ALGEBRAIC ANALYSIS AND RELATED TOPICS
BANACH CENTER PUBLICATIONS, VOLUME 53
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2000

NON-LEIBNIZ ALGEBRAS WITH LOGARITHMS DO NOT HAVE THE TRIGONOMETRIC IDENTITY

D. PRZEWORSKA-ROLEWICZ

Institute of Mathematics, Polish Academy of Sciences
Śniadeckich 8, 00-950 Warszawa, Poland
E-mail: rolewicz@impan.gov.pl

Abstract. Let X be a Leibniz algebra with unit e, i.e. an algebra with a right invertible linear operator D satisfying the Leibniz condition: $D(xy) = xDy + (Dx)y$ for x, y belonging to the domain of D. If logarithmic mappings exist in X, then cosine and sine elements $C(x)$ and $S(x)$ defined by means of antilogarithmic mappings satisfy the Trigonometric Identity, i.e. $[C(x)]^2 + [S(x)]^2 = e$ whenever x belongs to the domain of these mappings. The following question arises: Do there exist non-Leibniz algebras with logarithms such that the Trigonometric Identity is satisfied? We shall show that in non-Leibniz algebras with logarithms the Trigonometric Identity does not exist. This means that the above question has a negative answer, i.e. the Leibniz condition in algebras with logarithms is a necessary and sufficient condition for the Trigonometric Identity to hold.

Let X be a Leibniz algebra with unit e, i.e. an algebra with a right invertible linear operator D satisfying the Leibniz condition: $D(xy) = xDy + (Dx)y$ for x, y belonging to the domain of D. If logarithmic mappings exist in X, then cosine and sine elements $C(x)$ and $S(x)$ defined by means of antilogarithmic mappings satisfy the Trigonometric Identity, i.e. $[C(x)]^2 + [S(x)]^2 = e$ whenever x belongs to the domain of these mappings.

The following question has been posed in PR[2] (cf. also PR[3]):

Do there exist non-Leibniz algebras with logarithms such that the Trigonometric Identity is satisfied?

We shall show that in non-Leibniz algebras with logarithms the Trigonometric Identity does not exist. This means that the above open question has a negative answer, i.e.

The Leibniz condition in algebras with logarithms is a necessary and sufficient condition for the Trigonometric Identity to hold.

2000 *Mathematics Subject Classification*: 47H15, 34K30.

Key words and phrases: algebraic analysis, algebra with unit, Leibniz condition, logarithmic mapping, cosine mapping, sine mapping.

The paper is in final form and no version of it will be published elsewhere.

1. Preliminaries. We recall here some definitions and theorems (without proofs), which are fundamental for Algebraic Analysis (cf. PR[1]).

Let X be a linear space (in general, without any topology) over a field $\mathbb{F}$ of scalars of characteristic zero. We use the following notations:

- $L(X)$ is the set of all linear operators with domains and ranges in X;
- $\operatorname{dom} A$ is the domain of an $A \in L(X)$;
- $\ker A = \{x \in \operatorname{dom} A : Ax = 0\}$ is the kernel of $A \in L(X)$;
- $L_0(X) = \{A \in L(X) : \operatorname{dom} A = X\}$.
- $\mathcal{I}(X)$ is the set of all invertible operators belonging to $L(X)$.

Here the invertibility of an operator $A \in L(X)$ means that the equation $Ax = y$ has a unique solution for every $y \in X$.

An operator $D \in L(X)$ is said to be *right invertible* if there is an operator $R \in L_0(X)$ such that $RX \subset \operatorname{dom} D$ and $DR = I$, where I denotes the identity operator. The operator R is called a *right inverse* of D. By $R(X)$ we denote the set of all right invertible operators in $L(X)$. For $D \in R(X)$ we denote by $\mathcal{R}_D$ the set of all right inverses for D, i.e. $\mathcal{R}_D = \{R \in L_0(X) : DR = I\}$. We have $\operatorname{dom} D = RX \oplus \ker D$, independently of the choice of $R \in \mathcal{R}_D$.

Elements of $\ker D$ are said to be *constants*, since by definition, $Dz = 0$ if and only if $z \in \ker D$. The kernel of D is said to be the *space of constants*. We should point out that, in general, constants are different from scalars, since they are elements of the space X. Clearly, if $\ker D \neq \{0\}$ then the operator D is right invertible, but not invertible. If two right inverses commute, then they are equal. Let

$$\mathcal{F}_D = \{F \in L_0(X) : F^2 = F; FX = \ker D \text{ and } \exists_{R \in \mathcal{R}_D}\ FR = 0\}.$$

Any $F \in \mathcal{F}_D$ is said to be an *initial* operator for D corresponding to R. One can prove that *any* projection F' onto $\ker D$ is an initial operator for D corresponding to a right inverse $R' = R - F'R$ independently of the choice of $R \in \mathcal{R}_D$. It is enough to know one right inverse in order to determine all right inverses and all initial operators.

If two initial operators commute, then they are equal. Thus this theory is essentially *noncommutative*. An operator F is initial for D if and only if there is an $R \in \mathcal{R}_D$ such that $F = I - RD$ on $\operatorname{dom} D$.

Note that the superposition (if exists) of a finite number of right invertible operators is again a right invertible operator.

If $T \in L(X)$ belongs to the set $\Lambda(X)$ of all left invertible operators, then $\ker T = \{0\}$. If $D \in \mathcal{I}(X)$ then $\mathcal{F}_D = \{0\}$ and $\mathcal{R}_D = \{D^{-1}\}$.

Write for $A \in L(X)$

$$v_{\mathbb{F}} A = \{0 \neq \lambda \in \mathbb{F} : I - \lambda A \text{ is invertible}\}. \tag{1.1}$$

This means that $0 \neq \lambda \in v_{\mathbb{F}} A$ if and only if $1/\lambda$ is a regular value of A.

By $V(X)$ we denote the set of all *Volterra operators* belonging to $L(X)$, i.e. the set of all operators $A \in L(X)$ such that $I - \lambda A$ is invertible for all scalars λ. Clearly, $A \in V(X)$ if and only if $v_{\mathbb{F}} A = \mathbb{F} \setminus \{0\}$ (cf. Formula (1.1)).

If X is an algebra over $\mathbb{F}$ with a $D \in L(X)$ such that $x, y \in \operatorname{dom} D$ implies $xy, yx \in \operatorname{dom} D$, then we shall write $D \in \mathbf{A}(X)$. The set of all *commutative* algebras belonging to $\mathbf{A}(X)$ will be denoted by $\mathsf{A}(X)$. If $D \in \mathbf{A}(X)$ then

$$f_D(x, y) = D(xy) - c_D[xDy + (Dx)y] \text{ for } x, y \in \operatorname{dom} D,$$

where c_D is a scalar dependent on D only. Clearly, f_D is a bilinear form which is symmetric when X is commutative, i.e. when $D \in \mathsf{A}(X)$. This form is called a *non-Leibniz component* (cf. PR[3]). Non-Leibniz components have been introduced for right invertible operators $D \in \mathsf{A}(X)$ (cf. PR[1]). If $D \in \mathbf{A}(X)$ then the product rule in X can be written as follows:

$$D(xy) = c_D[xDy + (Dx)y] + f_D(x, y) \quad \text{for } x, y \in \operatorname{dom} D.$$

There are recurrence formulae which permit one to calculate non-Leibniz components for D^n and αD ($n \in \mathbb{N}$, $\alpha \in \mathbb{F}$).

If $D \in \mathbf{A}(X)$ and if D satisfies the *Leibniz condition*:

$$D(xy) = xDy + (Dx)y \quad \text{for } x, y \in \operatorname{dom} D,$$

then X is said to be a *Leibniz algebra*. This means that in Leibniz algebras $c_D = 1$ and $f_D = 0$. The Leibniz condition implies that $xy \in \operatorname{dom} D$ whenever $x, y \in \operatorname{dom} D$. If X is a Leibniz algebra with unit e then $e \in \ker D$, i.e. D is not left invertible.

Let $D \in \mathbf{A}(X)$. Then

- $I(X)$ is the set of all invertible elements belonging to X;
- $M(X)$ is the set of all multiplicative mappings (not necessarily linear) with domains and ranges in X:

$$M(X) = \{A : A(xy) = (Ax)(Ay) \text{ whenever } x, y \in \operatorname{dom} A \subset X\}.$$

We shall now show an approach to the trigonometric identity in Leibniz D-algebras with unit e (but not necessarily with logarithms). Clearly, without additional assumptions we cannot expect too much.

Proposition 1.1. *Suppose that X is a Leibniz D-algebra with unit e, $x \in \operatorname{dom} D^2$ and x, Dx are not zero divisors. If*

$$x^2 + (Dx)^2 = e, \tag{1.2}$$

then

$$\alpha x + \beta Dx \in \ker(D^2 + I) \quad \textit{for every } \alpha, \beta \in \mathbb{F}. \tag{1.3}$$

Proof. Let $y = -Dx$. Then $Dy = -D^2x$ and

$$0 = De = D[x^2 + (Dx)^2] = 2xDx - 2(Dx)D^2x = 2(Dx)(x - D^2x) = 2y(x - Dy).$$

Since $y = -Dx$ is not a zero divisor, we have $x - 2y = 0$. Hence $Dy = x$ and $y = -Dx = -D^2y$, which implies $y \in \ker(D^2 + I)$. On the other hand, $x = Dy = -D^2x$, which implies $x \in \ker(D^2 + I)$. ∎

PROPOSITION 1.2. *Suppose that all assumptions of Proposition 1.1 are satisfied. If Condition (1.3) holds for x and Dx and*

$$u = x^2 + (Dx)^2, \tag{1.4}$$

then $u \in \ker D$.

PROOF. Define u by (1.4). Then

$$Du = 2xDx + 2(Dx)Dx = 2(Dx)(x + Dx) = 2(Dx)(D^2 + I)x = 0,$$

which implies $u \in \ker D$. ∎

COROLLARY 1.1. *Suppose that all assumptions of Proposition 1.1 are satisfied, Condition (1.3) holds for x, Dx, $F \in \mathcal{F}_D \cap M(X)$, $Fx = e$, $FDx = 0$ and u is defined by (1.4). Then $u = e$ and x, Dx satisfy (1.2).*

PROOF. Since F is a multiplicative initial operator and $Fx = e$, $FDx = 0$, we find

$$u = F[x^2 + (Dx)^2] = (Fx)^2 + (FDx)^2 = e^2 + 0 = e,$$

which implies (1.2). ∎

PROPOSITION 1.3. *Suppose that all assumptions of Proposition 1.1 are satisfied, Condition (1.2) holds for x, Dx, $F \in \mathcal{F}_D \cap M(X)$ and $Fx = e$. Then $FDx = 0$.*

PROOF. By our assumptions,

$$e = Fe = F[x^2 + (Dx)^2] = (Fx)^2 + (FDx)^2 = e + (FDx)^2,$$

which implies $(FDx)^2 = 0$. Hence $FDx = 0$. ∎

PROPOSITION 1.4. *Suppose that all assumptions of Proposition 1.3 are satisfied. If $x_\pm \in \ker(D \pm iI)$ and $x = \frac{1}{2}(x_+ + x_-)$, $y = \frac{1}{2i}(x_+ - x_-)$, then*

(i) $x, y \in \ker(D^2 + I)$, $Dx = -y$, $Dy = x$ and $\frac{1}{2}(x \pm y) \in \ker(D^2 \mp iI)$;
(ii) $x^2 + y^2 = x_+x_- \in \ker D$.

PROOF. (i) is proved by checking. In order to prove (ii), observe that, by the Leibniz condition and our assumptions,

$$D(x_+x_-) = x_+Dx_- + x_-Dx_+ = ix_+x_- - ix{+}x_- = 0. \ ∎$$

Observe that $x_\pm$ are eigenvectors of the operator D corresponding to the eigenvalues $\mp i$, respectively (cf. Section 2).

2. Exponential elements in linear spaces. Trigonometric elements in algebras. Here and below we assume that $\mathbb{F}$ is an algebraically closed field of scalars. For instance, $\mathbb{F} = \mathbb{C}$. Following PR[1], we have (with some proofs slightly simpler than in PR[1])

DEFINITION 2.1. If $\lambda \in \mathbb{F}$ is an eigenvalue of an operator $D \in L(X)$ then every eigenvector x_λ corresponding to that eigenvalue is said to be an *exponential element* (briefly: an *exponential*). This means that x_λ is an exponential if and only if $x_\lambda \neq 0$ and $x_\lambda \in \ker(D - \lambda I)$.

PROPOSITION 2.1. *Suppose that* $D \in R(X)$. *If* $0 \neq x_\lambda \in \ker(I - \lambda R)$ *for an* $R \in \mathcal{R}_D$ *and a* $\lambda \in \mathbb{F}$ *then* $x_\lambda \in \ker(D - \lambda I)$, *i.e.* x_λ *is an exponential.*

PROOF. By our assumption, $(D - \lambda I)x_\lambda = (D - \lambda DR)x_\lambda = D(I - \lambda R)x_\lambda = 0$. ■

By induction we prove

PROPOSITION 2.2. *Suppose that* $D \in R(X)$ *and* $\{\lambda_n\} \subset \mathbb{F}$ *is a sequence of eigenvalues such that* $\lambda_i \neq \lambda_j$ *for* $i \neq j$. *Then for an arbitrary* $n \in \mathbb{N}$ *the exponentials* $x_{\lambda_1}, \ldots, x_{\lambda_n}$ *are linearly independent.*

PROPOSITION 2.3. *Suppose that* $D \in R(X)$, F *is an initial operator for* D *corresponding to an* $R \in \mathcal{R}_D$ *and* x_λ *is an exponential. Then* x_λ *is an eigenvector for* R *corresponding to the eigenvalue* $1/\lambda$ *if and only if* $Fx_\lambda = 0$, *i.e.* R *is not a Volterra operator.*

PROOF. *Sufficiency.* Since $Dx_\lambda = \lambda x_\lambda$ and $Fx_\lambda = 0$, we get $x_\lambda = x_\lambda - Fx_\lambda = (I - F)x_\lambda = RDx_\lambda = \lambda Rx_\lambda$. Hence $x_\lambda \in \ker(I - \lambda R)$. Since $x_\lambda \neq 0$, we conclude that x_λ is an eigenvector for R corresponding to $1/\lambda$.

Necessity. Suppose that $1/\lambda$ is an eigenvalue of R and the corresponding eigenvector x_λ is an exponential. Then $Fx_\lambda = (I - RD)x_\lambda = (I - \lambda R)x_\lambda = -\lambda(R - \frac{1}{\lambda}I)x_\lambda = 0$. ■

THEOREM 2.1. *Suppose that* $D \in R(X)$, $\ker D \neq \{0\}$, $R \in \mathcal{R}_D$ *and* $\lambda \in v_{\mathbb{F}}R$. *Then*

(i) λ *is an eigenvalue of* D *and the corresponding exponential is*

$$x_\lambda = e_\lambda(z), \quad \text{where } e_\lambda = (I - \lambda R)^{-1}, \ z \in \ker D; \tag{2.1}$$

whenever $e_\lambda = (I - \lambda R)^{-1}$ *exists, it is said to be an exponential operator;*

(ii) the dimension of the eigenspace X_λ *corresponding to the eigenvalue* λ *is equal to the dimension of the space of constants, i.e.* $\dim X_\lambda = \dim \ker D \neq 0$;

(iii) if $\lambda \neq 0$ *then there exist non-trivial exponentials:* $e_\lambda(z) \neq 0$.

(iv) exponentials are uniquely determined by their initial values, i.e. if F *is an initial operator for* D *corresponding to* R *then* $F[e_\lambda(z)] = z$;

(v) if R *is a Volterra operator then every* $\lambda \in \mathbb{F}$ *is an eigenvalue of* D, *i.e. for every* $\lambda \in \mathbb{F}$ *there exist exponentials.*

PROOF. (i) By definition, $(I - \lambda R)e_\lambda(z) = (I - \lambda R)(I - \lambda R)^{-1}z = z$, where $z \in \ker D$. Thus $e_\lambda(z) = z + \lambda Re_\lambda(z)$, which implies $De_\lambda(z) = Dz + \lambda DRe_\lambda(z) = e_\lambda(z)$.

(ii) Since by our assumptions, the operator $e_\lambda = I - \lambda R$ is invertible, $\dim X_\lambda = \dim\{e_\lambda(z) : z \in \ker D\} = \dim\{(I - \lambda R)^{-1}z : \ker D\} = \ker D \neq 0$.

(iii) If $\lambda \neq 0$ and $e_\lambda(z) = (I - \lambda R)^{-1}z = 0$ then $z = (I - \lambda R)e_\lambda(z) = 0$, This contradicts our assumption that $\ker D \neq \{0\}$.

(iv) By definitions and (i), we have $Fe_\lambda(z) = (I - RD)e_\lambda(z) = (I - \lambda R)e_\lambda(z) = z$.

(v) If $R \in V(X)$ then $v_{\mathbb{F}}R = \mathbb{F} \subset \{0\}$. Clearly, for $\lambda = 0$ the operator $I - \lambda R$ is also invertible. Hence, by (i), every scalar λ is an eigenvalue of D. ■

DEFINITION 2.2. Let $\mathbb{F} = \mathbb{C}$. Suppose that $D \in R(X)$, $\ker D \neq 0$ and $R \in \mathcal{R}_D \cap V(X)$. Then the operators

$$c_\lambda = \frac{1}{2}(e_{\lambda i} + e_{-\lambda i}), \quad s_\lambda = \frac{1}{2i}(e_{\lambda i} - e_{-\lambda i}) \qquad (\lambda \in \mathbb{R}) \tag{2.2}$$

are said to be *cosine* and *sine operators*, respectively (or: *trigonometric operators*). The elements $c_\lambda(z)$, $s_\lambda(z)$, where $z \in \ker D$, are said to be *cosine* and *sine elements*, respectively (or: *trigonometric elements*). ■

THEOREM 2.2. *Suppose that all assumptions of Definition 2.2 are satisfied. Then*

$$c_\lambda = (I + \lambda^2 R^2)^{-1}, \quad s_\lambda = \lambda R(I + \lambda^2 R^2)^{-1} \qquad (\lambda \in \mathbb{R}), \tag{2.3}$$

$$Dc_\lambda = -\lambda s_\lambda, \quad Ds_\lambda = \lambda c_\lambda \qquad (\lambda \in \mathbb{R}), \tag{2.4}$$

$$c_0(z) = z, \quad s_0(z) = 0, \quad Fs_\lambda(z) = 0 \quad \textit{for } z \in \ker D,\ \lambda \in \mathbb{R}. \tag{2.5}$$

Moreover, whenever $z \in \ker D$, $\lambda \in \mathbb{R}$, the element $c_\lambda(z)$ is even with respect to λ and the element s_λ is odd with respect to λ.

PROOF. By the first formula of (2.2), for $\lambda \in \mathbb{R}$ we get

$$\begin{aligned} c_\lambda &= \frac{1}{2}[(I - \lambda iR)^{-1} + (I + \lambda iR)^{-1} = \frac{1}{2}(I - \lambda iR)^{-1}(I + \lambda iR)(I + \lambda iR + I - \lambda iR) \\ &= \frac{1}{2}(I + \lambda^2 R^2)^{-1} 2I = (I + \lambda^2 R^2)^{-1}. \end{aligned}$$

A similar proof for s_λ. By definitions, if $\lambda \in \mathbb{R}$, then

$$Dc_\lambda = \frac{1}{2}(e_{\lambda i} + e_{-\lambda i}) = \frac{1}{2}(\lambda i e_{\lambda i} + \lambda i e_{-\lambda i}) = \frac{1}{2}\lambda i(e_{\lambda i} + e_{-\lambda i}) = -\frac{\lambda}{2i}(e_{\lambda i} + e_{-\lambda i}) = -\lambda s_\lambda.$$

Since $DR = I$, we have $Ds_\lambda = \lambda DR(I + \lambda^2 R^2)^{-1} = \lambda(I + \lambda^2 R^2)^{-1} = \lambda c_\lambda$.

Let $z \in \ker D$. Let $\lambda = 0$. Then $c_0(z) = z$, $s_0(z) = 0$. Since $FR = 0$, for every $\lambda \in \mathbb{R}$ we have $Fs_\lambda(z) = \lambda FR(I + \lambda^2 R^2)^{-1} = 0$. Let $z \in \ker D$. Then

$$\begin{aligned} c_{-\lambda}(z) &= [I + (-\lambda)^2 R^2)^{-1}](z) = (I + \lambda^2 R^2)^{-1} z = c_\lambda(z); \\ s_{-\lambda}(z) &= -\lambda R[I + (-\lambda)^2 R^2)^{-1}](z) = -\lambda R(I + \lambda^2 R^2)^{-1} z = -s_\lambda(z). \ ■ \end{aligned}$$

Consider now trigonometric elements in algebras.

PROPOSITION 2.4. *Suppose that $D \in \mathsf{A}(X) \cap R(X)$, $\ker D \neq \{0\}$ and $R \in \mathcal{R}_D \cap V(X)$. Then*

$$[c_\lambda(z)]^2 + [s_\lambda(z)]^2 = e_{\lambda i}(z) e_{-\lambda i}(z) \quad \textit{for all } z \in \ker D,\ \lambda \in \mathbb{R}. \tag{2.6}$$

PROOF. By checking. ■

PROPOSITION 2.5. *Suppose that $D \in \mathsf{A}(X) \cap R(X)$ and x_λ, x_μ are eigenvectors of D corresponding to the eigenvalues $\lambda, \mu \in \mathbb{C}$, respectively. Then*

$$D[x_\lambda x_\mu] = [c_D(\lambda + \mu)] x_\lambda x_\mu + f_D(x_\lambda, x_\mu) \quad \textit{for all } \lambda \in \mathbb{R}. \tag{2.7}$$

In particular, if $f_D(u,v) = d(Du)(Dv) + auv$ whenever $u, v \in \operatorname{dom} D$ $(d, a \in \mathbb{C})$, then

$$D[x_\lambda x_\mu] = [c_D(\lambda + \mu) + d\lambda\mu + a] x_\lambda x_\mu \quad \textit{for all } \lambda \in \mathbb{R}, \quad \textit{i.e.}$$

$$x_\lambda x_\mu = x_{c_D(\lambda+\mu) + d\lambda\mu + a}. \tag{2.8}$$

PROOF. By checking. ■

PROPOSITION 2.6. *Suppose that $D \in \mathsf{A}(X) \cap R(X)$, $\ker D \neq \{0\}$ and $R \in \mathcal{R}_D \cap V(X)$. Then*

$$D[e_{\lambda i}(z) e_{-\lambda i}(z)] = f_D(e_{\lambda i}(z), e_{-\lambda i}(z)) \quad \textit{for all } z \in \ker D,\ \lambda \in \mathbb{R}. \tag{2.9}$$

Proof. Put in Formula (2.7) λi instead of λ and $-\lambda i$ instead of μ. Then $\lambda+\mu=0$, which implies (2.9). ∎

Corollary 2.1. *Suppose that X is a Leibniz D-algebra,* $\ker D \neq \{0\}$ *and* $R \in \mathcal{R}_D \cap V(X)$. *Then*

$$D[e_{\lambda i}(z)e_{-\lambda i}(z)] = 0 \quad \textit{for all } z \in \ker D,\ \lambda \in \mathbb{R}. \tag{2.10}$$

Proof. Since X is a Leibniz D-algebra, we have $c_D = 1$ and $f_D = 0$. Hence (2.9) implies (2.10). ∎

Corollary 2.2. *Suppose that X is a Leibniz D-algebra,* $\ker D \neq \{0\}$ *and* $R \in \mathcal{R}_D \cap V(X)$. *Then the Trigonometric Identity holds, i.e.*

$$[c_\lambda(z)]^2 + [s_\lambda(z)]^2 = z \quad \textit{for all } z \in \ker D,\ \lambda \in \mathbb{R}. \tag{2.11}$$

Proof. This follows immediately from formulae (2.9) and (2.10). ∎

There is still

Open Question 2.1. *Do there exist non-Leibniz algebras with the Trigonometric Identity (2.11) ?*

Formula (2.8) shows that a sufficient condition for (2.11) to be satisfied is that $c_D = 1$, $d = a = 0$.

3. Logarithmic and antilogarithmic mappings. Suppose that $D \in \mathbf{A}(X)$. Let $\Omega_r, \Omega_l : \operatorname{dom} D \longrightarrow 2^{\operatorname{dom} D}$ be multifunctions defined as follows:

$$\Omega_r u = \{x \in \operatorname{dom} D : Du = uDx\}, \quad \Omega_l u = \{x \in \operatorname{dom} D : Du = (Dx)u\} \tag{3.1}$$

for $u \in \operatorname{dom} D$. The equations

$$Du = uDx \quad \text{for } (u,x) \in \text{graph } \Omega_r\ , \quad Du = (Dx)u \quad \text{for } (u,x) \in \text{graph } \Omega_l \tag{3.2}$$

are said to be the *right* and *left basic equations*, respectively. Clearly,

$$\Omega_r^{-1} x = \{u \in \operatorname{dom} D : Du = uDx\}, \quad \Omega_l^{-1} x = \{u \in \operatorname{dom} D : Du = (Dx)u\} \tag{3.3}$$

for $x \in \operatorname{dom} D$. The multifunctions Ω_r, Ω_l are well-defined and $\operatorname{dom}\Omega_r \cap \operatorname{dom}\Omega_l \supset \ker D$.

Suppose that $(u_r, x_r) \in \text{graph } \Omega_r$, $(u_l, x_l) \in \text{graph } \Omega_l$, L_r, L_l are selectors of Ω_r, Ω_l, respectively, and E_r, E_l are selectors of Ω_r^{-1}, Ω_l^{-1}, respectively. By definitions, $L_r u_r \in \operatorname{dom}\Omega_r^{-1}$, $E_r x_r \in \operatorname{dom}\Omega_r$, $L_l u_l \in \operatorname{dom}\Omega_l^{-1}$, $E_l x_l \in \operatorname{dom}\Omega_l$ and the following equations are satisfied:

$$Du_r = u_r D L_r u_r, \quad DE_r x_r = (E_r x_r) Dx_r;$$
$$Du_l = (DL_l u_l) u_l, \quad DE_l x_l = (Dx_l)(E_l x_l).$$

Definition 3.1 (cf. PR[3]). Any invertible selector L_r of Ω_r is said to be a *right logarithmic mapping* and its inverse $E_r = L_r^{-1}$ is said to be a *right antilogarithmic mapping*. If $(u_r, x_r) \in \text{graph } \Omega_r$ and L_r is an invertible selector of Ω_r then the element $L_r u_r$ is said to be a *right logarithm* of u_r and $E_r x_r$ is said to be a *right antilogarithm* of x_r. By $G[\Omega_r]$ we denote the set of all pairs (L_r, E_r), where L_r is an invertible selector of Ω_r and $E_r = L_r^{-1}$. Respectively, any invertible selector L_l of Ω_l is said to be a *left logarithmic mapping* and its inverse $E_l = L_l^{-1}$ is said to be a *left antilogarithmic mapping*.

If $(u_l, x_l) \in$ graph Ω_l and L_l is an invertible selector of Ω_l then the element $L_l u$ is said to be a *left logarithm* of u_l and $E_l x_l$ is said to be a *left antilogarithm* of x_l. By $G[\Omega_l]$ we denote the set of all pairs (L_l, E_l), where L_l is an invertible selector of Ω_l and $E_l = L_l^{-1}$.

If $D \in \mathsf{A}(X)$ then $\Omega_r = \Omega_l$ and we write $\Omega_r = \Omega$ and $L_r = L_l = L$, $E_r = E_l = E$, $(L, E) \in G[\Omega]$. The selectors L, E are said to be *logarithmic* and *antilogarithmic* mappings, respectively. For any $(u, x) \in$ graph Ω the elements Lu, Ex are said to be the *logarithm* of u and *antilogarithm* of x, respectively. ■

Clearly, by definition, for all $(L_r, E_r) \in G[\Omega_r]$, $(u_r, x_r) \in$ graph Ω_r, $(L_l, E_l) \in G[\Omega_l]$, $(u_l, x_l) \in$ graph Ω_l we have

$$E_r L_r u_r = u_r,\ L_r E_r x_r = x_r; \quad E_l L_l u_l = u_l,\ L_l E_l x_l = x_l; \tag{3.4}$$

$$\begin{aligned} DE_r x_r &= (E_r x_r) D x_r, \quad Du_r = u_r D L_r u_r; \\ DE_l x_l &= (D x_l)(E_l x_l), \quad Du_l = (D L_l u_l) u_l. \end{aligned} \tag{3.5}$$

A right (left) logarithm of zero is not defined. If $(L_r, E_r) \in G[\Omega_r]$, $(L_l, E_l) \in G[\Omega_l]$ then $L_r(\ker D \setminus \{0\}) \subset \ker D$, $E_r(\ker D) \subset \ker D$, $L_l(\ker D \setminus \{0\}) \subset \ker D$, $E_l(\ker D) \subset \ker D$. In particular, $E_r(0)$, $E_l(0) \in \ker D$.

If $D \in R(X)$ then logarithms and antilogarithms are uniquely determined up to a constant. These constants are additive for right (left) logarithms and logarithms and multiplicative for right (left) antilogarithms and antilogarithms. Let F be an initial operator for D corresponding to an $R \in \mathcal{R}_D$. Then there are $(L, E) \in G[\Omega]$ ($(L_r, E_r) \in G[\Omega_r]$, $(L_l, E_l) \in G[\Omega_l]$) such that $FD^j L = 0$ ($FD^j L_r = 0$, $FD^j L_l = 0$) ($j = 0, 1, \ldots, m-1$; $m \in \mathbb{N}$). We then say that (L, E) ((L_r, E_r), (L_l, E_l), respectively) is *m-normalized by R* and we write $(L, E) \in G_{R,m}[\Omega]$ ($(L_r, E_r) \in G_{R,m}[\Omega_r]$, $(L_l, E_l) \in G_{R,m}[\Omega_l]$, respectively). By definition, a 1-normalized logarithm (right, left logarithm, respectively) has a fixed constant.

If X is a *Leibniz algebra* with unit e and D is right invertible, then $e \in \ker D \subset \operatorname{dom} \Omega_r \cap \operatorname{dom} \Omega_l$.

If X is a Leibniz algebra with unit e and $(L, E) \in G_{R,1}[\Omega]$, $u \in I(X) \cap \operatorname{dom} \Omega$ ($(L_r, E_r) \in G_{R,1}[\Omega_r]$, $u \in I(X) \cap \operatorname{dom} \Omega_r$ or $(L_l, E_l) \in G_{R,1}[\Omega_l]$, $u \in I(X) \cap \operatorname{dom} \Omega_l$) then

$$L(-u) = Lu, \quad L_r(-u) = L_r u, \quad L_l(-u) = L_l u.$$

If $D \in R(X)$, X is a Leibniz algebra with unit e, $(L_r, E_r) \in G_{R,1}[\Omega_r]$, $(L_l, E_l) \in G_{R,1}[\Omega_l]$ for an $R \in \mathcal{R}_D$ and $u \in I(X) \cap \operatorname{dom} D$, then $u \in \operatorname{dom} \Omega_r \cap \operatorname{dom} \Omega_l$ and

$$L_r u^{-1} + L_l u = 0, \quad \text{i.e.} \quad L_r u^{-1} = -L_l u.$$

Similarly,

$$(E_r x) E_l(-x) = E_l(-x) E_r x = e, \quad \text{i.e.} \quad E_l(-x) = (E_r x)^{-1}$$

whenever $x \in \operatorname{dom} \Omega_r^{-1}$, $-x \in \operatorname{dom} \Omega_l^{-1}$.

In particular, if X is commutative and $(L, E) \in G_{R,1}[\Omega]$ then $Lu^{-1} = -Lu$ for $u \in I(X) \cap \operatorname{dom} \Omega$ and $E(-x) = (Ex)^{-1}$ whenever $x, -x \in \operatorname{dom} \Omega^{-1}$.

A right logarithmic mapping L_r (a left logarithmic mapping L_l) is said to be of the *exponential type* if $L_r(uv) = L_r u + L_r v$ for $u, v \in \operatorname{dom} \Omega_r$ ($L_l(uv) = L_l u + L_l v$ for $u, v \in \operatorname{dom} \Omega_l$, respectively). If L_r (L_l, respectively) is of the exponential type then

$E_r(x+y) = (E_r x)(E_r y)$ for $x, y \in \operatorname{dom}\Omega_r^{-1}$ ($E_l(x+y) = (E_l x)(E_l y)$ for $x, y \in \operatorname{dom}\Omega_l^{-1}$, respectively). We have proved that a logarithmic mapping L is of the exponential type if and only if X is a commutative *Leibniz algebra*. This means that, in general, not all antilogarithms are *exponentials*, i.e. eigenvectors of the operator D.

In commutative Leibniz algebras with a right invertible operator D a necessary and sufficient condition for $u \in \operatorname{dom}\Omega$ is that $u \in I(X)$. In the non-commutative case this theorem is false, as is shown by a result of Di Bucchianico (cf. DB[1], also [PR[3]).

By $\mathbf{Lg_r}(D)$ ($\mathbf{Lg_l}(D)$, $\mathbf{Lg}(D)$, respectively) we denote the class of those algebras with unit $e \in \operatorname{dom}\Omega$ for which $D \in R(X)$ and there exist invertible selectors of Ω_r (Ω_l, Ω, respectively), i.e. there exist $(L_r, E_r) \in G[\Omega_r]$ ($(L_l, E_l) \in G[\Omega_l]$, $(L, E) \in G[\Omega]$, respectively).

By $\mathbf{Lg}_{\#}(D)$ we denote the class of those commutative algebras with a left invertible D for which there exist invertible selectors of Ω, i.e. there exist $(L, E) \in G[\Omega]$. Clearly, if D is left invertible then $\ker D = \{0\}$. Thus the multifunction Ω is single-valued and we may write: $\Omega = L$. On the other hand, if $\ker D = \{0\}$ then either X is not a Leibniz algebra or X has no unit.

Suppose that either $X \in \mathbf{Lg_r}(D)$ or $X \in \mathbf{Lg_l}(D)$ or $X \in \mathbf{Lg}(D)$. If $c_D = 0$ (in particular, if D is multiplicative), then the intersections of the domains of the multifunctions Ω_r, Ω_l, Ω with lines passing through zero consist of one point only.

In order to show relations between results in algebras with and without logarithms, let us consider the following condition:

[L] $X \in \mathbf{Lg}(D)$ *is a Leibniz D-algebra with unit e*

(i.e. a commutative Leibniz algebra with unit and with $D \in R(X)$).

Proposition 3.1 (cf. PR[3]). *Suppose that Condition* **[L]** *holds. Then $\lambda g = Re \in \operatorname{dom}\Omega^{-1}$ for every $R \in \mathcal{R}_D$ and $\lambda \in v_{\mathbb{F}}R$ and there are $(L, E) \in G[\Omega]$ such that*

$$E(\lambda g) = (I - \lambda R)^{-1} z = e_\lambda(z) \in \ker(D - \lambda I) \quad \text{for all } z \in \ker D.$$

Proof. Let $R \in \mathcal{R}_D$ be fixed. Elements of the form $u = e_\lambda(z) = (I - \lambda R)^{-1} z$ are well-defined for all $z \in \ker D$ and $(D - \lambda I)u = D(I - \lambda R)u = Dz = 0$. Moreover, $Du = \lambda u = u\lambda e = u\lambda DRe = uD(\lambda g)$, which implies that $\lambda g \in \operatorname{dom}\Omega^{-1}$ and there are $(L, E) \in G[\Omega]$ such that $e_\lambda(z) = u = E(\lambda g)$. ∎

Recall that elements of the form $e_\lambda(z)$ are called *exponentials* for D (cf. Section 2; also PR[1]). Indeed, these elements belong to $\ker(D - \lambda I)$, hence they are eigenvectors of D.

4. Trigonometric mappings and elements in algebras with logarithms

Definition 4.1 (cf. PR[3]). Suppose that $\mathbb{F} = \mathbb{C}$, $X \in \mathbf{Lg_r}(D) \cap \mathbf{Lg_l}$ and $\mathbf{E}_1 = \operatorname{dom}\Omega_r^{-1} \cap \operatorname{dom}\Omega_l^{-1}$ is *symmetric*, i.e. $-x \in \mathbf{E}_1$ whenever $x \in \mathbf{E}_1$. Let $(L_r, E_r) \in G[\Omega_r]$, $(L_l, E_l) \in G[\Omega_l]$. Write

$$Cx = \frac{1}{2}[E_l(ix) + E_r(-ix)], \quad Sx = \frac{1}{2i}[E_l(ix) - E_r(-ix)] \quad \text{for } ix \in \mathbf{E}_1. \tag{4.1}$$

In particular, if $(L,E) \in G[\Omega]$, then $\mathbf{E}_1 = \operatorname{dom}\Omega^{-1}$ and $Cx = \frac{1}{2}[E(ix) + E(-ix)]$, $Sx = \frac{1}{2i}[E(ix) - E(-ix)]$. The mappings C and S are said to be *cosine* and *sine* mappings or *trigonometric* mappings. Elements Cx and Sx are said to be *cosine* and *sine* elements or *trigonometric* elements.

Clearly, trigonometric mappings and elements have properties representing those of the classical cosine and sine functions. Namely, we have (proofs can be found in PR[3]):

PROPOSITION 4.1 (cf. PR[3]). *Suppose that all assumptions of Definition 4.1 are satisfied. Let $(L_r, E_r) \in G[\Omega_r]$, $(L_l, E_l) \in G[\Omega_l]$, $(L, E) \in G[\Omega]$. Then trigonometric mappings C and S are well-defined for all $ix \in \mathbf{E}_1$ and*

(i) the De Moivre *formulae hold:*

$$E_r(ix) = Cx + iSx, \quad E_l(-ix) = Cx - iSx \quad \textit{for } ix \in \mathbf{E}_1; \tag{4.2}$$

in particular, if $X \in \mathbf{Lg}(D)$ is a commutative Leibniz algebra then

$$(Cx + iSx)^n = C(nx) + iS(nx) \quad \textit{for } ix \in \mathbf{E}_1 \textit{ and } n \in \mathbb{N}; \tag{4.2$'$}$$

(ii) $C(0), S(0) \in \ker D$;

(iii) if $X \in \mathbf{Lg}(D)$, i.e. $E_r = E_l = E$, then C and S are even and odd functions of their argument, respectively, i.e. $C(-x) = Cx$, $S(-x) = -Sx$;

(iv) for all $ix \in \mathbf{E}_1$

$$(Cx)^2 + (Sx)^2 = \frac{1}{2}[E_l(ix)E_r(-ix) + E_r(-ix)E_l(ix)]. \tag{4.3}$$

COROLLARY 4.1 (cf. PR[3]). *Suppose that all assumptions of Proposition 4.1 are satisfied. Then the mappings C', S' defined as follows: $C'x = C(x+z)$, $S'x = S(x+z)$ for $ix \in E_0$, $z \in \ker D$ also satisfy assertions (i)-(iv) of that Proposition.*

EXAMPLE 4.1 (cf. PR[3]). Suppose that all assumptions of Proposition 4.1 are satisfied. For $ix \in \mathbf{E}_1$ write

$$\widetilde{C}x = \frac{1}{2}[E_r(ix) + E_l(-ix)], \quad \widetilde{S}x = \frac{1}{2i}[E_r(ix) - E_l(-ix)].$$

In particular, if $(L,E) \in G[\Omega]$ then

$$\widetilde{C}x = \frac{1}{2}[E(ix) + E(-ix)], \quad \widetilde{S}x = \frac{1}{2i}[E(ix) - E(-ix)].$$

It is easy to verify that $\widetilde{C}x = C(-x)$, $\widetilde{S}x = -S(-x)$. If $X \in \mathbf{Lg}(D)$ then, by Proposition 4.1(iii), we get $\widetilde{C}x = C(-x) = Cx$, $\widetilde{S}x = -S(-x) = Sx$. This shows that trigonometric mappings are uniquely determined by the choice of right and left antilogarithms and antilogarithms. ■

PROPOSITION 4.2 (cf. PR[3]). *Suppose that all assumptions of Definition 4.1 are satisfied and $(L,E) \in G[\Omega]$. Then for all $ix \in \operatorname{dom}\Omega^{-1}$ we have*

$$(Cx)^2 + (Sx)^2 = E(ix)(E(-ix); \tag{4.4}$$

$$DCx = -(Sx)Dx, \quad DSx = (Cx)Dx. \tag{4.5}$$

EXAMPLE 4.2. We shall show connections of the trigonometric mappings and elements with the trigonometric operators and elements induced by a right inverse of a $D \in R(X)$

in linear spaces (cf. Section 2; also PR[1], Sections 2.3 and 6.2). Suppose then that Condition $[\mathbf{C}]_{\mathbf{1}}$ holds, $R \in \mathcal{R}_D$ and $\lambda \in v_{\mathbb{C}}R$. Let $g = Re$. Then there are $(L, E) \in G[\Omega]$ such that

$$E(\lambda g) = e_\lambda(z) \in \ker(D - \lambda I)\ , \quad \text{where } z \in \ker D \text{ is arbitrary.} \tag{4.7}$$

Indeed, elements of the form $u = e_\lambda(z) = (I - \lambda R)^{-1}z$ are well-defined for all $z \in \ker D$ and called *exponentials* for D since they are eigenvectors for D corresponding to the eigenvalue λ: $(D - \lambda I)u = D(I - \lambda R)u = Dz = 0$ (cf. PR[1]). Moreover, $Du = \lambda u = u\lambda e = u\lambda DRe = uD(\lambda g)$, which implies that $\lambda g \in \operatorname{dom}\Omega^{-1}$ and there are $(L, E) \in G[\Omega]$ such that $e_\lambda(z) = u = E(\lambda g)$. Clearly, $Lu = \lambda g$.

Suppose now that $\lambda i, -\lambda i \in v_{\mathbb{C}}R$. The operators

$$c_\lambda = (I + \lambda^2 R^2); \quad s_\lambda = \lambda R(I + \lambda^2 R^2)$$

are said to be *cosine* and *sine operators*, respectively. Let $z \in \ker D$. Then $c_\lambda z$ and $s_\lambda z$ are said to be *cosine* and *sine elements*, respectively. It is not difficult to verify that

$$c_\lambda = \frac{1}{2}(e_{\lambda i} + e_{-\lambda i}); \quad s_\lambda = \frac{1}{2i}(e_{\lambda i} - e_{-\lambda i}).$$

Thus

$$c_\lambda z = C(\lambda g); \quad s_\lambda z = S(\lambda g).$$

We therefore conclude that c_λ and s_λ have all properties listed in Proposition 4.1. Clearly, proofs in the book PR[1] are different, since they follow just from definitions. Also the assumption made there was much stronger. Namely, we have assumed that R is a *Volterra* operator, i.e. $I - \lambda R$ is invertible for all scalars λ (cf. also Section 2).

Corollary 4.2 (cf. PR[3]). *Suppose that all assumptions of Proposition 4.1 are satisfied and X is a Leibniz D-algebra with unit e. Then the* Trigonometric Identity *holds, i.e.*

$$(Cx)^2 + (Sx)^2 = e \quad \textit{whenever } ix \in \mathbf{E}_1. \tag{4.8}$$

On the other hand, we have

Proposition 4.3 (cf. PR[3]). *Suppose that all assumptions of Definition 4.1 are satisfied, $(L, E) \in G[\Omega]$ and the trigonometric identity (4.8) holds. Then*

(i) $E(ix)$, $E(-ix) \in I(X)$ and $E(-ix) = [E(ix)]^{-1}$ for all $ix \in \operatorname{dom}\Omega^{-1}$;

(ii) $E(0) = e$, hence $Le = 0$;

(iii) $e \in \ker D$;

(iv) if X is an almost Leibniz *D-algebra, i.e. if $f_D(x, z) = 0$ for $x \in \operatorname{dom} D$, $z \in \ker D$, then $c_D = 1$;*

(v) if $c_D = 1$ then $f_D(u, e) = 0$ for all $u \in I(X) \cap \operatorname{dom}\Omega$, i.e. $g_D(u) = u^{-1}f_D(u, e) = 0$.

Proposition 4.3 shows that commutative algebras with the trigonometric identity are very "similar" to Leibniz algebras.

The following question arises: *Do there exist non-Leibniz algebras with the Trigonometric Identity (4.8) ?*

We shall show that the answer to this question is negative.

In order to do it, observe that $c_D \neq 0$ by our assumption that $\operatorname{dom}\Omega^{-1}$ is symmetric. For instance, we have

PROPOSITION 4.4. *Suppose that all assumptions of Definition 4.1 are satisfied,* $(L, E) \in G_{R,1}[\Omega]$ *for an* $R \in \mathcal{R}_D$ *and* $D \in M(X)$ *(i.e.* $c_D = 0$*). Then the Trigonometric Identity (4.8) does not hold.*

PROOF. Suppose that $D \in M(X)$. Then $L, E \in M(X)$. Suppose that the Trigonometric Identity (4.8) holds. Then, by (4.4) and Proposition 4.3(ii), for all $ix \in \operatorname{dom}\Omega^{-1}$ we have

$$e = (Cx)^2 + (Sx)^2 = E(ix)E(-ix) = E[(ix)(-ix)] = E(-i^2x^2) = E(x^2),$$

which implies $x^2 = LE(x^2) = Le = 0$. Hence $x = 0$ and $ix = 0 \in \operatorname{dom}\Omega^{-1}$, which contradicts Proposition 6.2(vi) of PR[3]. Hence (4.8) does not hold. ∎

Note that in several cases an operator D under consideration may be reduced by a substitution to a multiplicative one. It is so, for instance, for various difference operators (cf. PR[3]). In general, if $c_D = 0$, then we have

PROPOSITION 4.5. *Suppose that all assumptions of Definition 4.1 are satisfied,* $(L, E) \in G_{R,1}[\Omega]$ *for an* $R \in \mathcal{R}_D$ *and* $c_D = 0$*. Then the Trigonometric Identity (4.8) does not hold.*

PROOF. Suppose that $c_D = 0$ and (4.8) holds. Then $D(uv) = f_D(u, v)$ for $u, v \in \operatorname{dom} D$. Hence $Du = D(ue) = f_D(u, e)$. Let $u = E(ix)$, where $ix \in \operatorname{dom}\Omega^{-1}$ is arbitrary. Then $e = (Cx)^2 + (Sx)^2 = E(ix)E(-ix)$, which implies $u \in I(X) \cap \operatorname{dom}\Omega$ and $u^{-1} = [E(ix)]^{-1} = E(-ix)$. Thus $f_D(e, e) = De = D(uu^{-1}) = f_D(u, u^{-1})$. The arbitrariness of $ix = Lu$ implies that the mapping f_D is constant. Thus Du is independent of the value of u, i.e. $u = z \in \ker D$. Hence $f_D(u, e) = Du = Dz = 0$ and the mapping $g_D(u, e) = u^{-1}f_D(u, e) = 0$ for all $u \in I(X) \cap \operatorname{dom}\Omega = I(X) \cap \ker D$. By Theorem 4.3 of PR[3], this implies $c_D = 1 \neq 0$, a contradiction. Thus the Trigonometric Identity does not hold. ∎

THEOREM 4.1. *Suppose that all assumptions of Definition 4.1 are satisfied,* $c_D \neq 0$*, and* $(L, E) \in G_{R,1}[\Omega]$ *for an* $R \in \mathcal{R}_D$*. If* X *is a non-Leibniz algebra, then the Trigonometric Identity (4.8) does not hold in* X*.*

PROOF. Suppose that $c_D \neq 0$, X is a non-Leibniz algebra and (4.8) holds. Let F be an initial operator for D corresponding to R. By (4.4), for all $ix \in \operatorname{dom}\Omega^{-1}$ we have

$$e = (Cx)^2 + (Sx)^2 = E(ix)E(-ix) = E(-ix)E(ix).$$

Let $u = E(ix)$. By definition, $u \in I(X) \cap \operatorname{dom}\Omega$. Then $Lu = ix$ and

$$u^{-1} = [E(ix)]^{-1} = E(-ix).$$

By Proposition 4.3(iii),

$$\begin{aligned} 0 = De = D[E(ix)E(-ix)] &= \\ = c_D[E(ix)DE(-ix) + E(-ix)DE(ix)] + f_D(E(ix), E(-ix)) &= \\ = c_D[E(ix)E(-ix)D(-ix) + E(-ix)E(ix)D(ix)] + f_D(E(ix), E(-ix)) &= \end{aligned}$$

$= c_D E(ix)E(-ix)D(-ix+ix) + f_D(E(ix), E(-ix)) = f_D(E(ix), E(-ix)) = f_D(u, u^{-1})$.
Since $(L, E) \in G_{R,1}[\Omega]$, we get $FL = 0$. Hence

$$\begin{aligned} e = E(ix)E(-ix) &= E\big\{c_D ix + R[c_D D(ix) + f_D((E(ix), E(-ix))E(ix)^{-1}E(-ix)^{-1}]\big\} = \\ &= E\big\{c_D ix + R[c_D D(ix) + f_D((E(ix), E(-ix))]\big\} = E[c_D ix + c_D RD(ix)] = \\ &= E[c_D(I + RD)Lu] = E[c_D(2I - F)Lu] = E(2c_D Lu), \end{aligned}$$

which, by Proposition 4.3(ii), implies

$$2c_D Lu = LE(2c_D Lu) = Le = 0.$$

If $c_D \neq 0$, we obtain $Lu = 0$, i.e. $u = e$. This contradicts our assumption on the arbitrariness of $Lu = ix \in \operatorname{dom} \Omega^{-1}$, which implies the arbitrariness of $u \in I(X) \cap \operatorname{dom} \Omega$. Hence $c_D = 0$. ∎

COROLLARY 4.4. *Suppose that all assumptions of Theorem 4.1 are satisfied. Then a necessary and sufficient condition for the Trigonometric Identity (4.8) to be satisfied is that the operator D satisfies the Leibniz condition, i.e. X is a Leibniz algebra.*

References

A. DI BUCCHIANICO

DB[1] *Banach algebras, logarithms and polynomials of convolution type*, J. Math. Anal. Appl. 156 (1991), 253–273.

D. PRZEWORSKA-ROLEWICZ

PR[1] *Algebraic Analysis*, PWN-Polish Scientific Publishers and D. Reidel, Warszawa-Dordrecht, 1988.

PR[2] *Consequences of the Leibniz condition*, in: Different Aspects of Differentiability. Proc. Conf. Warsaw, September 1993. Ed. D. Przeworska-Rolewicz. Dissertationes Math. 340 (1995), 289–300.

PR[3] *Logarithms and Antilogarithms. An Algebraic Analysis Approach*, With Appendix by Z. Binderman. Kluwer Academic Publishers, Dordrecht, 1998.

PR[4] *Linear combinations of right invertible operators in commutative algebras with logarithms*, Demonstratio Math. 31 (1998), 887–898.

PR[5] *Postmodern logarithmo-technia*, International Journal of Computers and Mathematics with Applications (to appear).

PR[6] *Some open questions in Algebraic Analysis*, in: Unsolved Problems on Mathematics for the 21th Century—A Tribute to Kiyoshi Iséki's 80^{th} Birthday, IOS Press, Amsterdam (to appear).