Decompositions of Graphs into Fans and Single Edges

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Abstract

Given two graphs G and H, an *H-decomposition* of G is a partition of the edge set of G such that each part is either a single edge or forms a graph isomorphic to H. Let $\phi(n,H)$ be the smallest number ϕ such that any graph G of order n admits an H-decomposition with at most ϕ parts. Pikhurko and Sousa conjectured that $\phi(n,H)=\text{ex}(n,H)$ for $\chi(H)\geq 3$ and all sufficiently large n, where $\text{ex}(n,H)$ denotes the maximum number of edges in a graph on n vertices not containing H as a subgraph. Their conjecture has been verified by Özkahya and Person for all edge-critical graphs H. In this article, the conjecture is verified for the k-fan graph. The *k-fan graph*, denoted by F_k, is the graph on $2k+1$ vertices consisting of k triangles which intersect in exactly one common vertex called the *centre* of the k-fan.

1 Introduction

Given two graphs G and H, an *H-decomposition* of G is a partition of the edge set of G such that each part is either a single edge or forms a graph isomorphic to H.

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Let $\phi(G,H)$ be the smallest possible number of parts in an H-decomposition of G. It is easy to see that, for non-empty H, we have $\phi(G,H) = e(G) - p_H(G)(e(H) - 1)$, where $p_H(G)$ is the maximum number of pairwise edge-disjoint copies of H that can be packed into G and $e(G)$ denotes the number of edges in G. In this paper, we study the function

$$\phi(n,H) = \max\{\phi(G,H) \mid v(G) = n\},$$

which is the smallest number ϕ such that any graph G of order n admits an H-decomposition with at most ϕ parts.

This function was first studied, in 1966, by Erdős, Goodman, and Pósa [6], who were motivated by the problem of representing graphs by set intersections. They proved that $\phi(n,K_3) = \mathrm{ex}(n,K_3)$, where K_s denotes the complete graph of order s and $\mathrm{ex}(n,H)$ denotes the maximum number of edges in a graph on n vertices not containing H as a subgraph. A decade later, Bollobás [2] proved that $\phi(n,K_r) = \mathrm{ex}(n,K_r)$, for all $n \geq r \geq 3$.

General graphs H were only considered recently by Pikhurko and Sousa [8]. They proved the following result.

Theorem 1.1 (See Theorem 1.1 from [8]). *Let H be any fixed graph of chromatic number $r \geq 3$. Then,*

$$\phi(n,H) = \mathrm{ex}(n,H) + o(n^2).$$

Pikhurko and Sousa also made the following conjecture.

Conjecture 1.2. [8] *For any graph H of chromatic number $r \geq 3$, there exists $n_0 = n_0(H)$ such that $\phi(n,H) = \mathrm{ex}(n,H)$ for all $n \geq n_0$.*

A graph H is *edge-critical* if there exists an edge $e \in E(H)$ such that $\chi(H) > \chi(H - e)$, where $\chi(H)$ denotes the *chromatic number* of H. For $r \geq 4$ a *clique-extension of order* r is a connected graph that consists of a K_{r-1} plus another vertex, say v, adjacent to at most $r-2$ vertices of K_{r-1}. Conjecture 1.2 has been verified by Sousa for some edge-critical graphs, namely, clique-extensions of order $r \geq 4$ ($n \geq r$) [12] and the cycles of length 5 ($n \geq 6$) and 7 ($n \geq 10$) [11, 10]. Later, Özkahya and Person [7] verified the conjecture for all edge-critical graphs with chromatic number $r \geq 3$. Recall that the Turán graph $T_{r-1}(n)$ is the complete balanced $(r-1)$-partite graph on n vertices and does not contain K_r as a subgraph. Their result is the following.

Theorem 1.3 (See Theorem 3 from [7]). *For any edge-critical graph H with chromatic number $r \geq 3$, there exists $n_0 = n_0(H)$ such that $\phi(n,H) = \mathrm{ex}(n,H)$, for all $n \geq n_0$. Moreover, the only graph attaining $\mathrm{ex}(n,H)$ is the Turán graph $T_{r-1}(n)$.*

Recently, as an extension of Özkahya and Person's work, Allen, Böttcher, and Person [1] improved the error term obtained by Pikhurko and Sousa in Theorem 1.1. In fact, they proved that the error term $o(n^2)$ can be replaced by $O(n^{2-\alpha})$ for some $\alpha > 0$. Furthermore, they also showed that this error term has the correct order of magnitude. Their result is indeed an extension of Theorem 1.3 since the error term $O(n^{2-\alpha})$ that they obtained vanishes for every edge-critical graph H.

Here we will verify Conjecture 1.2 for the k-fan graph. The *k-fan graph*, denoted by F_k, is the graph on $2k+1$ vertices consisting of k triangles which intersect in exactly one common vertex, called the *centre* of F_k. Observe that $\chi(F_k) = 3$ and for $k \geq 2$ the graph F_k is not edge-critical.

In 1995, Erdős, Füredi, Gould, and Gunderson [5] have determined the value of the function $\mathrm{ex}(n, F_k)$ as well as the F_k-extremal graphs for every fixed k and whenever n is large. They have proved the following result.

Theorem 1.4. [5] *Let $\mathcal{F}_{n,k}$ be the following family of graphs.*

- *If k is odd and $n \geq 4k-1$, then a member of $\mathcal{F}_{n,k}$ is a Turán graph $T_2(n)$ with two vertex-disjoint copies of K_k added into one class.*
- *If k is even and $n \geq 4k-3$, then a member of $\mathcal{F}_{n,k}$ is a $T_2(n)$ with a graph having $2k-1$ vertices, $k^2 - \frac{3}{2}k$ edges and maximum degree $k-1$ added into one class.*

For $k \geq 1$ and $n \geq 50k^2$, we have

$$\mathrm{ex}(n, F_k) = \left\lfloor \frac{n^2}{4} \right\rfloor + g(k) = \begin{cases} \left\lfloor \frac{n^2}{4} \right\rfloor + k^2 - k & \text{if } k \text{ is odd,} \\ \left\lfloor \frac{n^2}{4} \right\rfloor + k^2 - \frac{3}{2}k & \text{if } k \text{ is even.} \end{cases}$$

Moreover, the only F_k-free graphs with $\mathrm{ex}(n, F_k)$ edges are the members of $\mathcal{F}_{n,k}$.

Here we will prove the following result.

Theorem 1.5. *For $k \geq 1$, there exists $n_0 = n_0(k)$ such that $\phi(n, F_k) = \mathrm{ex}(n, F_k)$ for all $n \geq n_0$. Moreover, the only graphs attaining $\mathrm{ex}(n, F_k)$ are the members of $\mathcal{F}_{n,k}$.*

The lower bound $\phi(n, F_k) \geq \mathrm{ex}(n, F_k)$ follows immediately by considering any member of $\mathcal{F}_{n,k}$. The upper bound will be proved in Section 2.

Our notations throughout the paper are fairly standard. Let $G = (V, E)$ be a graph, $U \subset V$ and v a vertex of G. We denote by $\delta(G)$ and $\Delta(G)$ the minimum and the maximum degree of G, respectively. The subgraph of G induced by U is denoted by $G[U]$ and $e_G(U) = e(G[U])$. We write $\deg_G(v)$ for the degree of v in G and $\deg_G(v, U)$ for the number of neighbours that v has in U. If it is clear which graph is being considered we simply write $e(U)$, $\deg(v)$ and $\deg(v, U)$. Finally, for two disjoint subsets $U, W \subset V$, $e(U, W)$ denotes the number of edges of G with one endpoint in U and the other in W.

2 Proof of Theorem 1.5

In this section we will prove the upper bound in Theorem 1.5. In outline, the proof is the following. Suppose we have a graph G on n vertices such that $\phi(G, F_k) \geq \mathrm{ex}(n, F_k)$. We first apply a stability type result (Lemma 2.1) to deduce that G must be a dense and near-balanced bipartite graph with $m = o(n^2)$ edges inside the classes. Then, we find too many edge-disjoint copies of F_k in G which would imply that $\phi(G, F_k) \leq \mathrm{ex}(n, F_k)$, a contradiction to our initial assumption on $\phi(G, F_k)$. Such approach (stability method) has been widely used to study various problems in extremal graph theory. Our proof generally follows that of Özkahya and Person [7] except for the case when $m = O(k^2)$, when some further detailed analysis will be required.

Before presenting the proof we need to introduce the tools. Firstly, recall the following stability type result about graphs G on n vertices with $\phi(G, H) \geq \mathrm{ex}(n, H) - o(n^2)$ due to Özkahya and Person [7]. Their result follows from a result of Pikhurko and Sousa ([8], Theorem 1.1) and an application of a stability result of Erdős [4] and Simonovits [9].

Lemma 2.1 (See Lemma 4 in [7]). *Let H be a graph with $\chi(H) = r \geq 3$ and $H \neq K_r$. Then, for every $\gamma > 0$ there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that for every graph G on $n \geq n_0$ vertices the following is true. If*

$$\phi(G, H) \geq \mathrm{ex}(n, H) - \varepsilon n^2$$

then there exists a partition $V(G) = V_1 \dot\cup \cdots \dot\cup V_{r-1}$ such that $\sum_{i=1}^{r-1} e(V_i) < \gamma n^2$.

Secondly, let $f(\nu, \Delta) = \max\{e(G) \mid \nu(G) \leq \nu \text{ and } \Delta(G) \leq \Delta\}$, where $\nu(G)$ is the size of a maximum matching in G. We will need the following result of Chvátal and Hanson [3].

Theorem 2.2. [3] *For $\nu, \Delta \geq 1$, we have*

$$f(\nu, \Delta) = \nu\Delta + \left\lfloor \frac{\Delta}{2} \right\rfloor \left\lfloor \frac{\nu}{\lceil \Delta/2 \rceil} \right\rfloor \leq \nu\Delta + \nu.$$

We are now able to complete the proof of Theorem 1.5.

Proof of the upper bound in Theorem 1.5. The case $k = 1$ is the result of Erdős, Goodman, and Pósa [6], so assume $k \geq 2$. We choose $\gamma = \frac{1}{(288k)^2}$ and let $n_0(k) = \max(n_0, 505k^4) + \binom{n_0}{2}$ where n_0 is given by Lemma 2.1. Furthermore, suppose that there exists a graph G on $n \geq n_0(k)$ vertices such that $\phi(G, F_k) \geq \mathrm{ex}(n, F_k)$ and $G \notin \mathcal{F}_{n,k}$. We will derive a contradiction by finding sufficiently many edge-disjoint copies of F_k in G, which will give

$$\phi(G, F_k) = e(G) - p_{F_k}(G)(e(F_k) - 1) < \mathrm{ex}(n, F_k).$$

We first prove the following claim. Although the proof is similar to that of Claim 7 in [7], we include it for the sake of completeness.

Claim 2.3. *Let $m \geq 0$ and $\phi(G, F_k) = \mathrm{ex}(n, F_k) + m$. Then, there is a graph G' on $n' = n - i$ vertices that is obtained by deleting i vertices from G, for some $0 \leq i < n - n_0$, such that $\delta(G') \geq \lfloor \frac{n'}{2} \rfloor$ and $\phi(G', F_k) \geq \mathrm{ex}(n', F_k) + m + i$.*

Proof. If $\delta(G) \geq \lfloor \frac{n}{2} \rfloor$, then set $i = 0$. Otherwise, there exists $v \in V(G)$ with $\deg_G(v) < \lfloor \frac{n}{2} \rfloor$. Then we delete v from G obtaining $G_1 := G - v$ with

$$\begin{aligned}\phi(G_1, F_k) &\geq \phi(G, F_k) - \deg_G(v) \geq \mathrm{ex}(n, F_k) + m - \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ &= \mathrm{ex}(n-1, F_k) + m + 1,\end{aligned}$$

since $\mathrm{ex}(n, F_k) - \mathrm{ex}(n-1, F_k) = \lfloor \frac{n}{2} \rfloor$ by Theorem 1.4. If $\delta(G_1) < \lfloor \frac{n-1}{2} \rfloor$, then we iterate this procedure until we arrive at a graph G' that has $n-i$ vertices, $\delta(G') \geq \lfloor \frac{n-i}{2} \rfloor$ and $\phi(G', F_k) \geq \mathrm{ex}(n-i, F_k) + m + i$, or we stop when G' has n_0 vertices. But the latter case cannot occur since $\phi(G', F_k) > \binom{n_0}{2}$, which is a contradiction. □

By Claim 2.3, we may assume that $\delta(G) \geq \lfloor \frac{n}{2} \rfloor$. Otherwise, we can consider the graph G' instead of G. Note that if $G' \neq G$, then $\phi(G', F_k) > \mathrm{ex}(n', F_k)$ so that $G' \notin \mathcal{F}_{n',k}$.

Let $V_0 \dot\cup V_1$ be a partition of $V(G)$ such that $e(V_0, V_1)$ is maximised and let $m = e(V_0) + e(V_1)$. Observe that

$$m = e(G) - e(V_0, V_1) \geq \mathrm{ex}(n, F_k) - \left\lfloor \frac{n^2}{4} \right\rfloor = g(k),$$

and that

$$e(G) = m + e(V_0, V_1) \leq m + \left\lfloor \frac{n^2}{4} \right\rfloor = \mathrm{ex}(n, F_k) + m - g(k). \tag{2.1}$$

By Lemma 2.1, we also have $m < \gamma n^2$.

The following claim says that the partition $V(G) = V_0 \dot\cup V_1$ is very close to being balanced.

Claim 2.4. *For $i = 0, 1$, we have*

$$\frac{n}{2} - \sqrt{\gamma} n \leq |V_i| \leq \frac{n}{2} + \sqrt{\gamma} n. \tag{2.2}$$

Proof. Without loss of generality, let $|V_0| \leq |V_1|$ and $|V_0| = \frac{n}{2} - a$, where $a \geq 0$. By Lemma 2.1, we have

$$e(G) \leq |V_0||V_1| + \gamma n^2 = \frac{n^2}{4} - a^2 + \gamma n^2.$$

Also, by Theorem 1.4, we have

$$e(G) \geq \mathrm{ex}(n, F_k) = \left\lfloor \frac{n^2}{4} \right\rfloor + g(k) \geq \frac{n^2}{4}.$$

Therefore, $\frac{n^2}{4} - a^2 + \gamma n^2 \geq \frac{n^2}{4}$, which implies that $a \leq \sqrt{\gamma} n$ and (2.2) holds. □

In order to obtain the required contradiction it suffices to show that we can find $\lfloor \frac{m-g(k)}{3k-1} \rfloor + 1$ edge-disjoint copies of F_k in G. In fact, together with (2.1) we obtain

$$\begin{aligned}\phi(G, F_k) &= e(G) - (e(F_k) - 1)p_{F_k}(G) \\ &\leq \mathrm{ex}(n, F_k) + m - g(k) - (3k-1)\Big(\Big\lfloor \frac{m-g(k)}{3k-1} \Big\rfloor + 1\Big) \\ &< \mathrm{ex}(n, F_k).\end{aligned}$$

For this purpose we will describe a procedure that will find the required number of edge-disjoint copies of F_k in G. We continue the proof by considering two different cases.

Case 1: $m \geq \frac{5}{2}k^2 - \frac{1}{2}k + 1$.

For $i = 0, 1$ and a vertex $v \in V_i$, we call v a *bad* vertex if $\deg(v, V_i) > \frac{n}{72k}$. Otherwise, v is a *good* vertex. Observe that the total number of bad vertices in G is at most

$$\frac{2\gamma n^2}{\frac{n}{72k}} = 144k\gamma n. \tag{2.3}$$

For each bad vertex $v \in V_i$, $i = 0, 1$, we may choose $k\lceil \frac{1}{2k} \deg(v, V_i) \rceil$ edges of G which connect v to good vertices in V_i. This is possible since the number of bad vertices is at most $144k\gamma n$. We keep these edges and delete the remaining edges in $G[V_i]$ incident with v. We repeat this procedure for each bad vertex in G. Let G_0 be the resulting graph. Writing $U_i \subset V_i$ for the set of good vertices in V_i, we have

$$\begin{aligned}e_{G_0}(V_i) &= e_{G_0}(U_i) + \sum_{v \text{ bad} \in V_i} \deg_{G_0}(v, V_i) \\ &= e_G(U_i) + \sum_{v \text{ bad} \in V_i} k \Big\lceil \frac{1}{2k} \deg_G(v, V_i) \Big\rceil \\ &\geq \frac{1}{2} e_G(U_i) + \frac{1}{2} \sum_{v \text{ bad} \in V_i} \deg_G(v, V_i) \geq \frac{1}{2} e_G(V_i),\end{aligned}$$

so that $e_{G_0}(V_0) + e_{G_0}(V_1) \geq \frac{m}{2}$.

We now find sufficiently many edge-disjoint copies of F_k in G_0, with each copy having exactly k edges in either V_0 or V_1. Each time we find a copy of F_k we delete its edges. Let $G_s \subset G_0$ be the graph obtained after we have removed s copies.

We define a threshold

$$t = \frac{n}{2} - \frac{n}{36k} - \sqrt{\gamma} n. \tag{2.4}$$

The purpose for the introduction of the threshold t is that in our procedure, for any good vertex $v \in V_i$, $i = 0, 1$, we will ensure that $\deg_{G_s}(v, V_{1-i})$ will not be very much less than t, for every subgraph $G_s \subset G$.

For a good vertex $v \in V_i$, $i = 0, 1$, we say that v is *active* (in G_s) if $\deg_{G_s}(v, V_{1-i}) \geq t$. Otherwise, v is *inactive*. Note that initially, all good vertices $v \in V_i$ are active in

G_0, since $\delta(G) \geq \lfloor \frac{n}{2} \rfloor$ and $\deg_G(v, V_i) \leq \frac{n}{72k}$, so that

$$\deg_{G_0}(v, V_{1-i}) \geq \left\lfloor \frac{n}{2} \right\rfloor - \frac{n}{72k} \geq t + \frac{n}{72k}. \tag{2.5}$$

We perform the following procedure to find edge-disjoint copies of F_k. If we cannot perform Step 1 then we proceed to Step 2.

Step 1. Let $s \geq 0$ and let $G_s \subset G_0$ be a subgraph at some point of the iteration. Suppose there exists a vertex $u \in V_i$ with $\deg_{G_s}(u, V_i) \geq k$, for some $i \in \{0, 1\}$. We take k neighbours $v_1, \ldots, v_k \in V_i$ of u in G_s. We then find good and active vertices $w_1, \ldots, w_k \in V_{1-i}$, where w_j is a common neighbour of u and v_j in G_s for $1 \leq j \leq k$. This gives a copy of F_k. We remove the copy of F_k and update the status of the good vertices (whether they are active or inactive) and let $G_{s+1} \subset G_s$ be the new subgraph. We perform Step 1 successively by considering first all bad vertices, followed by good vertices. Suppose that we have considered bad vertices for a iterates and that Step 1 stops after $b \geq a$ iterates. That is, Step 1 is performed a times for bad vertices and $b - a$ times for good vertices.

Step 2. When Step 1 is completed, we obtain the subgraph $G_b \subset G_0$ such that for $i = 0, 1$, we have $\deg_{G_b}(u, V_i) = 0$ for all bad vertices $u \in V_i$, and $\Delta(G_b[V_i]) < k$. Suppose that $G_s \subset G_b$ $(s \geq b)$ is a subgraph at a further point of the iteration, and there exists a matching in $G_s[V_i]$ of size k for some $i \in \{0, 1\}$. Let $v_1w_1, \ldots, v_kw_k$ be the matching. We find a good and active vertex $u \in V_{1-i}$ which is a common neighbour of $v_1, w_1, \ldots, v_k, w_k$ in G_s. As before, remove the resulting copy of F_k, let $G_{s+1} \subset G_s$ be the new subgraph and update the status of the good vertices. Step 2 is repeated until we have exhausted all such matchings. Let G^* be the subgraph obtained after Step 2 is terminated.

Claim 2.5. *Steps 1 and 2 can be successfully iterated.*

For $i = 0, 1$, we have $\deg_{G^*}(u, V_i) = 0$ for all bad vertices $u \in V_i$, since $\deg_{G_0}(u, V_i)$ is a multiple of k. Also, $\Delta(G^*[V_i]) < k$ and $G^*[V_i]$ does not contain a matching of size k. Thus, by Theorem 2.2, $e_{G^*}(V_i) \leq f(k-1, k-1) \leq (k-1)^2 + (k-1)$ implying that $e_{G^*}(V_0) + e_{G^*}(V_1) \leq 2k(k-1)$. Therefore, since $e_{G_0}(V_0) + e_{G_0}(V_1) \geq \frac{m}{2}$, for $m \geq 14k^2$, we have found and removed at least

$$\left\lfloor \frac{1}{k}\Big(\frac{m}{2} - 2k(k-1)\Big) \right\rfloor \geq \left\lfloor \frac{m - k^2 + \frac{3}{2}k}{3k-1} \right\rfloor + 1 \geq \left\lfloor \frac{m - g(k)}{3k-1} \right\rfloor + 1$$

edge-disjoint copies of F_k from G. Suppose now that $\frac{5}{2}k^2 - \frac{1}{2}k + 1 \leq m \leq 14k^2$ holds. Then $\Delta(G[V_i]) \leq 14k^2 \leq \frac{n}{72k}$ for $i = 0, 1$, so that G has no bad vertices and $G_0 = G$. In this case we have found and removed at least

$$\left\lfloor \frac{1}{k}\big(m - 2k(k-1)\big) \right\rfloor \geq \left\lfloor \frac{m - k^2 + \frac{3}{2}k}{3k-1} \right\rfloor + 1 \geq \left\lfloor \frac{m - g(k)}{3k-1} \right\rfloor + 1$$

edge-disjoint copies of F_k from G, as required. Therefore, to complete the proof of Case 1 it remains to prove Claim 2.5.

Proof of Claim 2.5. The maximality of $e(V_0, V_1)$ implies that for every $v \in V_i$, $i = 0, 1$ we have

$$\deg_G(v, V_{1-i}) \geq \max\Big(\deg_G(v, V_i), \Big\lfloor \frac{n}{4} \Big\rfloor\Big). \tag{2.6}$$

For Step 1, let $G_s \subset G_0$ be a subgraph at some point of the iteration, where $0 \leq s < b$. Suppose firstly that $0 \leq s < a$, so that we have a bad vertex $u \in V_i$ with neighbours $v_1, \ldots, v_k \in V_i$ in G_s, for some $i \in \{0, 1\}$. Note that $v_1, \ldots, v_k$ are good vertices. Then, u was involved in at most $\frac{1}{k} \deg_{G_0}(u, V_i)$ previous iterates, and for each iterate the number of edges that u sends to V_{1-i} was reduced by k. Therefore, by (2.6),

$$\begin{aligned} \deg_{G_s}(u, V_{1-i}) &\geq \deg_{G_0}(u, V_{1-i}) - k \cdot \frac{1}{k} \deg_{G_0}(u, V_i) \\ &\geq \deg_G(u, V_{1-i}) - \frac{1}{2} \deg_G(u, V_i) - k \\ &\geq \frac{1}{2} \deg_G(u, V_{1-i}) - k \geq \frac{1}{2} \Big\lfloor \frac{n}{4} \Big\rfloor - k. \end{aligned} \tag{2.7}$$

Also, for every $1 \leq j \leq k$, we have

$$\deg_{G_s}(v_j, V_{1-i}) \geq t - \Delta(F_k) - \deg_{G_0}(v_j, V_i) \geq t - 2k - \frac{n}{72k}, \tag{2.8}$$

since after v_j becomes inactive, the number of edges that v_j sends to V_{1-i} decreases by at most $\deg_{G_0}(v_j, V_i)$. Finally, note that there are $s \leq \frac{m}{k} < \frac{\gamma n^2}{k}$ previous iterates, and for each iterate, the number of edges that a good and active vertex of V_{1-i} sends to V_i is reduced by at most $\Delta(F_k) = 2k$. By (2.5), the number of inactive good vertices of G_s in V_{1-i} is at most

$$\frac{\frac{2k\gamma n^2}{k}}{\frac{n}{72k}} = 144k\gamma n. \tag{2.9}$$

Let $L(u, v_j) \subset V_{1-i}$ be the set of good and active common neighbours of u and v_j in G_s. Then using (2.3) and (2.9), we have

$$\begin{aligned} |L(u, v_j)| &\geq \deg_{G_s}(u, V_{1-i}) + \deg_{G_s}(v_j, V_{1-i}) - |V_{1-i}| - 144k\gamma n - 144k\gamma n, \\ &\geq \frac{1}{2} \Big\lfloor \frac{n}{4} \Big\rfloor - k + t - 2k - \frac{n}{72k} - \frac{n}{2} - \sqrt{\gamma} n - 288k\gamma n \geq \frac{n}{12}, \end{aligned} \tag{2.10}$$

where the second inequality follows from (2.2), (2.7) and (2.8), and the last one follows from (2.4). Hence, there exist $w_1, \ldots, w_k \in V_{1-i}$ such that for all $1 \leq j \leq k$, w_j is a good and active vertex of G_s, and is a common neighbour of u and v_j in G_s.

Next, suppose that $a \leq s < b$, so that $\deg_{G_s}(u, V_i) = 0$ for all bad vertices $u \in V_i$, $i = 0, 1$. We have a good vertex $u \in V_i$ with $\deg_{G_s}(u, V_i) \geq k$ for some $i \in \{0, 1\}$. Let $v_1, \ldots, v_k \in V_i$ be (good) neighbours of u in G_s, and $L(u, v_j) \subset V_{1-i}$ be the good and active common neighbours of u and v_j in G_s $(1 \leq j \leq k)$. Then, we have

$$\deg_{G_s}(u, V_{1-i}) \geq t - \Delta(F_k) - \deg_{G_0}(u, V_i) \geq t - 2k - \frac{n}{72k}, \tag{2.11}$$

since after u becomes inactive, in each subsequent iterate, the number of edges that u sends to V_i and V_{1-i} are both reduced by either 1 or k. Similarly, for $1 \le j \le k$ we have

$$\deg_{G_s}(v_j, V_i) \ge t - 2k - \frac{n}{72k}.$$

Therefore,

$$|L(u, v_j)| \ge 2\Big(t - 2k - \frac{n}{72k}\Big) - \frac{n}{2} - \sqrt{\gamma}n - 288k\gamma n \ge \frac{n}{3}.$$

As before, we conclude that the required vertices $w_1, \dots, w_k \in V_{1-i}$ do exist.

Now, consider an iterate of Step 2 and let G_s be the graph at some point of the iteration. Suppose that we have a matching M in $G_s[V_i]$ of size k, for some $i \in \{0, 1\}$. Let $v_1w_1, \dots, v_kw_k$ be the edges of M. Then, as in (2.11), we have

$$\deg_{G_s}(v_j, V_{1-i}) \ge t - 2k - \frac{n}{72k},$$
$$\deg_{G_s}(w_j, V_{1-i}) \ge t - 2k - \frac{n}{72k},$$

for $1 \le j \le k$.

Let $L(M) \subset V_{1-i}$ be the set of good and active vertices which are adjacent to $v_1, w_1, \dots, v_k, w_k$ in G_s. Then, similar to (2.10), we have

$$|L(M)| \ge 2k\Big(t - 2k - \frac{n}{72k}\Big) - (2k-1)\Big(\frac{n}{2} + \sqrt{\gamma}n\Big) - 288k\gamma n \ge \frac{n}{3}.$$

Hence, there exists a good and active vertex $u \in V_{1-i}$ which is adjacent to $v_1, w_1, \dots, v_k, w_k$ in G_s. This completes the proof of Claim 2.5 and Case 1 follows. □

Case 2: $g(k) \le m \le \frac{5}{2}k^2 - \frac{1}{2}k$.

We may assume that $|V_0| \le |V_1|$. For $i = 0, 1$, let $A_i \subset V_i$ be the set of vertices incident to at least one edge of $G[V_i]$ and $B_i = V_i \setminus A_i$ the isolated vertices of $G[V_i]$. Then, $|A_i| \le 2m \le 5k^2$ and since $|V_i| \ge \frac{n}{2} - \sqrt{\gamma}n$ by Claim 2.4, we must have $B_i \ne \emptyset$.

Since $\delta(G) \ge \lfloor \frac{n}{2} \rfloor$, by considering any vertex of B_1, it follows that G has the following structure. The partition $V(G) = V_0 \dot\cup V_1$ must be exactly balanced, so that $|V_0| = \lfloor \frac{n}{2} \rfloor$ and $|V_1| = \lceil \frac{n}{2} \rceil$. Moreover, if n is even, then for $v \in B_i$, v is adjacent to all vertices in V_{1-i}. If n is odd, then for $v \in B_1$, v is adjacent to all vertices of V_0, and for $v \in B_0$, all but at most one vertex of V_1 are neighbours of v.

Suppose that $e(G) \ge \mathrm{ex}(n, F_k) + s(3k-1)$ for some integer $s \ge 0$. Then by (2.1), we have

$$s(3k-1) \le e(G) - \mathrm{ex}(n, F_k) \le m - g(k) \le \frac{5}{2}k^2 - \frac{1}{2}k - g(k)$$
$$= \begin{cases} \frac{3}{2}k^2 + \frac{1}{2}k & \text{if } k \text{ is odd,} \\ \frac{3}{2}k^2 + k & \text{if } k \text{ is even,} \end{cases}$$

so that $s \le \frac{k}{2} + \frac{3}{8}$ if k is odd and $s \le \frac{k}{2} + \frac{3}{5}$ if k is even. Therefore, $s \le \lfloor \frac{k}{2} \rfloor$ since s is an integer. Our goal is to prove the following claim.

Claim 2.6. *Let G have the structure described above and let $\mathrm{ex}(n, F_k) + s(3k-1) \leq e(G) < \mathrm{ex}(n, F_k) + (s+1)(3k-1)$ for some integer $0 \leq s \leq \lfloor \frac{k}{2} \rfloor$. Then, G contains at least $s+1$ edge-disjoint copies of F_k.*

Claim 2.6 will then give us

$$\phi(G, F_k) \leq e(G) - (s+1)(e(F_k) - 1) < \mathrm{ex}(n, F_k),$$

and thus completing the proof of Case 2. Before we prove Claim 2.6, we need an auxiliary claim.

Claim 2.7. *Let G have the structure as described. Then for $i = 0, 1$ and any non-empty set $A \subset A_i$ with $|A| \leq 2k^2$, there exists a set $B \subset B_{1-i}$ with $|B| = 2k^2$, such that, G contains a complete bipartite subgraph with classes A and B.*

Proof. Note that for $i = 0, 1$, we have $|B_i| \geq \lfloor \frac{n}{2} \rfloor - 5k^2 \geq 2k^2$. Hence, the only non-trivial case is when n is odd and $i = 1$. For this case, assume that the claim does not hold. Then, the number of vertices of B_0 which are not adjacent to one vertex of A is at least $\lfloor \frac{n}{2} \rfloor - 5k^2 - 2k^2 = \lfloor \frac{n}{2} \rfloor - 7k^2$. Hence, there exists a vertex $u \in A$ which is not adjacent to at least $\frac{1}{2k^2}(\lfloor \frac{n}{2} \rfloor - 7k^2)$ vertices of B_0. We have

$$\deg_G(u) \leq \left\lfloor \frac{n}{2} \right\rfloor + \frac{5}{2}k^2 - \frac{1}{2}k - \frac{1}{2k^2}\left(\left\lfloor \frac{n}{2} \right\rfloor - 7k^2 \right) < \left\lfloor \frac{n}{2} \right\rfloor,$$

which contradicts $\delta(G) \geq \lfloor \frac{n}{2} \rfloor$. □

Proof of Claim 2.6. The claim holds for $s = 0$ since we have $e(G) \geq \mathrm{ex}(n, F_k)$ and $G \notin \mathcal{F}_{n,k}$, so that by Theorem 1.4 G contains a copy of F_k. Now, let $1 \leq s \leq \lfloor \frac{k}{2} \rfloor$. Note that since $e(G) \geq \mathrm{ex}(n, F_k) + s(3k-1)$ and $s < 3k$, we may apply Theorem 1.4 successively s times to obtain and delete s edge-disjoint copies of F_k from G. Moreover, we will first greedily delete $q \leq s$ copies of F_k that have a specific property, followed by a further $s - q$ copies. Let $G' \subset G$ be a subgraph on $V(G)$ obtained after deleting s edge-disjoint copies of F_k from G, so that $e(G') \geq \mathrm{ex}(n, F_k) - s$. We will show that we can always obtain G', so that G' contains a further copy of F_k, which will imply Claim 2.6.

We consider several cases. In each case, we shall find the graph G' and a vertex $u \in V(G')$ with $\deg_{G'}(u) \leq \lfloor \frac{n}{2} \rfloor - s$.

(i) Suppose that G contains a copy of F_{qk} with centre $u \in A_0 \cup A_1$ and $q \geq 2$. We choose u so that q is maximum. We have q copies of F_k and we are done if $q \geq s+1$. If $q \leq s$, then $\deg_G(u) < \lceil \frac{n}{2} \rceil + (q+1)k$, otherwise Claim 2.7 implies that there exists a copy of $F_{(q+1)k}$ with centre u, contradicting the choice of q. Obtain G' by deleting the copy of F_{qk}, followed by a further $s - q$ copies of F_k. We have $\deg_{G'}(u) \leq \lceil \frac{n}{2} \rceil + (q+1)k - 1 - 2qk \leq \lfloor \frac{n}{2} \rfloor - (q-1)k < \lfloor \frac{n}{2} \rfloor - s$.

(ii) Suppose that there are s vertices $u_1, \ldots, u_s \in A_0 \cup A_1$ such that $\deg_G(u_j, A_i) \geq \lfloor \frac{3}{2}k \rfloor - 1$, if $u_j \in A_i$. Without loss of generality we may assume that for $q \geq \lceil \frac{s}{2} \rceil$ we have $u_1, \ldots, u_q \in A_i$, for some $i \in \{0, 1\}$. However, in the special case when $q = \lceil \frac{s}{2} \rceil$

and s is even, we will consider that $u_1, \dots, u_q \in A_0$. For each $u_j \in A_i$, $1 \le j \le q$, we can choose k neighbours of u_j, say $v_{j,1}, \dots, v_{j,k} \in A_i \backslash \{u_1, \dots, u_q\}$. This is possible since there are at least $\lfloor \frac{3}{2}k \rfloor - 1 - (q-1) \ge k$ such neighbours. We may further assume that $v_{j,1} \ne v_{\ell,1}$ for $1 \le j < \ell \le q$. By Claim 2.7, we can find and delete q copies of F_k with centres $u_1, \dots, u_q$ as follows. For $1 \le j \le q$, the copy with centre u_j has triangles $u_j v_{j,1} u, u_j v_{j,2} w_{j,2}, \dots, u_j v_{j,k} w_{j,k}$, for some $u, w_{j,2}, \dots, w_{j,k} \in B_{1-i}$, with the vertices $w_{j,p}$ distinct for $1 \le j \le q$ and $2 \le p \le k$. Obtain the subgraph G' by deleting a further $s-q$ copies of F_k. Then, $\deg_{G'}(u) \le \deg_G(u) - 2q$. In the special case when s is even and $q = \lceil \frac{s}{2} \rceil$ we must have $u \in B_1$, therefore $\deg_{G'}(u) \le \lfloor \frac{n}{2} \rfloor - 2q \le \lfloor \frac{n}{2} \rfloor - s$. In all other cases we have $\deg_{G'}(u) \le \lceil \frac{n}{2} \rceil - 2q = \lfloor \frac{n}{2} \rfloor - s$, as required.

(iii) Suppose that (i) and (ii) do not hold. We obtain G' by deleting any s copies of F_k. If some copy has centre $u \in B_0 \cup B_1$, then $\deg_{G'}(u) \le \lceil \frac{n}{2} \rceil - 2k < \lfloor \frac{n}{2} \rfloor - s$. Otherwise, all the centres lie in $A_0 \cup A_1$, and must be distinct. Moreover, at most $s-1$ vertices $v \in A_0 \cup A_1$ satisfy $\deg_G(v, A_i) \ge \lfloor \frac{3}{2}k \rfloor - 1$ if $v \in A_i$. Hence, there exists a centre $u \in A_i$ with $\deg_G(u, A_i) \le \lfloor \frac{3}{2}k \rfloor - 2$, for some $i \in \{0, 1\}$. We have $\deg_{G'}(u) \le \lceil \frac{n}{2} \rceil + \lfloor \frac{3}{2}k \rfloor - 2 - 2k < \lfloor \frac{n}{2} \rfloor - s$.

Now, $\mathrm{ex}(n, F_k) - \mathrm{ex}(n-1, F_k) = \lfloor \frac{n}{2} \rfloor$ by Theorem 1.4. Therefore in every case,

$$\begin{aligned} e(G' - u) = e(G') - \deg_{G'}(u) &\ge \mathrm{ex}(n, F_k) - s - \left\lfloor \frac{n}{2} \right\rfloor + s \\ &= \mathrm{ex}(n-1, F_k). \end{aligned} \tag{2.12}$$

Observe that equality in (2.12) can only happen in (ii). However, in this case, we have $\deg_{G'-u}(w_{1,2}) \le \lceil \frac{n}{2} \rceil - 2 < \lfloor \frac{n-1}{2} \rfloor$. Thus, $G' - u \notin \mathcal{F}_{n-1,k}$, since otherwise one must have $\delta(G' - u) \ge \lfloor \frac{n-1}{2} \rfloor$. Therefore, by Theorem 1.4, $G' - u$ contains a copy of F_k. This completes the proof of Claim 2.6 and Case 2 follows. □

The proof of Theorem 1.5 is now complete. □

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