x_ix_kp^{(ik)}. \tag{25}$$

Since A is symmetric its adjungate is symmetric, too. Hence the expression (25) is invariant with respect to the interchange of i and k. Summarizing the findings so far

$$\det(A^{(i|k)})=\begin{cases}(-1)^{k+i-1}\lambda x_ix_kp^{(ik)}, & i\neq k\\ p^{(i)}+\lambda\Sigma_{\ell\neq i}x_\ell^2p^{(i\ell)}, & i=k\end{cases} \tag{26}$$

and consequently

$$(A)_{ki}^{\mathrm{ad}}=\begin{cases}-\lambda x_ix_kp^{(ik)}, & i\neq k\\ p^{(i)}+\lambda\Sigma_{\ell\neq i}x_\ell^2p^{(i\ell)}, & i=k\end{cases}. \tag{27}$$

In the case that λ is an eigenvalue of A then

$$p+\lambda\sum_{\ell=1}^{n}x_\ell^2p^{(\ell)}=0 \tag{28}$$

which may be rewritten as

$$p_i p^{(i)}+\lambda x_i^2 p^{(i)}+\lambda \sum_{\ell \neq i} x_\ell^2 p^{(i\ell)} p_i=0 \tag{29}$$

or equivalently

$$p_i\left(p^{(i)}+\lambda \sum_{\ell \neq i} x_\ell^2 p^{(i\ell)}\right)=-\lambda x_i^2 p^{(i)}. \tag{30}$$

If, in addition, λ is not eigenvalue of D then $p_i \neq 0$ for all $i=1,\ldots,n$. Hence one can replace the diagonal of A^{ad} by $-\lambda x_i^2 p^{(i)}/p_i$ yielding

$$A^{\mathrm{ad}}=-\lambda p D^{-1} \mathbf{x}\mathbf{x}^{\mathsf{T}} D^{-1}. \tag{31}$$

Note, that (31) implies trace $(A^{\mathrm{ad}})=-\lambda p\|D^{-1}\mathbf{x}\|^2$, which suggests that the adjungate of A is zero only under certain conditions. The following theorem clarifies the situation.

Theorem: *Let D and A be defined as in Eqs. (8) and (11), respectively. If none of the eigenvalues of D are eigenvalues of A then if A has a repeated eigenvalue then A is defective*

Proof: If λ is eigenvalue of A then $p+\lambda\sum_{i=1}^n x_i^2 p^{(i)}=0$. Since λ is not an eigenvalue of D, $p_i \neq 0$ and hence $x_i \neq 0$ for all $i=1,\ldots,n$. Otherwise the matrix $\mathbf{x}\mathbf{x}^{\mathsf{T}}$ would contain a zero column and row and one eigenvalue of A would be an eigenvalue of D. One may also conclude that neither $p^{(i)}$ nor $p^{(ik)}$ are zero. Hence for $i \neq k$ the minor $m_{ik}=(-1)^{i+k+1}\lambda x_i x_k p^{(ik)}$ can only become zero if $\lambda=0$. But this is not possible because the determinant of A would lead to $p=0$, which means that λ is an eigenvalue of D in contrast to the assumption. Hence $A^{\mathrm{ad}} \neq 0$ and thus A has rank $n-1$ and therefore the system is defective.

In the next section some examples are presented.

2 Repeated Pole Placement

To calculate examples the inverse problem has to be solved to find a vector $\mathbf{x}$ such that the system A has given eigenvalues. Since the eigenvalue of interest is repeated there are two conditions: the determinant and the derivative of the determinant with respect to the eigenvalue have to be zero, that is

$$p+\lambda \sum_{i=1}^{n} x_i^2 p^{(i)}=0 \tag{32}$$

$$p'+\sum_{i=1}^{n} x_i^2(\lambda p^{(i)\prime}+p^{(i)})=0, \tag{33}$$

where $(\ldots)'$ denotes the derivative with respect to λ, and all polynomials and derivatives are evaluated at the repeated eigenvalue. Note that these equations are linear in x_i^2. Hence one may write

$$\underbrace{\begin{bmatrix} \lambda p^{(1)} & \cdots & \lambda p^{(n)} \\ p^{(1)}+\lambda p^{(1)\prime} & \cdots & p^{(n)}+\lambda p^{(n)\prime} \end{bmatrix}}_{=:H^{\mathsf{T}(\lambda)}} \underbrace{\begin{pmatrix} x_1^2 \\ \vdots \\ x_n^2 \end{pmatrix}}_{=:\mathbf{y}} = -\underbrace{\begin{pmatrix} p \\ p' \end{pmatrix}}_{\mathbf{g}(\lambda)} \tag{34}$$

Since the vector $\mathbf{y}$ is real-valued we may double the order of the equation to get

$$V^{\mathsf{T}}(\lambda)\mathbf{y}=-\mathbf{q}(\lambda), \tag{35}$$

where $V=[\mathrm{Re}\{H\},\mathrm{Im}\{H\}]\in\mathbb{R}^{n\times 4}$ and $\mathbf{q}^{\mathsf{T}}=(\mathrm{Re}\{\mathbf{g}\}^{\mathsf{T}},\mathrm{Im}\{\mathbf{g}\}^{\mathsf{T}})$. Given $m \leq 2n$ complex scalars λ_i, $i=1,\ldots,m$ there exists a vector $\mathbf{x}$ such that $D(\lambda)+\lambda\mathbf{x}\mathbf{x}^T$ has all λ_i as repeated eigenvalues if and only if

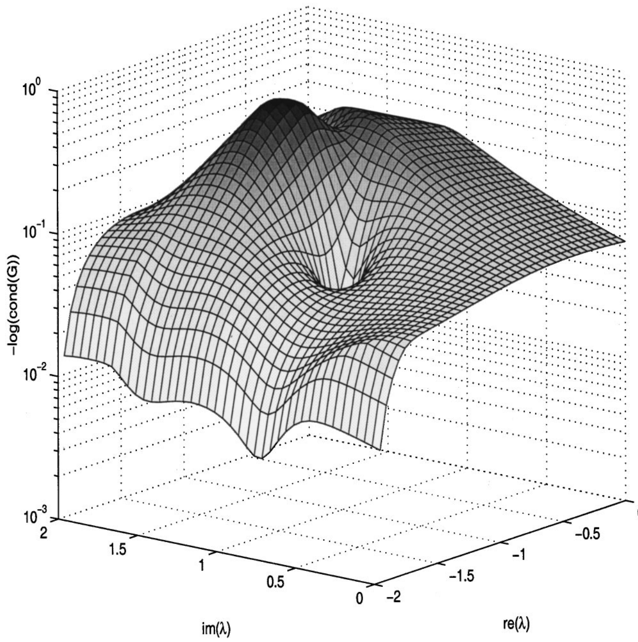

Fig. 1 The inverse condition number of *G* as a function of the eigenvalue λ

$$\mathrm{rank}\left\{\underbrace{\begin{bmatrix} V^{\mathsf{T}}(\lambda_1) \\ \vdots \\ V^{\mathsf{T}}(\lambda_m) \end{bmatrix}}_{=:T}, \underbrace{\begin{pmatrix} \mathbf{q}(\lambda_1) \\ \vdots \\ \mathbf{q}(\lambda_m) \end{pmatrix}}_{=:\mathbf{f}}\right\} \leq n. \tag{36}$$

Moreover the solution $\mathbf{x}$ is real-valued if a vector $\mathbf{z}$ can be chosen such that all components of the vector

$$\mathbf{y}(\mathbf{z})=-T^{+}\mathbf{f}+T_K\mathbf{z} \tag{37}$$

are non-negative. The columns of T_K form a basis of all vectors $\mathbf{u}$ such that $T\mathbf{u}=0$. The corresponding subspace is called the kernel of T. In practical applications one may introduce additional constraints on the remaining eigenvalues by expanding the linear inverse problem $T\mathbf{y}=-\mathbf{f}$ correspondingly. If condition (36) is satisfied it is straightforward to calculate the normal solution $\overline{\mathbf{y}}:=-T^{+}\mathbf{f}$ and a representation T_K of the kernel of T. The matrix T has a nontrivial kernel if $4m<n$. In this case there exists a solution space $\mathbf{y}(\mathbf{z})$ generated by the kernel of T. The problem to determine all z that leads to non-negative solutions $\mathbf{y}(\mathbf{z})$ is a common problem in linear programming [13].

2.1 Example 1. Consider the example presented in the introduction, i.e., the case $m=1$, $n=2$. Note that in this case $T=V^{\mathsf{T}}\in\mathbb{R}^{4\times 2}$ and $\mathbf{f}=\mathbf{q}\in\mathbb{R}^4$. A necessary condition for the existence of a solution $x\in\mathbb{R}^2$ is that at λ the rank of the matrix

$$G(\lambda):=[V^{\mathsf{T}}(\lambda),\mathbf{q}(\lambda)]\in\mathbb{R}^{4\times 3} \tag{38}$$

is two. In Fig. 1 the inverse condition number of G is plotted over the phase plane $(\mathrm{Re}\{\lambda\},\mathrm{Im}\{\lambda\})\in[-2,0]\times[0,2]$. There is a distinct minimum at $\lambda=-1+j$, that leads to the solution $\mathbf{x}=\alpha(1,-1)^{\mathsf{T}}$ (see Eq. (5)). Indeed, as stated by Theorem 1, the resulting non-proportionally damped system has no basis of eigenvectors.

2.2 Example 2. The following example covers the case of nontrivial kernel T_K. Let $\mathrm{diag}(\Gamma)=(1,2,1,3,2)/2$ and $\mathrm{diag}(\Omega^2)=(10,5,7,8,12)$, which leads to conjugate pairs of eigenvalues (rounded) $\mathrm{diag}(\Lambda_0)=(-1\pm 2j,\ -1.5\pm 2.4j,\ -0.5\pm 2.6j,\ -0.5\pm 3.12j,\ -1\pm 3.32j)$. In this case the matrix T has a nontrivial kernel of dimension one. Hence there exists a solution space, that is,

$$\mathbf{y}(z) = -T^{+}\mathbf{f} + T_K z, \tag{39}$$

where $TT_K = 0$. In order to find all solutions $\mathbf{x}$ for the repeated eigenvalue $\lambda = -0.8 + 3j$ one has to determine all scalars z such that the vector

$$\mathbf{y}(z) = \begin{pmatrix} 0.2263 \\ 0.5531 \\ 0.0535 \\ -1.3545 \\ 0.0185 \end{pmatrix} + \begin{pmatrix} 0.3330 \\ 0.7452 \\ 0.1474 \\ 0.3715 \\ 0.4172 \end{pmatrix} z, \tag{40}$$

has no negative components. Obviously every $z \geqslant 1.3545/0.3715 \approx 3.65$ lead to a non-negative solution. For instance the choice $\mathbf{z} = 3.75$ leads to $\mathbf{x} = (1.2145, 1.8297, 0.7786, 0.1962, 1.2581)^{\mathrm{T}}$. The modified system has four conjugate pairs $\{-1.45 \pm 2.4j, -0.61 \pm 2.59j, -0.8 \pm 3j, -0.8 \pm 3j\}$ and two real eigenvalues $\{-0.82, -7.83\}$, which correspond to overdamped modes.

Conclusions

This technical brief has highlighted that defective systems may be more common that many engineers believe. It has been proved that if a unit-rank modification is performed on a system with classical damping, and this results in repeated eigenvalues, then the modified system is defective. The theorem, as formulated, requires that the eigenvalues of all the modes affected by the modification are changed. The conditions on the modification may suggest that the theorem has limited applicability. However, there are two counter arguments to this. The first is that there are indeed many systems that do consist of a rank one modification, for example the addition of a discrete damper to a lightly damped structure. Second, it is not suggested that other modifications do not lead to defective systems, it is just that a proof of the conditions required in the case of general viscous damping has proved elusive. Indeed, the authors believe that many, if not most, nonclassically damped systems with repeated eigenvalues are defective. The challenge, which the authors are continuing to pursue, is to rigorously prove the conditions required to produce a defective system. We encourage other researchers to rise to this challenge.

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Exact Solutions for Longitudinal Vibration of Multi-Step Bars with Varying Cross-Section

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Using appropriate transformations, the equation of motion for free longitudinal vibration of a nonuniform one-step bar is reduced to an analytically solvable equation by selecting suitable expressions, such as power functions and exponential functions, for the area variation. Exact analytical solutions to determine the longitudinal natural frequencies and mode shapes for a one step nonuniform bar are derived and used to obtain the frequency equation of multi-step bars. The new exact approach is presented which combines the transfer matrix method and closed form solutions of one step bars. A numerical example demonstrates that the calculated natural frequencies and mode shapes of a television transmission tower are in good agreement with the corresponding experimental data, and the selected expressions are suitable for describing the area variation of typical high-rise structures. [S0739-3717(00)00302-0]

Introduction

A broad range of engineering problems involves longitudinal vibration analysis of uniform and nonuniform bars. A great deal of research on longitudinal vibration of structural components has taken place over the last decade, as reviewed by Timoshenko et al. [1], in which references traced back to one hundred years ago. The free longitudinal vibrations of uniform rods have been thoroughly discussed in the literature and the solutions are well known (e.g., Meirovitch [2]). However, in general, it is not possible or, at least, very difficult to get the exact solutions of differential equations of free longitudinal vibrations of bars with varying cross-section. These exact bar solutions are available only for certain bar shapes and boundary conditions. Analytical solutions for transverse vibration of beams with linearly varying section were proposed by Ward [3]. Conway et al. [4] obtained an exact solution for a conical beam in terms of Bessel functions. Wang [5] and Bapat [6] derived the closed-form solutions for the free longitudinal vibration of exponential and catenoidal bars. Eisenberger [7] found exact longitudinal natural frequencies of a variable cross-section rod with polynomial variation in the cross-sectional area and mass distribution along the member using exact element method. Lau [8] and Abrate [9] derived closed-form solutions for the free longitudinal vibration of rods whose cross-section varies as $A(x) = A_0(x/L)^2$ and $A(x) = A_0[1 + a(x/L)]^2$, respectively. Kumar and Sujith [10] obtained exact solutions for the longitudinal vibration of nonuniform rods whose cross-section varies as $A(x) = (a + bx)^n$ and $A(x) = A_0 \sin^2(ax + b)$.

The objective of this paper is to present exact solutions for the longitudinal vibration of one-step and multi-step bars with varying cross-section. In this paper, exact analytical solutions for free longitudinal vibration of a one step nonuniform bar are derived by selecting appropriate expression of area variation and using appropriate transformations. The free longitudinal vibration of

Contributed by the Technical Committee on Vibration and Sound for publication in the JOURNAL OF VIBRATION AND ACOUSTICS. Manuscript received Nov. 1999. Associate Technical Editor: A. Vakakis.

amulti-step bar is a complex problem, and the exact solution of this problem has not previously been obtained. Use of the exact solution of a one-step bar together with a transfer matrix method is presented in this paper in order to resolve this problem.

One-Step Bar

The equation for longitudinal vibration mode function, $U(x)$, of a nonuniform bar shown in Fig. 1 is as follows (Li et al. [11,12])

$$EA(x)\frac{d^2U}{dx^2}+E\frac{dA(x)}{dx}\frac{dU}{dx}+\rho A(x)\omega^2U=0 \quad (1)$$

where ω is the circular natural frequency and other parameters are defined in Fig. 1.

It is difficult to find the exact solutions of Eq. (1) for general cases, because the coefficients in the equation vary with the coordinate x. It is obvious that the exact solutions are dependent on the area variation, $A(x)$. Thus, the exact solutions of Eq. (1) may be obtained by means of reasonable selection for $A(x)$. In this paper, two important cases are considered and discussed.

Case A: Expression of $A(x)$ is a power function

$$A(x)=\alpha\left(1+\beta\frac{x}{l}\right)^n \quad (2)$$

in which α, β and n are constants which can be determined in terms of the values of $A(x)$ at three control sections.

Substituting Eq. (2) into Eq. (1) one obtains

$$\frac{d^2U}{d\zeta^2}+\frac{n}{\zeta}\frac{dU}{d\zeta}+\lambda^2U=0 \quad (3)$$

in which

$$\zeta=1+\beta\frac{x}{l},\quad \lambda^2=\frac{\rho\omega^2l^2}{E\beta^2} \quad (4)$$

Setting

$$U=(\lambda\zeta)^\nu Z,\quad \nu=\frac{1-n}{2} \quad (5)$$

and substituting Eq. (5) into Eq. (3) yield the Bessel's equation of the νth order as follows

$$\frac{d^2Z}{d\zeta^2}+\frac{1}{\zeta}\frac{dZ}{d\zeta}+\left(1-\frac{\nu^2}{\zeta^2}\right)Z=0 \quad (6)$$

When ν is not an integer, $U(x)$ is given by

$$U(x)=\left(1+\beta\frac{x}{l}\right)^\nu\left\{C_1J_\nu\left[\lambda\left(1+\beta\frac{x}{l}\right)\right]+C_2J_{-\nu}\left[\lambda\left(1+\beta\frac{x}{l}\right)\right]\right\} \quad (7)$$

The eigenvalue equation is

$$J_\nu(\lambda)J_{-(\nu-1)}(\lambda\theta)=-J_{-\nu}(\lambda)J_{\nu-1}(\lambda\theta)\quad \text{for a cantilever bar} \quad (8)$$

or

$$J_\nu(\lambda)J_{-\nu}(\lambda\theta)=J_{-\nu}(\lambda)J_\nu(\lambda\theta)\quad \text{for a fixed-fixed bar} \quad (9)$$

where

$$\theta=1+\beta$$

When ν is an integer, in Eqs. (7)–(9), $J_{-\nu}(\lambda)$ should be changed to $Y_\nu(\lambda)$ and $J_{-(\nu-1)}(\lambda)$ should be $-Y_{\nu-1}(\lambda)$.

Solving the eigenvalue equation one obtains the jth eigenvalue, λ_j $(j=1,2,3,\ldots)$, and substituting λ_j into Eq. (4) one yields the j-th circular natural frequency, ω_j, as follows

$$\omega_j=\frac{\lambda_j|\beta|}{l}\sqrt{\frac{E}{\rho}}\quad j=1,2,3,\ldots \quad (10)$$

When $n=2$, then $\nu=-\frac{1}{2}$, the eigenvalue equation is

$$tg\lambda\beta=\lambda(1+\beta)\quad \text{for a cantilever bar} \quad (11)$$

or

$$\left.\begin{aligned}\sin\lambda\beta=0\\ \lambda_j=\frac{j\pi}{\beta}\end{aligned}\right\}\quad \text{for a fixed-fixed bar} \quad (12)$$

The first six eigenvalues, $|\beta|\lambda_j$ $(j=1,2,3,4,5,6)$, of a cantilever bar with various values of β for the case $n=2$ are calculated and listed in Table 1 (for $\beta>0$) and Table 2 (for $\beta<0$).

Case B: Expression of $A(x)$ is an exponential function

$$A(x)=\alpha e^{-\beta(x/l)}, \quad (13)$$

The parameters, α, β can be determined in terms of the values of $A(x)$ at two control sections. Substituting Eq. (13) into Eq. (1) obtains a differential equation with constant coefficients as

$$\frac{d^2U}{dx^2}-\frac{\beta}{l}\frac{dU}{dx}+\eta^2U=0 \quad (14)$$

in which

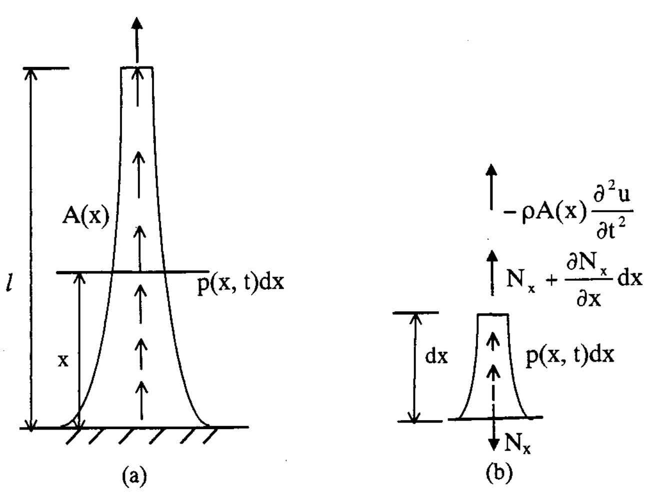

Fig. 1 A cantilever bar with varying cross-section

Table 1 The first six eigenvalues, $\lambda_j\beta$, of a cantilever bar with $A(x)=\alpha(1+\beta(x/l))^2$ for $\beta>0$

	Mode					
β	1	2	3	4	5	6
0	1.5708	4.7124	7.8540	10.9956	14.1372	17.2787
1	1.1653	4.6038	7.7893	10.9492	14.1010	17.2483
5	0.6940	4.5315	7.7463	10.9185	14.0770	17.2292
10	0.5210	4.5130	7.7390	10.9120	14.0720	17.2248

Table 2 The first six eigenvalues, $\lambda_j\beta$, of a cantilever bar with $A(x)=\alpha(1+\beta(x/l))^2$ for $\beta<0$

	Mode					
β	1	2	3	4	5	6
0	1.5708	4.7124	7.8540	10.9956	14.1372	17.2787
0.3	1.8038	4.8011	7.9076	11.0340	14.1661	17.3022
0.6	2.1745	5.0035	8.0379	11.1285	14.2061	17.3635
0.9	2.8362	5.7175	8.6585	11.6548	14.6856	17.7501

$$\eta = \omega \sqrt{\frac{\rho}{E}} \tag{15}$$

The general solution of $U(x)$ is found as

$$U(x) = e^{\beta x/2l}\left(C_1 \cos\frac{cx}{l} + C_2 \sin\frac{cx}{l}\right) \tag{16}$$

where

$$c^2 = \frac{l^2}{4}\left(4\eta^2 - \frac{\beta^2}{l^2}\right) \tag{17}$$

The eigenvalue equations and circular natural frequencies are as follows

$$\left.\begin{aligned} &\tan c = -\frac{2}{\beta}c \\ &\omega_j = \frac{1}{l}\sqrt{\frac{E}{\rho}\left[C_j^2 + \left(\frac{\beta}{2}\right)^2\right]}, \quad j = 1,2,\ldots \\ &\omega_j \approx \frac{C_j}{l}\sqrt{\frac{E}{\rho}}, \quad \text{for small } \beta \text{ and } j \geqslant 2 \end{aligned}\right\} \quad \text{for a cantilever bar} \tag{18}$$

or

$$\left.\begin{aligned} &\sin c = 0 \\ &\omega_j = \frac{1}{l}\sqrt{\frac{E}{\rho}\left[(j\pi)^2 + \left(\frac{\beta}{2}\right)^2\right]}, \quad j = 1,2,\ldots \\ &\omega_j \approx \frac{j\pi}{l}\sqrt{\frac{E}{\rho}}, \quad \text{for small } \beta \text{ and } j \geqslant 2 \end{aligned}\right\} \quad \text{for a fixed-fixed bar} \tag{19}$$

The j-th mode shape function for a cantilever bar and a fixed-fixed bar can be written as

$$U_j(x) = e^{\beta x/2l} \sin\frac{C_j x}{l} \tag{20}$$

The first six eigenvalues, C_j $(j = 1,2,3,4,5,6)$, of a cantilever bar with various values of β are calculated and listed in Table 3. It can be seen from the results presented in Table 1 that, like the case of a cantilever bar, the lowest natural frequency is affected most by the taper. For higher modes, the natural frequencies are close to those of corresponding uniform bars. The same conclusions can be draw from Table 2. It can be seen from Table 1 that all the natural frequencies, including the lowest and higher frequencies, are less than those of corresponding uniform bars when $\beta > 0$. However, the results presented in Table 2 show that all the natural frequencies for the case of $\beta < 0$ are greater than those of corresponding uniform bars. Table 3 shows that the variation of C_j with β is similar to that of $\lambda_j \beta$ with β presented in Table 2.

Multi-Step Bars

A multi-step bar is shown in Fig. 2. It is assumed that the area of section in each step varies with x. The equation of mode shape function of the i-th step bar is as follows

$$EA_i(x)\frac{d^2U_i}{dx^2} + E\frac{dA_i(x)}{dx}\frac{dU_i}{dx} + \rho A_i(x)\omega^2 U_i = 0 \tag{21}$$

Table 3 The first six eigenvalues, C_j, of a cantilever bar with $A(x) = \alpha e^{-\beta(x/l)}$

	Mode					
β	1	2	3	4	5	6
0	1.5708	4.7124	7.8540	10.9956	14.1372	17.2787
1.0	1.8350	4.8150	7.9164	11.0400	14.1713	17.3063
2.0	2.0261	4.9115	7.9780	11.0847	14.2063	17.3352
4.0	2.2878	5.0865	8.0962	11.1743	14.2753	17.3920

The general solution of mode shape function for the i-th step bar can be expressed as

$$U_i(x) = C_{i1}S_{i1}(x) + C_{i2}S_{i2}(x) \tag{22}$$

where i denotes the ith step and q is the total step number (Fig. 2), $S_{i1}(x)$ and $S_{i2}(x)$ are special solutions of mode shape functions of the ith step bar, which can be found from those derived in the last section for the Case A and Case B.

The transfer matrix method (TMM) is often used for structures composed of one-dimension elements. The applications of TMM are also limited by possible occurrence of numerical problems for certain cases, which have been extensively discussed by Yong and Lin [13]. But, it has some advantages in computations: ease of programming and small memory requirements, etc. Thus, the

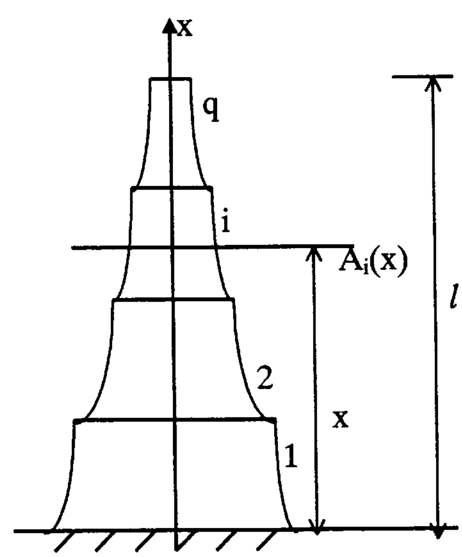

Fig. 2 A multi-step bar with varying cross-section

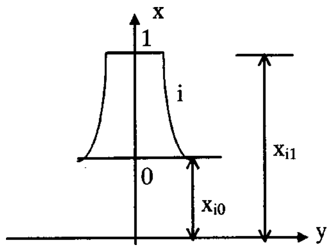

Fig. 3 The *i*th step

transfer matrix method is adopted herein to establish the equation of mode shape function and the eigenvalue equation for a multi-step bar.

The relationship between the parameters, U_{i1} (longitudinal displacement) and N_{i1} (axis force) at the end 1 and U_{i0}, N_{i0} at the end 0 of the ith step bar (Fig. 3) can be expressed as

$$\begin{bmatrix} U_{i1} \\ N_{i1} \end{bmatrix} = [T_i] \begin{bmatrix} U_{i0} \\ N_{i0} \end{bmatrix} \tag{23}$$

in which

$$[T_i] = [S(x_{i1})][S(x_{i0})]^{-1}$$

$$[S(x_{i0})] = \begin{bmatrix} S_{i1}(x_{i0}) & S_{i2}(x_{i0}) \\ EA_{i0}S'_{i1}(x_{i0}) & EA_{i0}S'_{i2}(x_{i0}) \end{bmatrix}$$

$$[S(x_{i1})] = \begin{bmatrix} S_{i1}(x_{i1}) & S_{i2}(x_{i1}) \\ EA_{i1}S'_{i1}(x_{i1}) & EA_{i1}S'_{i2}(x_{i1}) \end{bmatrix} \tag{24}$$

$$U_{i0} = U_i(x_{i0}) \quad U_{i1} = U_i(x_{i1})$$

$$N_{i1} = N_i(x_{i1}), \quad N_{i0} = N_i(x_{i0}), \quad EA_{i1} = EA_i(x_{i1}),$$

$$EA_{i0} = EA_i(x_{i0})$$

$[T_i]$ is called the transfer matrix because it transfers the parameters at the end 0 to those at the end 1 of the ith step bar.

Since

$$U_{i0} = U_{i-1,1}, \quad N_{i0} = N_{i-1,1} \tag{25}$$

The equation for the top step bar ($i = q$, Fig. 2) can be established by use of Eqs. (23) and (25) repeatedly as follows

$$\begin{bmatrix} U_{q1} \\ N_{q1} \end{bmatrix} = [T] \begin{bmatrix} U_{10} \\ N_{10} \end{bmatrix} \tag{26}$$

in which

$$[T] = [T_q][T_{q-1}]\cdots[T_1]$$

$[T]$ is a matrix which can be written as

$$[T] = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{11} \end{bmatrix} \tag{27}$$

If a multi-step bar has lumped masses as shown in Fig. 4, then, the transfer matrix $[T_i]$ should be replaced by $[T_{mi}]$,

$$[T_{mi}] = \begin{bmatrix} 1 & 0 \\ -\omega^2 m_i & 1 \end{bmatrix} [T_i] \tag{28}$$

According to the following boundary conditions of a multi-step cantilever bar (Fig. 2),

$$\left. \begin{aligned} x = 0, \quad U_{10} = 0 \\ x = l, \quad N_{q1} = 0 \end{aligned} \right\} \tag{29}$$

we obtain the eigenvalue equation as

$$T_{22} = 0 \tag{30}$$

According to the following boundary conditions for a multi-step fixed-fixed bar,

$$\left. \begin{aligned} x = 0, \quad U_{10} = 0 \\ x = l, \quad N_{q1} = 0 \end{aligned} \right\} \tag{31}$$

the eigenvalue equation is found as

$$T_{12} = 0 \tag{32}$$

After the natural frequencies of a multi-step bar have been found, the mode shapes of a multi-step bar can be determined by use of Eq. (22) and the general solutions of each step bar.

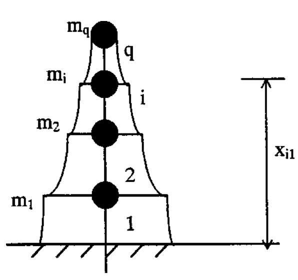

Fig. 4 A multi-step bar with lumped masses

Numerical Example

Wuhan Television Transmission Tower (Wuhan T.V. Tower) located in Wuhan, China, is a reinforced concrete tube structure, its geometric configuration is shown in Fig. 5. The top of the tower is 221 meters. The height of the main tower body (main structure) is 187 meters. The main structure was dynamically tested by Li et al. [14]. The objective of this numerical example is to determine the longitudinal natural frequencies and vibration mode shapes of the main structure of Wuhan T.V. Tower by the proposed method.

Wuhan T.V. Tower is treated as a cantilever multi-step bar with varying cross-section for free vibration analysis. Because the variation of cross-sectional area of the tower is complicated, the main structure is divided into 18 steps for computation. Since the main structure is a tube and the variation of its diameter in a step along the height is linear, the distribution of cross-sectional area in the ith step can be described by

$$A_i(x) = \alpha_i(1 + \beta_i x) \quad i = 1,2, \ldots ,18 \tag{33}$$

The mass per unit volume and Young's modulus of the main structure have been found as [14]

$$\rho_i = \rho = \text{constant} = 2.51 \times 10^3 \text{ kg/m}^3,$$

$$E_i = E = \text{constant} = 3.14 \times 10^7 \text{ kg/m}^2$$

Using the following formula

$$[T] = [T_1][T_2]\cdots[T_{18}]$$

and Eq. (23) obtains $[T]$ which can be written as Eq. (27). Substituting T_{22} into Eq. (30) yields the eigenvalue equation. Solving the eigenvalue equation obtains the first circular natural frequency, ω_1, as 42.16. In order to examine the accuracy of the methods proposed in this paper, the finite element method (FEM) is also employed to calculate the structural dynamic characteristics of this tower. This main structure was divided into 40 uniform steps (elements). The calculated value of ω_1 by the finite element method is 42.36. Although the result obtained by FEM is almost the same as that calculated by the present method, the computer consuming time of FEM is much more than that of the present method. Li et al. [14] reported that the field-measured value of the first circular natural frequency is 41.98. This illustrates that the

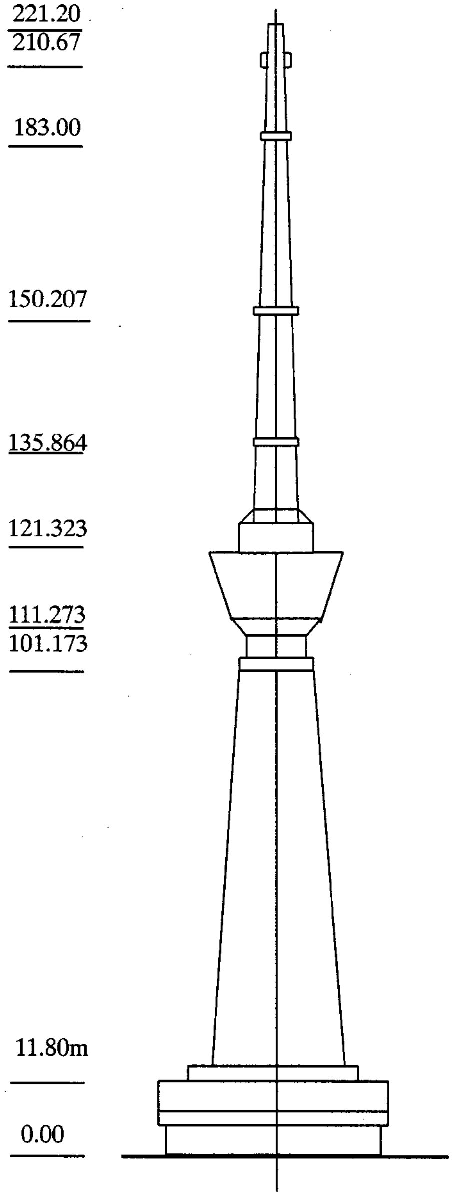

Fig. 5 Wuhan T.V. Tower

computed value of ω_1 by the present method is very close to the measured one. The assumptions of the area variation are thus justified.

Substituting the first natural frequency, ω_1, into Eq. (23), and setting $U_{10}=0$, $N_{10}=1$, and repeatedly using Eq. (23) one obtains the first mode shape. It should be noted that using the aforementioned procedure, the higher natural frequencies and corresponding mode shapes could also be determined.

Conclusions

The general solutions for free longitudinal vibration of one-step nonuniform bars are derived and used to obtain the eigenvalue equation of multi-step bars. The new exact approach is presented which combines the transfer matrix method and closed-form solutions of one step bars derived in this paper. The calculated results of the first six eigenvalues of cantilever nonuniform bars show that the fundamental natural frequencies of those bars are affected most by the taper, but, the higher natural frequencies are close to those of corresponding uniform bars.

The numerical example demonstrates that the calculated natural frequencies and mode shapes of Wuhan T.V. Tower are in good agreement with the corresponding experimental data and the results determined by FEM. This numerical example shows that one of the advantages of the present method is that the total number of the segments (steps or elements) required by the proposed method could be much less than that normally used in conventional finite element methods. Therefore, the proposed method has practical significance for free vibration analysis of nonuniform structures. It is also shown through the numerical example that the selected expressions are suitable for describing the variation of cross-sectional area of typical high-rise structures.

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