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On Some Nonlinear Integral Inequalities with an Advanced Argument

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Abstract

In this paper, we investigate some nonlinear integral inequalities with an advanced argument which provide explicit bounds on unknown functions. The inequalities given here can be used as tools in the qualitative theory of certain advanced nonlinear differential equations. In the end of this paper, an illustrated example is given.

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1. Introduction

It is well known that the integral inequalities involving functions of one and more than one independent variables which provide explicit bounds on unknown functions play a fundamental role in the development of the theory of differential equations. In the past few years, a number of integral inequalities had been established by many scholars, which are motivated by certain applications. For example, we refer the reader to literatures [1–10] and the references therein. However, it is much to be regretted that nobody studied the integral inequalities with an advanced argument, as far as we know.

Our aim in this paper is to study the following nonlinear integral inequalities with an advanced argument of the form

$$x^p(t) \leq a(t) + c(t) \int_t^{\infty} [f(s)x(s+\sigma) + g(s)]\mathrm{d}s, \;\; t \in R_+, \tag{1.1}$$

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$$x^p(t) \le a(t) + \int_t^\infty b(s)x^p(s)\mathrm{d}s + \int_t^\infty [f(s)x(s+\sigma) + g(s)]\mathrm{d}s, \;\; t \in R_+, \tag{1.2}$$

and

$$x^p(t) \le a(t) + \int_t^\infty b(s)x^p(s)\mathrm{d}s + \int_t^\infty L(s, x(s+\sigma))\mathrm{d}s, \;\; t \in R_+, \tag{1.3}$$

where $R_+ = [0, \infty)$ is the given subset of R, $p \ge 1$, $\sigma \in R_+$ are constants, and $L \in C(R_+^2, R_+)$.

Throughout this paper, all the functions which appear in the inequalities are assumed to be real–valued and all the integrals involved exist on the respective domains of their definitions, and $C(M, S)$ denotes the class of all continuous functions defined on set M with range in the set S.

2. Main Results

We firstly introduce two lemmas which are useful in our main results.

Lemma 2.1: Assume that $p \ge 1, a \ge 0$. Then

$$a^{\frac{1}{p}} \le \left(\frac{1}{p}k^{\frac{1-p}{p}}a + \frac{p-1}{p}k^{\frac{1}{p}}\right), \tag{2.1}$$

for any $k > 0$.

Lemma 2.2 [6]: Assume that $u(t), a(t)$ and $b(t)$ are nonnegative functions defined for $t \in R_+$, and $a(t)$ is nonincreasing for $t \in R_+$. If

$$u(t) \le a(t) + \int_t^\infty b(s)u(s)\mathrm{d}s, \;\; t \in R_+, \tag{2.2}$$

then

$$u(t) \le a(t) \exp\left(\int_t^\infty b(s)\mathrm{d}s\right), \;\; t \in R_+. \tag{2.3}$$

Nextly, we establish our main results.

Theorem 2.3: Assume that $x(t), a(t), c(t), f(t)$ and $g(t)$ are nonnegative functions defined for $t \in R_+$. If $a(t)$ and $c(t)$ are nonincreasing in R_+, then the inequality (1.1) implies

$$x(t) \le \left[a(t) + c(t)h(t)\exp\left(\int_t^\infty \frac{f(s)c(s)}{pk^{\frac{p-1}{p}}}\mathrm{d}s\right)\right]^{\frac{1}{p}}, \tag{2.4}$$

for any $k > 0, t \in R_+$, where

$$h(t) = \int_t^\infty \left[f(s)\left(\frac{p-1}{p}k^{\frac{1}{p}} + \frac{a(s)}{pk^{\frac{p-1}{p}}}\right) + g(s)\right]\mathrm{d}s. \tag{2.5}$$

Proof: Fixing any $\delta \geq 0$, we define a function $z(t)$ by

$$z(t) = \left\{ a(t) + \delta + c(t) \int_t^\infty [f(s)x(s+\sigma) + g(s)]\mathrm{d}s \right\}^{\frac{1}{p}}, \ \ t \in R_+. \tag{2.6}$$

It is easy to see that $z(t)$ is a nonnegative and nonincreasing function, and

$$x(t) \leq z(t), \ \ t \in R_+. \tag{2.7}$$

Therefore, we obtain

$$x(t+\sigma) \leq z(t+\sigma) \leq z(t), \ \ t \in R_+. \tag{2.8}$$

It follows from (2.6) and (2.8) that

$$z^p(t) \leq a(t) + \delta + c(t) \int_t^\infty [f(s)z(s) + g(s)]\mathrm{d}s, \ \ t \in R_+. \tag{2.9}$$

Taking $\delta \to 0$ in (2.9), we have

$$z^p(t) \leq a(t) + c(t) \int_t^\infty [f(s)z(s) + g(s)]\mathrm{d}s, \ \ t \in R_+. \tag{2.10}$$

Define a function $u(t)$ by

$$u(t) = \int_t^\infty [f(s)z(s) + g(s)]\mathrm{d}s, \ \ t \in R_+. \tag{2.11}$$

Then (2.10) can be restated as

$$z^p(t) \leq a(t) + c(t)u(t). \tag{2.12}$$

Using Lemma 2.1, from (2.12), for any $k > 0$, we easily obtain

$$z(t) \leq \Big(a(t) + c(t)u(t) \Big)^{\frac{1}{p}} \leq \frac{p-1}{p} k^{\frac{1}{p}} + \frac{a(t)}{pk^{\frac{p-1}{p}}} + \frac{c(t)u(t)}{pk^{\frac{p-1}{p}}}. \tag{2.13}$$

Combining (2.11) and (2.13), we have

$$\begin{aligned} u(t) &\leq \int_t^\infty \left[f(s) \left(\frac{p-1}{p} k^{\frac{1}{p}} + \frac{a(s)}{pk^{\frac{p-1}{p}}} + \frac{c(s)u(s)}{pk^{\frac{p-1}{p}}} \right) + g(s) \right] \mathrm{d}s \\ &= h(t) + \int_t^\infty \frac{f(s)c(s)u(s)}{pk^{\frac{p-1}{p}}} \mathrm{d}s, \ \ t \in R_+, \end{aligned} \tag{2.14}$$

where $h(t)$ is defined by (2.5). Obviously, $h(t)$ is nonnegative and nonincreasing for $t \in R_+$.

(i) If $h(t) > 0$ for $t \in R_+$. From (2.14), we easily observe that

$$\frac{u(t)}{h(t)} \leq 1 + \int_t^\infty \frac{f(s)c(s)}{pk^{\frac{p-1}{p}}} \cdot \frac{u(s)}{h(s)} \mathrm{d}s.$$

Setting

$$y(t) = 1 + \int_t^\infty \frac{f(s)c(s)}{pk^{\frac{p-1}{p}}} \cdot \frac{u(s)}{h(s)} \mathrm{d}s, \tag{2.15}$$

we obtain

$$y(t) = -\frac{f(t)c(t)}{pk^{\frac{p-1}{p}}} \cdot \frac{u(t)}{h(t)} \geq -\frac{f(t)c(t)}{pk^{\frac{p-1}{p}}} y(t). \tag{2.16}$$

Noting that $\lim_{t\to\infty} y(t) = 1$, it follows from (2.16) that

$$y(t) \leq \exp\left(\int_t^\infty \frac{f(s)c(s)}{pk^{\frac{p-1}{p}}} \mathrm{d}s\right).$$

Therefore,

$$u(t) \leq h(t) \exp\left(\int_t^\infty \frac{f(s)c(s)}{pk^{\frac{p-1}{p}}} \mathrm{d}s\right). \tag{2.17}$$

It is easy to see that the desired inequality (2.4) follows from (2.7), (2.12) and (2.17).

(ii) If $h(t)$ is nonnegative for $t \in R_+$, we carry out the above procedure with $h(t) + \varepsilon$ instead of $h(t)$, where $\varepsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\varepsilon \to 0$ to obtain (2.4).

This completes the proof of Theorem 2.3. ■

Theorem 2.4: Assume that $x(t), a(t), b(t), f(t)$ and $g(t)$ are nonnegative functions for $t \in R_+$, and $a(t)$ is nonincreasing in R_+. Then the inequality (1.2) implies

$$x(t) \leq B(t)\left[a(t) + F(t) \exp\left(\int_t^\infty \frac{f(s)B(s)}{pk^{\frac{p-1}{p}}} \mathrm{d}s\right)\right]^{\frac{1}{p}}, \tag{2.18}$$

for any $k > 0, t \in R_+$, where

$$B(t) = \left[\exp\left(\int_t^\infty b(s)\mathrm{d}s\right)\right]^{\frac{1}{p}}, \tag{2.19}$$

$$F(t) = \int_t^\infty \left[f(s)B(s)\left(\frac{p-1}{p}k^{\frac{1}{p}} + \frac{a(s)}{pk^{\frac{p-1}{p}}}\right) + g(s)\right] \mathrm{d}s. \tag{2.20}$$

Proof: Fixing any $\delta \geq 0$, we define a function $z(t)$ by

$$z(t) = \left\{a(t)+\delta+\int_t^\infty b(s)x^p(s)\mathrm{d}s+\int_t^\infty [f(s)x(s+\sigma)+g(s)]\mathrm{d}s\right\}^{\frac{1}{p}}, \ t \in R_+. \quad (2.21)$$

Using a similar way in the proof of Theorem 2.3, we easily see that $z(t)$ is nonincreasing,

$$x(t) \leq z(t), \ \ t \in R_+, \quad (2.22)$$

and

$$x(t+\sigma) \leq z(t+\sigma) \leq z(t), \ \ t \in R_+.$$

Therefore,

$$z^p(t) \leq a(t) + \int_t^\infty b(s)z^p(s)\mathrm{d}s + \int_t^\infty [f(s)z(s) + g(s)]\mathrm{d}s, \ \ t \in R_+. \quad (2.23)$$

Define a function $u(t)$ by

$$u(t) = a(t) + v(t), \quad (2.24)$$

where

$$v(t) = \int_t^\infty [f(s)z(s) + g(s)]\mathrm{d}s. \quad (2.25)$$

Then (2.23) can be restated as

$$z^p(t) \leq u(t) + \int_t^\infty b(s)z^p(s)\mathrm{d}s. \quad (2.26)$$

Obviously, $u(t)$ is a nonnegative and nonincreasing function for $t \in R_+$. Therefore, using Lemma 2.2 to (2.29), we obtain

$$z^p(t) \leq u(t) \exp\left(\int_t^\infty b(s)\mathrm{d}s\right),$$

i.e.,

$$z(t) \leq B(t)[a(t) + v(t)]^{\frac{1}{p}}, \quad (2.27)$$

where $B(t)$ is defined by (2.19). Using Lemma 2.1, for any $k > 0$, it follows from (2.27) that

$$z(t) \leq B(t)[a(t) + v(t)]^{\frac{1}{p}} \leq B(t)\left[\frac{p-1}{p}k^{\frac{1}{p}} + \frac{a(t)}{pk^{\frac{p-1}{p}}} + \frac{v(t)}{pk^{\frac{p-1}{p}}}\right]. \quad (2.28)$$

Combining (2.25) and (2.28), we obtain

$$\begin{aligned} v(t) &\leq \int_t^\infty \left\{f(s)B(s)\left(\frac{p-1}{p}k^{\frac{1}{p}} + \frac{a(s)}{pk^{\frac{p-1}{p}}} + \frac{v(s)}{pk^{\frac{p-1}{p}}}\right) + g(s)\right\}\mathrm{d}s \\ &= F(t) + \int_t^\infty \frac{f(s)B(s)v(s)}{pk^{\frac{p-1}{p}}}\mathrm{d}s, \quad (2.29) \end{aligned}$$

where $F(t)$ is defined by (2.20).

We easily see that $F(t)$ is nonnegative and nonincreasing for $t \in R_+$. Using Lemma 2.2, it follows from (2.29) that

$$v(t) \le F(t) \exp\left(\int_t^\infty \frac{f(s)B(s)}{pk^{\frac{p-1}{p}}} \mathrm{d}s \right). \tag{2.30}$$

Combining (2.27) and (2(30), we have

$$z(t) \le B(t) \left[a(t) + F(t) \exp\left(\int_t^\infty \frac{f(s)B(s)}{pk^{\frac{p-1}{p}}} \right) \mathrm{d}s \right]^{\frac{1}{p}}. \tag{2.31}$$

Therefore, the desired inequality (2.18) follows from (2.22) and (2.31). The proof is complete. ■

Theorem 2.5: Assume that $x(t), a(t)$ and $b(t)$ are nonnegative functions for $t \in R_+$, and $a(t)$ is nonincreasing in R_+. If

$$0 \le L(t,x) - L(t,y) \le K(t,y)(x-y), \tag{2.32}$$

for $x \ge y \ge 0$, where $K \in C(R_+^2, R)$, then the inequality (1.3) implies

$$x(t) \le B(t) \left[a(t) + G(t) \exp\left(\int_t^\infty K\left(s, B(s) \left(\frac{p-1}{p} k^{\frac{1}{p}} + \frac{a(s)}{pk^{\frac{p-1}{p}}} \right) \right) \frac{B(s)}{pk^{\frac{p-1}{p}}} \mathrm{d}s \right) \right]^{\frac{1}{p}}, \tag{2.33}$$

for any $k > 0, t \in R_+$, where

$$G(t) = \int_t^\infty Ł\left(s, B(s) \left(\frac{p-1}{p} k^{\frac{1}{p}} + \frac{a(s)}{pk^{\frac{p-1}{p}}} \right) \right) \mathrm{d}s, \tag{2.34}$$

and $B(t)$ is defined by (2.19).

Proof: Fixing any $\delta \ge 0$, we define a function $z(t)$ by

$$z(t) = \left\{ a(t) + \delta + \int_t^\infty b(s)x^p(s)\mathrm{d}s + \int_t^\infty L(s, x(s+\sigma))\mathrm{d}s \right\}^{\frac{1}{p}}, \quad t \in R_+. \tag{2.35}$$

Using a similar way in the proof of Theorem 2.3, we easily obtain that $z(t)$ is a nonnegative and nonincreasing function,

$$x(t) \le z(t), \quad t \in R_+, \tag{2.36}$$

and

$$x(t+\sigma) \le z(t+\sigma) \le z(t), \quad t \in R_+. \tag{2.37}$$

Noting the condition (2.32) and combining (2.35)–(2.37), we have

$$z^p(t) \le a(t) + \int_t^\infty b(s)z^p(s)\mathrm{d}s + \int_t^\infty L(s, z(s))\mathrm{d}s, \ \ t \in R_+. \tag{2.38}$$

Define a function $u(t)$ by

$$u(t) = a(t) + w(t),$$

where

$$w(t) = \int_t^\infty L(s, z(s))\mathrm{d}s. \tag{2.39}$$

Then (2.38) can be restated as

$$z^p(t) \le u(t) + \int_t^\infty b(s)z^p(s)\mathrm{d}s. \tag{2.40}$$

We easily see that $u(t)$ is a nonnegative and nonincreasing function for $t \in R_+$. Therefore, it follows from (2.40) that

$$z(t) \le B(t)[a(t) + w(t)]^{\frac{1}{p}}. \tag{2.41}$$

Using Lemma 2.1, for any $k > 0$, from (2.41) we have

$$z(t) \le B(t)\left[\frac{p-1}{p}k^{\frac{1}{p}} + \frac{a(t)}{pk^{\frac{p-1}{p}}} + \frac{w(t)}{pk^{\frac{p-1}{p}}}\right]. \tag{2.42}$$

Combining (2.32), (2.39) and (2.42), we obtain

$$\begin{aligned} w(t) \le \int_t^\infty \Bigg\{ & L\Bigg(s, B(s)\bigg(\frac{p-1}{p}k^{\frac{1}{p}} + \frac{a(s)}{pk^{\frac{p-1}{p}}} + \frac{w(s)}{pk^{\frac{p-1}{p}}}\bigg)\Bigg) \\ & - L\Bigg(s, B(s)\bigg(\frac{p-1}{p}k^{\frac{1}{p}} + \frac{a(s)}{pk^{\frac{p-1}{p}}}\bigg)\Bigg) \\ & + L\Bigg(s, B(s)\bigg(\frac{p-1}{p}k^{\frac{1}{p}} + \frac{a(s)}{pk^{\frac{p-1}{p}}}\bigg)\Bigg)\Bigg\}\mathrm{d}s \\ \le G(t) + \int_t^\infty & K\Bigg(s, B(s)\bigg(\frac{p-1}{p}k^{\frac{1}{p}} + \frac{a(s)}{pk^{\frac{p-1}{p}}}\bigg)\Bigg)\frac{B(s)w(s)}{pk^{\frac{p-1}{p}}}\mathrm{d}s, \end{aligned} \tag{2.43}$$

where $G(t)$ is defined by (2.34).

It is obvious that $G(t)$ is nonnegative and nonincreasing for $t \in R_+$. Using Lemma 2.2, from (2.43), we have

$$w(t) \le G(t)\exp\Bigg(\int_t^\infty K\Bigg(s, B(s)\bigg(\frac{p-1}{p}k^{\frac{1}{p}} + \frac{a(s)}{pk^{\frac{p-1}{p}}}\bigg)\Bigg)\frac{B(s)}{pk^{\frac{p-1}{p}}}\mathrm{d}s\Bigg). \tag{2.44}$$

It follows from (2.41) and (2.44) that

$$z(t) \le B(t)\Bigg[a(t) + G(t)\exp\Bigg(\int_t^\infty K\Bigg(s, B(s)\Bigg(\frac{p-1}{p}k^{\frac{1}{p}} + \frac{a(s)}{pk^{\frac{p-1}{p}}}\Bigg)\Bigg)\frac{B(s)}{pk^{\frac{p-1}{p}}}\mathrm{d}s\Bigg)\Bigg]^{\frac{1}{p}}. \tag{2.45}$$

Therefore, the desired inequality (2.33) follows from (2.36) and (2.45). The proof is complete. ∎

3. An Application

In this section, using Theorem 2.3, we obtain the bound on the solution of a nonlinear differential equation with an advanced argument.

Example: Consider the final value problems of differential equation with an advanced argument

$$\begin{cases} px^{p-1}(t)x'(t) = H(t, x(t+\sigma)) + q(t), \ t \in R_+, \\ x(\infty) = x_\infty, \end{cases} \tag{3.1}$$

where $p \ge 1$, $\sigma \in R_+$ are constants, $H \in C(R_+ \times R, R)$, $q \in C(R_+, R)$, and $x_\infty \in R_+$.

Assume that

$$\begin{cases} |H(t, x(t+\sigma))| \le f(t)|x(t+\sigma)| + g(t), \\ \left|x_\infty - \int_t^\infty q(s)\mathrm{d}s\right| \le a(t), \ t \in R_+. \end{cases} \tag{3.2}$$

If $x(t)$ is a solution of the equation (3.1). Then

$$|x(t)| \le \left[a(t) + h(t)\exp\left(\int_t^\infty \frac{f(s)}{pk^{\frac{p-1}{p}}}\mathrm{d}s\right)\right]^{\frac{1}{p}}, \tag{3.3}$$

for any $k > 0, t \in R_+$, where $a(t), f(t), g(t)$ and $h(t)$ are defined as in Theorem 2.3.

In fact, if $x(t)$ is a solution of the equation (3.1), then, by using the ideas in [1], the equation (3.1) can be written as

$$x^p(t) = x_\infty^p - \int_t^\infty q(s)\mathrm{d}s - \int_t^\infty H(s, x(s+\sigma))\mathrm{d}s. \tag{3.4}$$

Therefore,

$$|x^p(t)| \le \left|x_\infty^p - \int_t^\infty q(s)\mathrm{d}s\right| + \int_t^\infty |H(s, x(s+\sigma))|\mathrm{d}s. \tag{3.5}$$

Noting the assumption (3.2), we easily obtain

$$|x^p(t)| \le a(t) + \int_t^\infty \Big(f(s)|x(s+\sigma)| + g(s)\Big)\mathrm{d}s. \tag{3.6}$$

Now a suitable application of Theorem 2.3 to (3.6) immediately yields (3.3).

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