Lattice Type Fuzzy Order and Closure Operators in Fuzzy Ordered Sets

Radim Bělohlávek
Dept. Computer Science, Palacký University
Tomkova 40, CZ-779 00 Olomouc, Czech Republic
radim.belohlavek@upol.cz

Inst. Fuzzy Modeling, University of Ostrava
30. dubna 22, CZ-701 03 Ostrava, Czech Republic

Abstract

Complete lattices and closure operators in ordered sets are considered from the point of view of fuzzy logic. A typical example of a fuzzy order is the graded subsethood of fuzzy sets. Graded subsethood makes the set of all fuzzy sets in a given universe into a completely lattice fuzzy ordered set (i.e. a complete lattice in fuzzy setting). Another example of a completely lattice fuzzy ordered set is the set of all so-called fuzzy concepts in a given fuzzy context; the respective fuzzy order is the graded subconcept/superconcept relation. Conversely, each completely lattice fuzzy ordered set is isomorphic to some fuzzy ordered set of fuzzy concepts of a given fuzzy context. These natural examples motivate us to investigate some general properties of complete lattice-type fuzzy order. Particularly, in this paper we focus mainly on closure operators in fuzzy ordered sets.

1. Preliminaries

The notion of a (partial) order plays a central role in mathematics and its applications. It goes back to 19-th century investigations in logic [15]. The origins are in the study of hierarchy of concepts, i.e. the relation of being a subconcept of a superconcept. From the point of view of fuzzy logic, a natural question arises of whether it is reasonable to consider the notion of a fuzzy order. The first paper on fuzzy order is Zadeh's [18]; there are several further papers devoted to fuzzy order (see e.g. [11]). In [3, 4], we introduced a general notion (we use complete residuated lattices as the structure of truth values) of a fuzzy order and of a complete lattice fuzzy order; the main motivation being to find an appropriate axiomatic characterization of the graded hierarchical structure of so-called fuzzy concepts induced by a fuzzy context (so-called fuzzy concept lattices, see later on). This paper brings some new results on fuzzy order and, particularly, focuses on closure operators in fuzzy ordered sets. The emphasis on closure operators is motivated by the fact that fuzzy concept lattices are, up to an isomorphism, all completely lattice fuzzy ordered sets and that a fuzzy concept lattice is exactly a set of fixed points of a fuzzy closure operator.

We recall some necessary notions: It turned out in the investigations of fuzzy logic that an important structure of truth values is that of (complete) residuated lattice [7, 9, 10], i.e. an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that (1) $\langle L, \wedge, \vee, 0, 1\rangle$ is a (complete) lattice with the least element 0 and the greatest element 1, (2) $\langle L, \otimes, 1\rangle$ is a commutative monoid, (3) $\otimes, \rightarrow$ form an adjoint pair, i.e. we have $x \otimes y \leq z$ iff $x \leq y \rightarrow z$. for all $x, y, z \in L$. Examples of residuated lattices are: BL-algebras, MV-algebras, Boolean algebras, Heyting algebras, BL-algebras, Girard monoids. In particular, if $\otimes$ is a left-continuous t-norm then putting $a \rightarrow b := \bigvee\{c \mid a \otimes c \leq b\}$, $\langle [0,1], \min, \max, \otimes, \rightarrow, 0, 1\rangle$ is a complete residuated lattice.

A nonempty subset $K \subseteq L$ is called an $\leq$-*filter* if for every $a, b \in L$ such that $a \leq b$ we have that $b \in K$ whenever $a \in K$. An $\leq$-filter K is called a *filter* if $a, b \in K$ implies $a \otimes b \in K$. Unless stated otherwise, in what follows we denote by $\mathbf{L}$ a complete residuated lattice and by K a $\leq$-filter in $\mathbf{L}$ (both $\mathbf{L}$ and K possilbly with indicies).

Basic notions of fuzzy sets: An $\mathbf{L}$-*set* (or fuzzy set, if $\mathbf{L}$ is obvious or not important) [17, 7] A in a universe set X is any map $A : X \to L$, $A(x)$ being interpreted as the truth degree of the fact "x belongs to A". By L^X we denote the set of all **L**-sets in X. The concept of an **L**-relation is defined obviously; we will use both prefix and infix notation (thus, the truth degrees to which elements x and y are related by an **L**-relation R are denoted by $R(x,y)$ or (xRy)). Operations on L extend pointwise to L^X, e.g. $(A \vee B)(x) = A(x) \vee B(x)$ for $A, B \in L^X$. Following common usage, we write $A \cup B$ instead of $A \vee B$, etc. Given $A, B \in L^X$, the subsethood degree [7] $\mathrm{S}(A,B)$ of A in B is defined by $\mathrm{S}(A,B) = \bigwedge_{x \in X} A(x) \to B(x)$. We write $A \subseteq B$ if $\mathrm{S}(A,B) = 1$. Analogously, the equality degree $(A \approx B)$ of A and B is defined by $(A \approx B) = \bigwedge_{x \in X}(A(x) \leftrightarrow B(x))$ where $a \leftrightarrow b := (a \to b) \wedge (b \to a)$. It is immediate that $(A \approx B) = \mathrm{S}(A,B) \wedge \mathrm{S}(B,A)$. For $A \in L^X$ and $a \in L$, the set ${}^aA = \{x \in X \mid A(x) \geq a\}$ is called the a-cut of A. For $x \in X$ and $a \in L$, $\{{}^a/x\}$ is the **L**-set in X defined by $\{{}^a/x\}(x) = a$ and $\{{}^a/x\}(y) = 0$ for $y \neq x$.

A binary **L**-relation $\approx$ on X is called an **L**-equality if it is an **L**-equivalence, i.e. satisfies $(x \approx x) = 1$ (reflexivity), $(x \approx y) = (y \approx x)$ (symmetry), $(x \approx y) \otimes (y \approx z) \leq (x \approx z)$ (transitivity), and, moreover, $(x \approx y) = 1$ implies $x = y$. Note that **2**-equality on X is precisely the usual equality (identity) id_X (i.e. $\mathrm{id}_X(x,y) = 1$ for $x = y$ and $\mathrm{id}_X(x,y) = 0$ for $x \neq y$).

2. Fuzzy order and lattice type fuzzy order

We say that a binary **L**-relation R between X and Y is *compatible* w.r.t. **L**-equivalence relations $\approx_X$ and $\approx_Y$ (on X and Y) if $R(x_1, y_1) \otimes (x_1 \approx_X x_2) \otimes (x_2 \approx_Y y_2) \leq R(y_1, y_2)$ for any $x_i \in X$, $y_i \in Y$ $(i = 1, 2)$. By $L^{\langle X, \approx_X \rangle \times \langle Y, \approx_Y \rangle}$ we denote the set of all **L**-relations between X and Y compatible w.r.t. $\approx_X$ and $\approx_Y$. Analogously, $A \in L^X$ is compatible w.r.t. $\approx_X$ if $A(x_1) \otimes (x_1 \approx_X x_2) \leq A(x_2)$. Note that $L^X = L^{\langle X, \mathrm{id}_X \rangle}$. An **L**-set $A \in L^{\langle X, \approx \rangle}$ is called an $\approx$-*singleton* if there is some $x_0 \in X$ such that $A(x) = (x_0 \approx x)$ for any $x \in X$. Clearly, an $\approx$-singleton is the least **L**-set A compatible w.r.t. $\approx$ such that $A(x_0) = 1$. For $\mathbf{L} = \mathbf{2}$, singletons coincide with one-elements sets. For an **L**-set A in X and an **L**-equivalence $\approx$ on X we define the **L**-set $\mathrm{C}_{\approx}(A)$ by $\mathrm{C}_{\approx}(A)(x) = \bigvee_{x' \in X} A(x') \otimes (x' \approx x)$. It is easy to see that $\mathrm{C}_{\approx}(A)$ is the smallest (w.r.t. $\subseteq$) **L**-set in X that is compatible with $\approx$ and contains A. For $A \in L^X$ and $a \in L$, the a-cut of A is the set ${}^aA = \{x \in X \mid a \leq A(x)\}$.

Definition 1 *An* **L**-order *on a set* X *with an* **L**-*equality relation* $\approx$ *is a binary* **L**-*relation* $\preceq$ *which is compatible w.r.t.* $\approx$ *and satisfies*

$$\begin{array}{rcll} x \preceq x & = & 1 & \textit{(reflexivity)} \\ (x \preceq y) \wedge (y \preceq x) & \leq & x \approx y & \textit{(antisymmetry)} \\ (x \preceq y) \otimes (y \preceq z) & \leq & x \preceq z & \textit{(transitivity).} \end{array}$$

If $\preceq$ is an **L**-order on a set X with an **L**-equality $\approx$, we call the pair $\mathbf{X} = \langle \langle X, \approx \rangle, \preceq \rangle$ an **L**-ordered set. Clearly, if $\mathbf{L} = \mathbf{2}$, the notion of **L**-order coincides with the usual notion of crisp (partial) order.

We say that **L**-ordered sets $\langle \langle X, \approx_X \rangle, \preceq_X \rangle$ and $\langle \langle Y, \approx_Y \rangle, \preceq_Y \rangle$ are isomorphic if there is a bijective mapping $h : X \to Y$ such that $(x \approx_X x') = (h(x) \approx_Y h(x'))$ and $(x \preceq_X x') = (h(x) \preceq_Y h(x'))$ is true for all $x, x' \in X$.

Example 2 *(1) For any set* $X \neq \emptyset$ *and any subset* $\emptyset \neq M \subseteq L^X$, $\langle \langle M, \approx \rangle, \mathrm{S} \rangle$ *is an* **L**-*ordered set.*

(2) For a residuated lattice **L** *define* $\approx$ *and* $\preceq$ *by* $(x \approx y) := (x \to y) \wedge (y \to x)$ *and* $(x \preceq y) := x \to y$. *Then* $\langle \langle L, \approx \rangle, \preceq \rangle$ *is an* **L**-*ordered set.*

Polarities and fuzzy concept lattices Let X and Y be sets with **L**-equalities $\approx_X$ and $\approx_Y$, respectively; I be an **L**-relation between X and Y which is compatible w.r.t. $\approx_X$ and $\approx_Y$. For $A \in L^X$ and $B \in L^Y$ let $A^{\uparrow} \in L^Y$ and $B^{\downarrow} \in L^Y$ be defined by

$$A^{\uparrow}(y) = \bigwedge_{x \in X} A(x) \to I(x,y) \tag{1}$$

and

$$B^{\downarrow}(x) = \bigwedge_{y \in Y} B(y) \to I(x,y). \tag{2}$$

Clearly, $A^{\uparrow}(y)$ is the truth degree to which "for each x from A, x and y are in I", and similarly for $B^{\downarrow}(x)$; the pair $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ is called an **L**-*polarity* induced by I (and is denoted also by $\langle {}^{\uparrow_I}, {}^{\downarrow_I} \rangle$). The one-to-one relationship between polarities and Galois connections [13] generalizes as follows: An **L**-Galois connection between $\langle X, \approx_X \rangle$ and $\langle Y, \approx_Y \rangle$ is a pair $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ of mappings ${}^{\uparrow} : L^{\langle X, \approx_X \rangle} \to L^{\langle Y, \approx_Y \rangle}$, ${}^{\downarrow} : L^{\langle Y, \approx_Y \rangle} \to L^{\langle X, \approx_X \rangle}$ satisfying

$$\mathrm{S}(A_1, A_2) \leq \mathrm{S}(A_2^{\uparrow}, A_1^{\uparrow}) \tag{3}$$

$$\mathrm{S}(B_1, B_2) \leq \mathrm{S}(B_2^{\uparrow}, B_1^{\uparrow}) \tag{4}$$

$$A \subseteq A^{\uparrow\downarrow} \tag{5}$$

$$B \subseteq B^{\downarrow\uparrow} \tag{6}$$

for any $A, A_1, A_2 \in L^X$, $B, B_1, B_2 \in L^Y$. Now, there is an one-to-one relationship between **L**-polarities and **L**-Galois connections (see [1]). Furthermore, denote $\mathcal{B}(X, Y, I) = \{\langle A, B \rangle \in L^X \times$

$L^Y \mid A^{\uparrow} = B, B^{\downarrow} = A\}$ the set of all fixed points of $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ (which is induced by I). Consider the following interpretation: X is a set of objects, Y is a set of attributes, and I is the relation "to have" (i.e. $I(x, y)$ is the truth degree to which the object x has the attribute y). Then $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$ iff A is the collection of all objects sharing all the attributes from B and, conversely, B is the collection of all attributes shared by all the objects from A; thus $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$ satisfy the verbal definition of a concept that is due to so-called traditional logic (see [3, 4, 6]). The triple $\langle X, Y, I\rangle$ is called an (formal) **L**-context, pairs $\langle A, B\rangle$ are called (formal) **L**-concepts; $\mathcal{B}(X, Y, I)$ is called an **L**-concept lattice (the term lattice will be justified later). We can naturally introduce the following **L**-relations on $\mathcal{B}(X, Y, I)$: put $(\langle A_1, B_1\rangle \approx \langle A_2, B_2\rangle) := (A_1 \approx A_2)$ (equivalently, $:= (B_1 \approx B_2)$); put $(\langle A_1, B_1\rangle \preceq \langle A_2, B_2\rangle) := (A_1 \mathrm{S} A_2)$ (equivalently, $:= (B_2 \mathrm{S} B_1)$). Then $\approx$ is an **L**-equality on $\mathcal{B}(X, Y, I)$ and $\preceq$ is an **L**-order, thus $\langle\langle \mathcal{B}(X, Y, I), \approx\rangle \preceq\rangle$ is an **L**-ordered set (easy by Example 2 (1)).

An **L**-order $\preceq$ on $\langle X, \approx\rangle$ is a binary **L**-relation between $\langle X, \approx\rangle$ and $\langle X, \approx\rangle$. Therefore, $\preceq$ induces an **L**-Galois connection $\langle {}^{\uparrow_\preceq}, {}^{\downarrow_\preceq}\rangle$ between $\langle X, \approx\rangle$ and $\langle X, \approx\rangle$. Clearly, for an **L**-set A in X, $A^{\uparrow_\preceq}$ ($A^{\downarrow_\preceq}$) can be verbally described as the **L**-set of elements which are greater (smaller) than all elements of A. Therefore, we call $A^{\uparrow_\preceq}$ and $A^{\downarrow_\preceq}$ the *upper cone* and the *lower cone* of A, respectively. For $\mathbf{L} = \mathbf{2}$, we get the usual notions of upper and lowe cone. Thus, following the common usage in the theory of ordered sets, we denote $A^{\uparrow_\preceq}$ by $U(A)$ and $A^{\downarrow_\preceq}$ by $\mathcal{L}(A)$, and write $\mathcal{UL}(A)$ instead of $\mathcal{U}(\mathcal{L}(A))$ etc. We now introduce the notion of an infimum and supremum in an **L**-ordered set, and the notion of an completely lattice **L**-ordered set.

Definition 3 *(1) For an* **L**-*ordered set* $\langle\langle X, \approx\rangle, \preceq\rangle$ *and* $A \in L^X$ *we define the* **L**-*sets* $\inf(A)$ *and* $\sup(A)$ *in* X *by*

$$\begin{aligned}(\inf(A))(x) &= (\mathcal{L}(A))(x) \wedge (\mathcal{UL}(A))(x),\\ (\sup(A))(x) &= (\mathcal{U}(A))(x) \wedge (\mathcal{LU}(A))(x).\end{aligned}$$

$\inf(A)$ *and* $\sup(A)$ *are called the* infimum *and* supremum *of* A*, respectively.*

(2) An **L**-*ordered set* $\langle\langle X, \approx\rangle, \preceq\rangle$ *is said to be completely lattice* **L**-*ordered if for any* $A \in L^X$ *both* $\sup(A)$ *and* $\inf(A)$ *are* $\approx$*-singletons.*

Remark (1) The notions of infimum and supremum are generalizations of the classical notions. Indeed, if $\mathbf{L} = \mathbf{2}$, $(\inf(A))(x)$ is the truth value of the fact that x belongs to both the lower cone of A and the upper cone of the lower cone of A, i.e. x is the greatest lower bound of A; similarly for $\sup(A)$.

(2) It can be shown that in an **L**-ordered set, $(\inf(A))(x) = 1$ and $(\inf(A))(y) = 1$ implies $x = y$ (and similarly for $\sup(A)$).

(3) It is easy to see that in a completely lattice **L**-ordered set **X**, supremum $\sup(A)$ of $A \in L^X$ is uniquely determined by the the element $x \in X$ such that $(\sup(A))(x) = 1$ (and similarly for infima). Therefore, we can denote the unique element x for which $(\sup(A))(x) = 1$ directly by $\sup(A)$ (and similarly for $\inf(A)$).

Note that for an **L**-order $\preceq$, ${}^1\!\preceq$ (the 1-cut of $\preceq$, i.e. ${}^1\!\preceq = \{\langle x, y\rangle \in X \times X \mid (x \preceq y) = 1\}$) is a binary relation on X.

Theorem 4 ([3, 4]) *For an* **L**-*ordered set* $\mathbf{X} = \langle\langle X, \approx\rangle, \preceq\rangle$*, the relation* ${}^1\!\preceq$ *is an order on* X*. Moreover, if* **X** *is completely lattice* **L**-*ordered then* ${}^1\!\preceq$ *is a lattice order on* X*.*

Remark (1) Theorem 4 has the following consequence: If **X** is a completely lattice **L**-ordered set, we may speak about the infimum (supremum) of a (crisp) subset A of X w.r.t. ${}^1\!\preceq$.

(2) Note, however, that a completely lattice **L**-ordered set $\mathbf{X} = \langle\langle X, \approx\rangle, \preceq\rangle$ is in general not determined by ${}^1\!\preceq$. Indeed, consider the following example: Let **L** be the Gödel algebra on $[0, 1]$ (i.e. $a \otimes b = \min(a, b)$), let $X = \{x, y\}$. Consider the bivalent order $\leq = \{\langle x, x\rangle, \langle x, y\rangle, \langle y, y\rangle\}$ on X. Then $\mathbf{X}_1 = \langle\langle X_1, \approx_1\rangle, \preceq_1\rangle$ and $\mathbf{X}_2 = \langle\langle X_2, \approx_2\rangle, \preceq_2\rangle$ defined by $(x \preceq_1 x) = 1$, $(x \preceq_1 y) = 1$, $(y \preceq_1 x) = 0.6$, $(y \preceq_1 y) = 1$; $(x \preceq_2 x) = 1$, $(x \preceq_2 y) = 1$, $(y \preceq_2 x) = 0.8$, $(y \preceq_2 y) = 1$; and $(u \approx_i v) = \min((u \preceq_i v), (v \preceq_i u))$ (for $u, v \in X$, $i = 1, 2$), are two different completely lattice **L**-ordered sets such that $\leq$ equals both ${}^1\!\preceq_1$ and ${}^1\!\preceq_2$.

We now recall a theorem characterizing **L**-concept lattices and showing that they are precisley the completely lattice **L**-ordered sets. Recall that for an **L**-set A in U and $a \in L$, $a \otimes A$ and $a \to A$ denote the **L**-sets such that $(a \otimes A)(u) = a \otimes A(u)$ and $(a \to A)(u) = a \to A(u)$, respectively.

If $\mathcal{M}$ is an **L**-set in Y and each $y \in Y$ is an **L**-set in X, we define the **L**-sets $\bigcap \mathcal{M}$ and $\bigcup \mathcal{M}$ in X by

$$(\bigcap \mathcal{M})(x) = \bigwedge_{A \in Y} \mathcal{M}(A) \to A(x)$$

$$(\bigcup \mathcal{M})(x) = \bigvee_{A \in Y} \mathcal{M}(A) \otimes A(x).$$

Clearly, $\bigcap \mathcal{M}$ and $\bigcup \mathcal{M}$ are generalizations of an intersection and a union of a system of sets, respectively. For an **L**-set $\mathcal{M}$ in $\mathcal{B}(X, Y, I)$, we put $\bigcap_X \mathcal{M} = \bigcap \mathrm{pr}_X(\mathcal{M})$, $\bigcup_X \mathcal{M} = \bigcup \mathrm{pr}_X(\mathcal{M})$, $\bigcap_Y \mathcal{M} = \bigcap \mathrm{pr}_Y(\mathcal{M})$, $\bigcup_Y \mathcal{M} = \bigcup \mathrm{pr}_Y(\mathcal{M})$, where $\mathrm{pr}_X(\mathcal{M})$ is an **L**-set in the set $\{A \in L^X \mid A = A^{\uparrow\downarrow}\}$ of all extents of $\mathcal{B}(X,Y,I)$ defined by $(\mathrm{pr}_X\mathcal{M})(A) = \mathcal{M}(A, A^{\uparrow})$ and, similarly, $\mathrm{pr}_Y(\mathcal{M})$ is an **L**-set in the set $\{B \in L^Y \mid B = B^{\uparrow\downarrow}\}$ of all intents of $\mathcal{B}(X,Y,I)$ defined by $(\mathrm{pr}_Y\mathcal{M})(B) = \mathcal{M}(B^{\downarrow}, B)$. Thus, $\bigcap_X \mathcal{M}$ is the "intersection of all extents from $\mathcal{M}$" etc.

Let **X** be a completely lattice **L**-ordered set, $L' \subseteq L$. A subset $K \subseteq X$ is called L'-infimally dense in **X** (L'-supremally dense in **X**) if for each $x \in X$ there is some $A \in L'^X$ such that $A(y) = 0$ for all $y \notin K$ and $(\inf(A))(x) = 1$ $((\sup(A))(x) = 1)$.

The following result shows, among others, that completely lattice fuzzy ordered sets are precisely of the form $\mathcal{B}(X,Y,I)$.

Theorem 5 *Let $\langle X,Y,I\rangle$ be an* **L***-context. (1) $\langle\langle\mathcal{B}(X,Y,I),\approx\rangle,\preceq\rangle$ is completely lattice* **L***-ordered set in which infima and suprema can be described as follows: for an* **L***-set $\mathcal{M}$ in $\mathcal{B}(X,Y,I)$ we have*

$$^1\inf(\mathcal{M}) = \{\left\langle \bigcap_X \mathcal{M}, (\bigcap_X \mathcal{M})^{\uparrow} \right\rangle\} \quad (7)$$

$$^1\sup(\mathcal{M}) = \{\left\langle (\bigcap_Y \mathcal{M})^{\downarrow}, \bigcap_Y \mathcal{M} \right\rangle\} \quad (8)$$

(2) Moreover, a completely lattice **L***-ordered set* $\mathbf{V} = \langle\langle V,\approx\rangle,\preceq\rangle$ *is isomorphic to* $\langle\langle\mathcal{B}(X,Y,I),\approx\rangle,\preceq\rangle$ *iff there are mappings* $\gamma : X \times L \to V$, $\mu : Y \times L \to V$, *such that* $\gamma(X \times L)$ *is* $\{0,1\}$*-supremally dense in* **V**, $\mu(Y \times L)$ *is* $\{0,1\}$*-infimally dense in* **V**, *and* $((a \otimes b) \to I(x,y)) = (\gamma(x,a) \preceq \mu(y,b))$ *for all* $x \in X$, $y \in Y$, $a,b \in L$. *In particular,* **V** *is isomorphic to* $\mathcal{B}(V,V,\preceq)$.

Theorem 4 implies that $\mathcal{B}(X,Y,I)$, equipped with $^1\preceq$, is a complete lattice. The lattice structure of $\mathcal{B}(X,Y,I)$ is characterized in [4].

A natural question arises of whether any **L**-ordered set can be "embedded" into a completely lattice **L**-ordered set. The answer is positive; we have a generalization of the well-known Dedekind-MacNeille completion [12] to fuzzy setting: Let $\mathbf{X} = \langle\langle X,\approx_X\rangle,\preceq_X\rangle$ and $\mathbf{Y} = \langle\langle Y,\approx_Y\rangle,\preceq_Y\rangle$ be **L**-ordered sets. A mapping $g : X \to Y$ is called an embedding of **X** into **Y** if g is injective, $(x \preceq_X x') = (g(x) \preceq_Y g(x'))$, and $(x \approx_X x') = (g(x) \approx_Y g(x'))$ for every $x, x' \in X$. Therefore, the image of X under g is a "copy" of **X**. We say that an embedding $g : X \to Y$ preserves infima if for any $M \in L^X$ and $x \in X$ we have $(\mathcal{L}(M))(x) = (\mathcal{L}(g(M)))(g(x))$ and $(\mathcal{UL}(M))(x) = (\mathcal{UL}(g(M)))(g(x))$ (the case for suprema is dual) where $g(M) \in L^Y$ is defined by $(g(M))(y) = M(x)$ if $y = g(x)$ and $(g(M))(y) = 0$ otherwise. Clearly, the preservation of infima (suprema) implies that $(\inf(M))(x) = \inf(g(M))(g(x))$ $((\sup(A))(x) = \sup(g(A))(g(x)))$. For an **L**-ordered set **X** and $x \in X$ we put $(x] := \mathcal{L}(\{ {}^1/x\})$ and $[x) := \mathcal{U}(\{ {}^1/x\})$. Therefore, $((x])(y) = (y \preceq x)$ and $([x))(y) = (x \preceq y)$ for each $y \in X$.

Theorem 6 ([3, 4]) *Let* **X** *be an* **L***-ordered set. Then $g : x \mapsto \langle (x], [x)\rangle$ is an embedding of* **X** *into a completely* **L***-ordered set $\mathcal{B}(X,X,\preceq)$ which preserves infima and suprema. Moreover, if f is an embedding of* **X** *into a completely lattice* **L***-ordered set* **Y** *which preserves infima and suprema then there is an embedding h of $\mathcal{B}(X,X,\preceq)$ into* **Y** *such that $f = g \circ h$.*

3. Closure operators

Recall that a closure operator in a partially ordered set $\langle V,\leq\rangle$ is a mapping $c : V \to V$ satisfying (i) $v \leq c(v)$; (ii) $v_1 \leq v_2$ implies $c(v_1) \leq c(v_2)$; (iii) $c(v) = c(c(v))$ for all $v, v_1, v_2 \in V$. For $V = 2^X$ and $\leq$ being the subsethood relation, we get the notion of a closure operator in a set. Closure operatorsin a set have been studied also from the point of view of fuzzy logic (see e.g.[8] or [2]) We now introduce the notion of a closure operator in an **L**-ordered set.

Definition 7 *Let $\mathbf{V} = \langle\langle V,\approx\rangle,\leq\rangle$ be an* **L***-ordered set, K be a $\leq$-filter in* **L**. *An* $\mathbf{L}_K$-closure operator *in* **V** *is a mapping $c : V \to V$ satisfying*

(i) $(v \preceq c(v)) = 1$,

(ii) $(v_1 \preceq v_2) \leq (c(v_1) \preceq c(v_2))$ *whenever* $(v_1 \preceq v_2) \in K$,

(iii) $c(v) = c(c(v))$

for any $v, v_1, v_2 \in V$.

The role of K is to control the sensitivity of c w.r.t. the graded order.

Two extreme cases are important, $K = L$ and $K = \{1\}$. For $K = L$ we omit the subscript and write only **L**-closure operator instead of $\mathbf{L}_K$-closure operator.

Lemma 8 *Let c be an $\mathbf{L}_K$-closure operator in an* **L***-ordered set $\mathbf{V} = \langle\langle V,\approx\rangle,\leq\rangle$. Then c is a closure operator in a partially ordered set $\langle V, {}^1\preceq\rangle$.*

Proof. Immediate. □

For an $\mathbf{L}_K$-closure operator in $\mathbf{V} = \langle\langle V, \approx\rangle, \preceq\rangle$ denote $\mathcal{S}_c = \{v \in V \mid v = c(v)\}$ the set of all fixed points of c. Equipping $\mathcal{S}_c$ with the restrictions of $\approx$ and $\preceq$ (for simplicity, we denote the restrictions again by $\approx$ and $\preceq$), we get an $\mathbf{L}$-ordered set $\langle\langle \mathcal{S}_c, \approx\rangle, \preceq\rangle$.

Example 9 *(1) Put $V = L^X$, let $\approx$ be the graded equality of $\mathbf{L}$-sets in X, let $\preceq$ be the graded subsethood* S*. Then $\mathbf{L}_K$-closure operators in $\langle\langle V, \approx\rangle, \preceq\rangle$ are exactly the $\mathbf{L}_K$-closure operators in a set as introduced and studied in [2, 4] (where also other examples can be found). Particularly, for $V = L^X$ and $K = \{1\}$ we get what is usually known as fuzzy closure operators in a set [8, 14].*

(2) Let $\langle X, Y, I\rangle$ be an $\mathbf{L}$-context, $\langle^{\uparrow}, ^{\downarrow}\rangle$ be the pair of induced polarities. Then $^{\uparrow\downarrow} : L^X \to L^X$ and $^{\downarrow\uparrow} : L^Y \to L^Y$ are $\mathbf{L}$-closure operators in $\langle\langle L^X, \approx\rangle, \mathrm{S}\rangle$ and $\langle\langle L^Y, \approx\rangle, \mathrm{S}\rangle$, respectively. It is immediate that $\mathcal{S}_{\uparrow\downarrow}$, $\mathcal{S}_{\downarrow\uparrow}$, and $\mathcal{B}(X, Y, I)$ equipped with the respective $\mathbf{L}$-relations are pairwise isomorphic $\mathbf{L}$-ordered sets. Thus, any $\mathbf{L}$-concept lattice can be considered as the set of all fixed points of an $\mathbf{L}$-closure operator in an $\mathbf{L}$-ordered set.

The following lemma provides a single condition that replaces (ii) and (iii).

Lemma 10 *c is an $\mathbf{L}_K$-closure operator iff c satisfies (i) and the following condition: (iv) $(v_1 \preceq c(v_2)) \leq (c(v_1) \preceq c(v_2))$ whenever $(v_1, c(v_2)) \in K$.*

Proof. Let c be an $\mathbf{L}_K$-closure operator, let $(v_1 \preceq c(v_2)) \in K$. We have $(v_1 \preceq c(v_2)) \leq (c(v_1) \preceq c(c(v_2))) = (c(v_1) \preceq c(v_2))$, i.e. (iv) is true. Conversely, assume (i) and (iv). Suppose $(v_1 \preceq v_2) \in K$. Since $(v_2 \preceq c(v_2)) = 1$, we have $(v_1 \preceq c(v_2)) = (v_1 \preceq v_2) \otimes (v_2 \preceq c(v_2)) \leq (v_1 \preceq c(v_2))$, thus $(v_1 \preceq c(v_2)) \in K$ (K is an $\leq$-filter). (iv) now yields $(v_1 \preceq c(v_2)) \leq (c(v_1) \preceq c(v_2))$, thus $(v_1 \preceq v_2) \leq (c(v_1) \preceq c(v_2))$ proving (ii). Finally, $1 = (c(v) \preceq c(v)) \leq (c(c(v)) \preceq c(v))$, and thus $c(v) = c(c(v))$ proving (iii). □

Then next lemma shows that $\mathbf{L}_K$-closure operators "preserve equality."

Lemma 11 *Let c be an $\mathbf{L}_K$-closure operator on an $\mathbf{L}$-ordered set $\mathbf{V} = \langle\langle V, \approx\rangle, \preceq\rangle$. Then for any $u, v \in V$ we have $(u \approx v) \leq (c(u) \approx c(v))$ whenever $(u \approx v) \in K$.*

Proof. By [3, 4], $(u \approx v) = (u \preceq v) \wedge (v \preceq u)$. Therefore, $(u \approx v) \in K$ yields $(u \preceq v) \in K$ and $(v \preceq u) \in K$. By (i) we get $(u \preceq v) \leq (c(u) \preceq c(v))$ and $(v \preceq u) \leq (c(v) \preceq c(u))$, whence $(u \approx v) = (u \preceq v) \wedge (v \preceq u) \leq (c(u) \preceq c(v)) \wedge (c(v) \preceq c(u)) = (c(u) \approx c(v))$, proving the assertion. □

Note that for $K = L$ the foregoing lemma states that $(u \approx v) \leq (c(u) \approx c(v))$; this can be read as "if u and v are equal then $c(u)$ and $c(v)$ are equal."

Next we observe that, as in the classical case, ($\mathbf{L}_K$-)closure operators are induced by ($\mathbf{L}_K$-)Galois connections.

Definition 12 *Let $\mathbf{U} = \langle\langle U, \approx\rangle, \leq\rangle$, $\mathbf{V} = \langle\langle V, \approx\rangle, \leq\rangle$ be $\mathbf{L}$-ordered sets, K be a $\leq$-filter in $\mathbf{L}$. An $\mathbf{L}_K$-Galois connection between $\mathbf{U}$ and $\mathbf{V}$ is a pair $\langle^{\uparrow}, ^{\downarrow}\rangle$ of mappings $^{\uparrow} : U \to V$, $^{\uparrow} : V \to U$ satisfying (i) $(u_1 \preceq u_2) \leq (u_1^{\uparrow} \preceq u_2^{\uparrow})$ whenever $(u_1 \preceq u_2) \in K$; (ii) $(v_1 \preceq v_2) \leq (v_1^{\downarrow} \preceq v_2^{\downarrow})$ whenever $(v_1 \preceq v_2) \in K$; (iii) $(u \preceq u^{\uparrow\downarrow}) = 1$, $(v \preceq u^{\downarrow\uparrow}) = 1$ for any $u, u_i \in U$, $v, v_i \in V$ ($i = 1, 2$).*

Lemma 13 *Let $\langle^{\uparrow}, ^{\downarrow}\rangle$ be an $\mathbf{L}_K$-Galois connection between $\mathbf{U} = \langle\langle U, \approx\rangle, \leq\rangle$ and $\mathbf{V} = \langle\langle V, \approx\rangle, \leq\rangle$. Then the composed mappings $^{\uparrow\downarrow}$ and $^{\downarrow\uparrow}$ are $\mathbf{L}_K$-closure operators on $\mathbf{U}$ and $\mathbf{V}$, respectively.*

Proof. An easy exercise. □

Example 14 *Let $\mathbf{L}$ be a complete residuated lattice, take any $a \in L$. Define $\varphi : L \to L$ by $\varphi(x) := x \to a$. Then the pair $\langle\varphi, \varphi\rangle$ forms an $\mathbf{L}$-Galois connection between $\langle\langle L, \leftrightarrow\rangle, \to\rangle$ and $\langle\langle L, \leftrightarrow\rangle, \to\rangle$. Therefore, $\varphi \circ \varphi : x \mapsto (x \to a) \to a$ is an $\mathbf{L}$-closure operator in $\langle\langle L, \leftrightarrow\rangle, \to\rangle$.*

Our aim in the following is to concentrate on $\mathbf{L}$-closure operators and on the sets of their fixed points. In the rest of the paper, we denote by $\mathbf{V} = \langle\langle V, \approx\rangle, \preceq\rangle$ an arbitrary completely lattice $\mathbf{L}$-ordered set. We say that a subset $U \subseteq V$ is closed w.r.t. arbitrary infima if for any $A \in L^U$ we have that $\inf(A) \in U$ (that is, the unique element $x \in V$ for which $\inf(A)(x) = 1$ belongs to U). We have the following result.

Lemma 15 *For an $\mathbf{L}$-closure operator c on a completely lattice $\mathbf{L}$-ordered set $\mathbf{V}$, $\mathcal{S}_c$ is closed w.r.t. arbitrary infima.*

Proof. The proof is a tedious (but straightforward) verification of the fact that for any $\mathbf{L}$-set A in $\mathcal{S}_c$, $\inf(A)$ belongs to $\mathcal{S}_c$; the proof will be included in the full version of this paper. □

Let now $U \subseteq V$ and define a mapping $c : V \to V$ by

$$c_U(v) = \inf(A)$$

where $A(u) := (v \preceq u)$.

Remark It can be verified that the above definition of c_U is equivalento to putting $c_U(v) = \inf\{u \in V \mid (v \preceq u) = 1\}$.

Lemma 16 *If* $\mathbf{V}$ *is a completely lattice* $\mathbf{L}$*-ordered set and* $\mathcal{S} \subseteq V$ *is closed w.r.t. arbitrary infima then* $c_{\mathcal{S}}$ *is an* $\mathbf{L}$*-closure operator on* $\mathbf{V}$*.*

Proof. The proof is tedious (but straightforward) verification of conditions (i)–(iii) and will be omitted because of the limited scope (it will be included in the full version of this paper). □

Theorem 17 *Let* $\mathbf{V}$ *be a completely lattice* $\mathbf{L}$*-ordered set, c be an* $\mathbf{L}$*-closure operator on* $\mathbf{V}$*,* $\mathcal{S} \subseteq V$ *be closed w.r.t. arbitrary infima. Then (1)* $\mathcal{S}_c$ *is closed w.r.t. arbitrary infima; (2)* $c_{\mathcal{S}}$ *is an* $\mathbf{L}$*-closure operator on* $\mathbf{V}$*; (3)* $c = c_{\mathcal{S}_c}$ *and* $\mathcal{S} = \mathcal{S}_{c_{\mathcal{S}}}$*.*

Proof. By Lemma 15 and Lemma 16, it is sufficient to check (3). This is straightforward but tedious; we omit the proof (it will be presented in the full version of the paper). □

Remark In [2, 4], fuzzy closure operators in a set X (i.e. mappings $C : L^X \to L^X$) are studied. A one-to-one correspondence between fuzzy closure operators and special subsets of L^X is shown. Namely, it is shown that **L**-closure operators in X (which are, in our terms, exactly the **L**-closure operators in $\langle\langle L^X, \approx\rangle, \mathrm{S}\rangle$) correspond in a one-to-one way to subsets $\mathcal{S}$ of L^X which are closed w.r.t. to arbitrary intersections and to L-shifts (i.e. for any $A \in \mathcal{S}$ we have $a \to A \in \mathcal{S}$). Applying Theorem 17 we get that a subset $\mathcal{S} \subseteq L^X$ is closed w.r.t. arbitrary infima (i.e. infima of **L**-sets in $\mathcal{S}$) iff it is closed under arbitrary intersections and L-shifts.

4. Acknowledgment

Supported by grant no. 201/99/P060 of the GA ČR.

References

[1] Bělohlávek R.: Fuzzy Galois connections. *Math. Logic Quarterly* **45**,4 (1999), 497–504.

[2] Bělohlávek R.: Fuzzy closure operators. *J. Math. Anal. Appl.* (to appear).

[3] Bělohlávek R.: Concept lattices and order in fuzzy logic (submitted, 2000).

[4] Bělohlávek R.: *Fuzzy Concept Lattices and Related Structures.* Habilitation thesis, University of Ostrava, 2000, ix+130 pp.

[5] Birkhoff G.: *Lattice Theory, 3-rd edition.* AMS Coll. Publ. 25, Providence, R.I., 1967.

[6] Ganter B., Wille R.: *Formal concept analysis. Mathematical Foundations.* Springer-Verlag, Berlin, 1999.

[7] Goguen J. A.: L-fuzzy sets. *J. Math. Anal. Appl.* **18**(1967), 145–174.

[8] Gerla G.: *Fuzzy Logic. Mathematical Tools for Approximate Reasoning.* Kluwer (to appear).

[9] Hájek P.: *Metamathematics of Fuzzy Logic.* Kluwer, Dordrecht, 1998.

[10] Höhle U.: On the fundamentals of fuzzy set theory. *J. Math. Anal. Appl.* **201**(1996), 786–826.

[11] Klir G. J., Yuan B.: *Fuzzy Sets and Fuzzy Logic. Theory and Applications.* Prentice Hall, 1995.

[12] MacNeille H. M.: Partially ordered sets. *Trans. Amer. Math. Soc.* **42**(1937), 416–460.

[13] Ore O.: Galois connections. *Trans. Amer. Math. Soc.* **55**(1944), 493–513.

[14] Pavelka J.: On fuzzy logic I, II, III. *Zeit. Math. Log. Grungl. Math.* **25**(1979), 45–52, 119–134, 447–464.

[15] Schröder E.: Algebra der Logik I, II, III.

[16] Wille R.: Restructuring lattice theory: an approach based on hierarchies of concepts. In: Rival I.: *Ordered Sets.* Reidel, Dordrecht, Boston, 1982, 445—470.

[17] Zadeh L. A.: Fuzzy sets. *Inf. Control* **8**(3)(1965), 338–353.

[18] Zadeh L. A.: Similarity relations and fuzzy orderings. *Information Sciences* **3**(1971), 159–176.