Shortest-weight paths in random regular graphs

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Abstract

Consider a random regular graph with degree d and of size n. Assign to each edge an i.i.d. exponential random variable with mean one. In this paper we establish a precise asymptotic expression for the maximum number of edges on the shortest-weight paths between a fixed vertex and all the other vertices, as well as between any pair of vertices. Namely, for any fixed $d \geq 3$, we show that the longest of these shortest-weight paths has about $\widehat{\alpha} \log n$ edges where $\widehat{\alpha}$ is the unique solution of the equation $\alpha \log\left(\frac{d-2}{d-1}\alpha\right) - \alpha = \frac{d-3}{d-2}$, for $\alpha > \frac{d-1}{d-2}$.

1 Introduction

The focus of this paper is on first passage percolation on a random regular graph, namely on $G \sim \mathcal{G}(n,d)$, a graph uniformly distributed over the set of all graphs on n vertices $[n] := \{1,\dots,n\}$, in which every vertex has degree d, for $d \geq 3$ and n large. We assume that each edge in this graph has an i.i.d. exponential weight with mean one. We consider the shortest-weight paths between any pair of vertices of this graph, and establish that the longest of these shortest-weight paths has about $\widehat{\alpha} \log n$ edges for some positive constant $\widehat{\alpha}$ depending on d that we will shortly define. We also derive a similar precise asymptotic expression for the maximum number of edges on the shortest-weight paths between a fixed vertex and all the other vertices, see Theorem 1 for the exact statement.

Let $G = (V,E,w)$ be a weighted graph, defined as the data of a graph $G = (V,E)$ and a collection of weights $w = \{w_e\}_{e \in E}$ associated to each edge $e \in E$. For two vertices $a, b \in V$, the weighted distance between a and b is given by

$$\mathrm{dist}_w(a,b) = \min_{\pi \in \Pi(a,b)} \sum_{e \in \pi} w_e \,,$$

where the minimum is taken over the set $\Pi(a,b)$ of all paths between a and b in the graph. For $a, b \in V$ we denote by $\pi(a,b)$ the shortest-weight path between a and b.

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We define the function $f : \mathbb{R}^+ \to \mathbb{R}$ as follows

$$f(\alpha) := \alpha \log\left(\frac{d-2}{d-1}\alpha\right) - \alpha + \frac{1}{d-2}. \tag{1}$$

Note that $f'(\alpha) = \log\left(\frac{d-2}{d-1}\alpha\right)$ is positive for $\alpha > \frac{d-1}{d-2}$, and $f\left(\frac{d-1}{d-2}\right) = -1$.

We let α^* and $\widehat{\alpha}$ be respectively the unique solutions to $f(\alpha) = 0$ and $f(\alpha) = 1$ for $\alpha > \frac{d-1}{d-2}$.

The main result of this paper is the following theorem.

Theorem 1. *Fix $d \geq 3$ and let $G \sim \mathcal{G}(n, d)$ be a weighted random d-regular graph with n vertices and i.i.d. rate one exponential variables on its edges. Then, as $n \to \infty$, we have*

$$\frac{\max_{j\in[n]} |\pi(1, j)|}{\log n} \xrightarrow{p} \alpha^*, \tag{2}$$

and

$$\frac{\max_{i,j\in[n]} |\pi(i, j)|}{\log n} \xrightarrow{p} \widehat{\alpha}, \tag{3}$$

where $\xrightarrow{p}$ denotes the convergence in probability.

In order to compare our result with the existing ones, we reproduce here a result of Bhamidi, van der Hofstad and Hooghiemstra [8] concerning the number of edges in the shortest-weight path between two uniformly chosen nodes (as well as the weighted distance); see also [23] for the joint distribution of (weighted) distances in random regular graphs. Remark that, the following theorem is stated in [8] in a more general setting (random graphs with i.i.d. degrees).

Theorem 2 (Bhamidi, van der Hofstad and Hooghiemstra [8])**.** *Fix $d \geq 3$ and let $G \sim \mathcal{G}(n, d)$ be a random d-regular graph with n vertices and i.i.d. rate one exponential variables on its edges. Then, as $n \to \infty$,*

$$\frac{|\pi(1, 2)| - \gamma \log n}{\sqrt{\gamma \log n}} \xrightarrow{d} Z, \tag{4}$$

where Z has a standard normal distribution, $\gamma = \frac{d-1}{d-2}$, and $\xrightarrow{d}$ denotes the convergence in distribution. Furthermore, there exists a non-degenerate random variable W such that

$$\mathrm{dist}_w(1, 2) - \frac{1}{d-2}\log n \xrightarrow{d} W. \tag{5}$$

By the above theorem, the ratio of the length and the weight along a shortest-weight path between two (uniformly chosen) nodes is asymptotically $d - 1$ while this ratio for a minimum length path between two nodes is asymptotically 1. Our proof of Theorem 1 (see Section 4) implies that, there exists with high probability (that is, with probability tending to 1 as $n \to \infty$) shortest-weight paths of length about $\widehat{\alpha} \log n$ whose total weight is about $\frac{1}{d-2} \log n$ (typical weighted distance between two uniformly chosen nodes). This means that, for these paths, the ratio of the length and the weight is even larger, i.e., asymptotically $(d-2)\widehat{\alpha}$!

For completeness, we also include results of Ding, Kim, Lubetzky and Peres [13] concerning the weighted diameter in random regular graphs; see also [4] for a generalization.

Theorem 3 (Ding, Kim, Lubetzky and Peres [13]). *Fix $d \geq 3$ and let $G \sim \mathcal{G}(n,d)$ be a random d-regular graph with n vertices and i.i.d. rate one exponential variables on its edges. Then, as $n \to \infty$, we have*

$$\frac{\max_{j\in[n]} \mathrm{dist}_w(1,j)}{\log n} \xrightarrow{p} \frac{1}{d-2} + \frac{1}{d}, \tag{6}$$

and

$$\frac{\max_{i,j\in[n]} \mathrm{dist}_w(i,j)}{\log n} \xrightarrow{p} \frac{1}{d-2} + \frac{2}{d}. \tag{7}$$

In particular, the result of [13] implies that there exists with high probability shortest-weight paths of length about $\frac{d-1}{d-2}\log n$ (the same as the length between two uniformly chosen nodes, see Theorem 2) whose total weight is about $\left(\frac{1}{d-2} + \frac{2}{d}\right)\log n$. This result is used in [3] to analyze an asynchronous randomized broadcast algorithm for random regular graphs.

Related work. First passage percolation model has been mainly studied on lattices motivated by its subadditive property and its link to a number of other stochastic processes, see e.g., [17, 20, 18] for a more detailed discussion. First passage percolation with exponential weights has received substantial attention, in particular on the complete graph [16, 19, 2, 1, 12, 22], and more recently on random graphs [7, 8, 9, 13, 4, 5]. In particular, Janson [19] considered the case of the complete graph with fairly general i.i.d. weights on edges, including the exponential distribution with parameter one. It is shown that, when n goes to infinity, the asymptotic distance for two given points is $\log n/n$, that the maximum distance if one point is fixed and the other varies is $2\log n/n$, and the maximum distance over all pairs of points is $3\log n/n$. He also derives asymptotic results for the corresponding number of hops or hopcount (the number of edges on the paths with the smallest weight). It is shown that (when n goes to infinity) the number of hops is $\log n$ for two given nodes, and the maximum hops if one point is fixed and the other varies is $e\log n$. More recently, Addario-Berry, Broutin and Lugosi [1] showed that the longest of these shortest-weight paths in a complete graph has about $\widetilde{\alpha}\log n$ edges where $\widetilde{\alpha} \sim 3.5911$ is the unique solution of the equation $\alpha\log(\alpha) - \alpha = 1$, which answered a question posed by Janson [19]. Note that $\alpha^* \to e$ and $\widehat{\alpha} \to \widetilde{\alpha}$ as $d \to \infty$.

Organization of the paper. The remainder of the paper is organized as follows. In the next section we provide several preliminary facts on random regular graphs. We also consider in this section the exploration process for configuration model which consists in growing balls (neighborhoods) simultaneously from each vertex. In addition, the section provides some necessary notations and definitions that will be used throughout the paper. Sections 3 and 4 form the heart of the proof. We first prove that the above bound is an upper bound in Sections 3. The final section provides the corresponding lower bound using the second moment method, applied to a suitably defined set of shortest paths with special properties that make them amenable to analysis.

Basic notations. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued random variables on a sequence of probability spaces $\{(\Omega_n, \mathbb{P}_n)\}_{n\in\mathbb{N}}$. If $c \in \mathbb{R}$ is a constant, we write $X_n \xrightarrow{p} c$ to denote that X_n *converges in probability to* c. That is, for any $\varepsilon > 0$, we have $\mathbb{P}_n(|X_n - c| > \varepsilon) \to 0$ as $n \to \infty$. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers that tends to infinity as $n \to \infty$. We write $X_n = o_p(a_n)$ if $|X_n|/a_n$ *converges to 0 in probability.* Additionally, we write $X_n = O_p(a_n)$ to denote that for any positive-valued function $\omega(n) \to \infty$, as $n \to \infty$, we have $\mathbb{P}(|X_n|/a_n \geq \omega(n)) = o(1)$. If $\mathcal{E}_n$ is a measurable subset of Ω_n, for any $n \in \mathbb{N}$, we say that the sequence $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ occurs *with high probability (w.h.p.)* if $\mathbb{P}(\mathcal{E}_n) = 1 - o(1)$, as $n \to \infty$.

The notation $\mathsf{Bin}(k, p)$ denotes a binomially distributed random variable corresponding to the number of successes of a sequence of k independent Bernoulli trials each having probability of success equal to p.

We recall here that for two real-valued random variables A and B, we say A is stochastically dominated by B and write $A \leq_{st} B$ if for all x, we have $\mathbb{P}(A \geq x) \leq \mathbb{P}(B \geq x)$. If C is another random variable, we write $A \leq_{st} (B \,|\, C)$ if for all x, $\mathbb{P}(A \geq x) \leq \mathbb{P}(B \geq x \,|\, C)$ almost surely.

2 Preliminaries

2.1 Configuration model

We recall first the setup of the *configuration model* (CM), as introduced by Bender and Canfield[6] and Bollobás [10]. To construct a graph using this method, to each of the n (even) vertices allocate d distinct half-edges, and select a uniform perfect matching on these points. When a half-edge of i is paired with a half-edge of j, we interpret this as an edge between i and j.

The random graph obtained following this procedure may not be simple, i.e., may contain self-loops due to the pairing of two half-edges of i, and multi-edges due to the existence of more than one pairing between two given nodes. Conditional on the event that the graph produced is simple, it is uniformly distributed over the set of all d-regular graphs on n vertices. The probability of this event is uniformly bounded away from zero, equivalent to $(1 + o(1)) \exp\left(\frac{1-d^2}{4}\right)$ as n tends to infinity [24]. Hence, any event that holds w.h.p. for the graph obtained via the configuration model also holds w.h.p. for $G \sim \mathcal{G}(n, d)$.

Note that the assumption $d \geq 3$ implies that $G \sim \mathcal{G}(n, d)$ is connected with high probability [10, 24]. We will assume this in what follows.

The advantage of using the configuration model is that it allows one to construct the graph gradually, exposing the edges of the perfect matching one at a time. This way, each additional edge is uniformly distributed among all possible edges on the remaining (unmatched) half-edges.

2.2 Neighborhoods and tree excess

For $a, b \in V$, let $\mathrm{dist}(a,b) = \mathrm{dist}_G(a,b)$ denote the typical distance between a and b. For a vertex $a \in V$ and an integer number m, the m-step neighborhood of a, denoted by $B(a,m)$ and its boundary $\partial B(a,m)$, are defined as

$$B(a,m) := \{v \in V \mid \mathrm{dist}(a,v) \le m\}, \text{ and, } \partial B(a,m) := B(a,m) \backslash B(a,m-1). \tag{8}$$

For a vertex $a \in V$ and a real number $t > 0$, the t-radius neighborhood of a in the weighted graph, or the ball of radius t centered at a, is defined as

$$B_w(a,t) := \left\{ b, \ \mathrm{dist}_w(a,b) \le t \right\}.$$

The first time t where the ball $B_w(a,t)$ reaches size $k+1 \ge 1$ will be denoted by $T_k(a)$, i.e.,

$$T_k(a) = \min \left\{ t : |B_w(a,t)| \ge k+1 \right\}, \qquad T_0(a) = 0.$$

Note that there is a vertex in $B_w(a, T_k(a))$ which is not in any ball of smaller radius around a. When the weights are i.i.d. according to a random variable with continuous density, this vertex is in addition unique with probability one. We will assume this in what follows. Let $v_k(a)$ denote this node. Furthermore, let $H_k(a)$ denote the number of edges (hopcounts) in the shortest path between the node a and $v_k(a)$, i.e., the generation of $v_k(a)$.

For a connected graph F, the tree excess of F is denoted by $tx(F)$, which is the maximum number of edges that can be deleted from F while still keeping it connected. By an abuse of notation, for a subset $W \subseteq V$, we denote by $tx(W)$ the tree excess of the induced subgraph $G[W]$ of G on W. (If $G[W]$ is not connected, then $tx(W) := \infty$.)

We need the following lemma which demonstrates the well known locally tree-like properties of $G \sim G(n,d)$ for $d \ge 3$.

Lemma 4. *Let $G \sim G(n,d)$ for some fixed $d \ge 3$, and let $m = \lfloor \frac{1}{5} \log_{d-1} n \rfloor$. Then w.h.p.,* $\mathrm{tx}(B(u,m)) \le 1$ *for all $u \in V(G)$.*

Proof. See [21, Lemma 2.1] . □

Consider now the growing balls $B_w(a, T_k(a))$ for $0 \le k \le n-1$ centered at a and let $X_k(a)$ be the tree excess of $B_w(a, T_k(a))$, i.e.,

$$X_k(a) := tx\,(\,B_w(a, T_k(a))\,).$$

The number of edges crossing the boundary of the ball $B_w(a, T_k(a))$ is denoted by $S_k(a)$. A simple calculation shows that (for $G \sim \mathcal{G}(n,d)$)

$$S_k(a) = d + (d-2)k - 2X_k(a). \tag{9}$$

2.3 Shortest-weight paths on a tree

Assume we have positive integers $d_1, d_2, \ldots$. We consider the following construction of a branching process (with these degrees) in discrete time:

- At time 0, start with one alive vertex (the root);
- At each time step k, pick one of the alive vertices at random, this vertex dies giving birth to d_k children.

This type of random tree is known as (random) increasing trees which have been well-studied, see e.g. [11, 14, 15]. We will need the following basic result, the proof of which is easy and can be found for example in [8, Proposition 4.2]. Let $s_k := d_1 + \ldots + d_k - (k-1)$.

Lemma 5. *Pick an alive vertex at time $k \geq 1$ uniformly at random among all vertices alive at this time. Then, the generation of the k-th chosen vertex is equal in distribution to*

$$G_k \stackrel{d}{=} \sum_{i=1}^{k} I_i,$$

where $\{I_i\}_{i=1}^{\infty}$ are independent Bernoulli random variables with parameter

$$\mathbb{P}(I_i = 1) = \frac{d_i}{s_i}.$$

In what follows, instead of taking a graph at random and then analyzing the balls, we use a standard coupling argument in random graph theory which allows to build the balls and the graph at the same time. Fix two vertices, say u and v. We grow the balls around these vertices simultaneously at rate 1, so that at time t, $B_w(u,t)$ and $B_w(v,t)$ are the constructed balls from u and v. When these two balls intersect via the formation of an edge (u_v^*, v_u^*) between two vertices $u_v^* \in B_w(u,.)$ and $v_u^* \in B_w(v,.)$, then the shortest-weight path between the two vertices has been found. Furthermore, we have

$$|\pi(u,v)| = |\pi(u,u_v^*)| + |\pi(v,v_u^*)| + 1.$$

2.4 The exploration process

Fix a vertex a, and consider the following continuous-time exploration process. At time $t = 0$, we have a neighborhood consisting only of a, and for $t > 0$, the neighborhood is precisely $B_w(a,t)$. We now give an equivalent description of this process.

- Start with $B = \{a\}$, where a has d half-edges. For each half edge, decide (at random depending on the previous choices) if the half-edge is matched to a half-edge adjacent to a or not. Reveal the matchings consisting of those half-edges adjacent to a which are connected amongst themselves (creating self-loops at a) and assign weights independently at random to these edges. The remaining unmatched half-edges adjacent to a are stored in a list L. (See the next step including a more precise description of this first step.)

- Repeat the following exploration step as long as the list L is not empty.

  Given there are $\ell \geq 1$ half-edges in the current list, say $L = (h_1, \dots, h_\ell)$, let $\Psi \sim \mathrm{Exp}(\ell)$ be an exponential variable with mean ℓ^{-1}. After time Ψ select a half-edge from L uniformly at random, say h_i. Remove h_i from L and match it to a uniformly chosen half-edge in the entire graph excluding L, say h. Add the new vertex (connected to h) to B and reveal the matchings (and weights) of any of its half-edges whose matched half-edge is also in B. More precisely, let $2x$ be the number of already matched half-edges in B (including the matched half-edges h_i and h). There is a total of $dn - 2x$ unmatched half-edges. Consider one of the $d-1$ half-edges of the new vertex (excluding h which is connected to h_i); with probability $(\ell-1)/(dn-2x-1)$ it is matched with a half-edge in L and with the complementary probability it is matched with an unmatched half-edge outside L. In the first case, match it to a uniformly chosen half-edge of L and remove the corresponding half-edge from L. In the second case, add it to L. We proceed in the similar manner for all the $d-1$ half-edges of the new vertex.

To verify the validity of the above process, let $B_t(a)$ and $L(a,t)$ be respectively the set of vertices and the list generated by the above procedure at time t, where a is the initial vertex. Considering the usual configuration model and using the memoryless property of the exponential distribution, we have $B_w(a,t) = B_t(a)$ for all t. To see this, we can continuously grow the weights of the half-edges $h_1, \dots, h_\ell$ in L until one of their rate 1 exponential clocks fire. Since the minimum of ℓ i.i.d exponential variables with rate 1 is exponential with rate ℓ, this is the same as choosing uniformly a half-edge h_i after time Ψ (recall that by our conditioning, these ℓ half-edges do not pair within themselves). Note that the final weight of an edge is accumulated between the time of arrival of its first half-edge and the time of its pairing (except edges going back into B whose weights are revealed immediately). Then the equivalence follows from the memoryless property of the exponential distribution.

Note that $T_i(a)$ is the time of the i-th exploration step in the above continuous-time exploration process. Assuming $L(a, T_i(a))$ is not empty, at time $T_{i+1}(a)$, we match a uniformly chosen half-edge from the set $L(a, T_i(a))$ to a uniformly chosen half-edge among all other half-edges, excluding those in $L(a, T_i(a))$. Let $\mathcal{F}_t$ be the σ-field generated by the above process until time t. Given $\mathcal{F}_{T_i(a)}$, $T_{i+1}(a) - T_i(a)$ is an exponential random variable with rate $S_i(a)$ given by Equation (9) which is equal to $|L(a, T_i(a))|$ the size of the list consisting of unmatched half-edges in $B_{T_i(a)}(a)$. In other words,

$$\left(T_{i+1}(a) - T_i(a) \,|\, \mathcal{F}_{T_i(a)}\right) \stackrel{d}{=} \mathrm{Exp}(S_i(a)),$$

this is true since the minimum of k i.i.d. rate one exponential random variables is an exponential of rate k.

We will need the following coupling lemma the proof of which can be found in [8, Proposition 4.5].

Lemma 6 (Coupling shortest-weight graphs on a tree and CM)**.** *For a uniformly chosen vertex* *u, we have (for all $k \geq 1$)*

$$H_k(u) \stackrel{d}{=} \sum_{i=1}^{k} I_i,$$

where $\{I_i\}_{i=1}^{\infty}$ are independent Bernoulli random variables with parameter

$$\mathbb{P}(I_i = 1) = \frac{d-1}{S_i(u)},$$

and $S_i(u)$ is given by Equation 9.

3 Proof of the upper bound

In this section we present the proof of the upper bound for Theorem 1.

As described above, we grow the balls around each vertex simultaneously (at rate one) so that at time t, $B_t(a) = B_w(a,t)$ is the ball constructed from vertex a.

We let $q := \lfloor 2\sqrt{dn \log n} \rfloor$. The following lemma says that for all vertices u and v, the growing balls centered at u and v intersect w.h.p. provided that they contain each at least q nodes. More precisely,

Lemma 7. *We have with high probability*

$$B_w(u, T_q(u)) \cap B_w(v, T_q(v)) \neq \emptyset, \text{ for all } u \text{ and } v. \tag{10}$$

For the sake of readability, we postpone the proof of the lemma to the end of this section.

Fix two vertices u and v. Let

$$C(u,v) := \min\{k \geq 0 : B_w(u, T_k(u)) \cap B_w(v, T_k(v)) \neq \emptyset\},$$

be the first time that $B_w(u, T_\cdot(u))$ and $B_w(v, T_\cdot(v))$ share a vertex. Thus, by the above lemma w.h.p. $C(u,v) < q$ for all u and v. Let us denote by Q the following event:

$$Q := \{C(u,v) < q \text{ for all } u \text{ and } v\}.$$

Consider now the exploration process started at a vertex u. We will need to find lower bounds for $S_k(u)$ in the range $1 \leq k \leq q$. We let $r := \lfloor (\log n)^3 \rfloor$.

By the uniform choice of the matching, for every $k \geq 0$, the number of half-edges introduced by the new vertex at time $T_{k+1}(u)$ and connecting back to $B_w(u, T_k(u))$ (given $\mathcal{F}_{T_k(u)}$) is stochastically dominated by a binomial variable

$$\mathsf{Bin}(d-1, \alpha), \text{ where } \alpha = \frac{d + (d-2)(k+1)}{dn - 2k} \leq \frac{k+2}{n},$$

where the above inequality is valid for $k \leq \frac{n}{2} - 5$. Therefore, the tree excess of $B_w(u, T_k(u))$ is stochastically dominated by a binomial variable $\mathsf{Bin}(dk, \frac{k+2}{n})$.

We have (for large n)

$$\mathbb{P}(X_r(u) \geq 2) \leq \mathbb{P}\left(\mathsf{Bin}\left(dr, \frac{r+2}{n}\right) \geq 2\right) \leq O(\frac{r^4}{n^2}) = o(n^{-3/2}). \tag{11}$$

Moreover, for any k satisfying $r \leq k \leq 2q$, we have by Chernoff's inequality

$$\mathbb{P}(\{X_k(u) \geq k/\sqrt{r}\}) \leq \mathbb{P}\left(\mathsf{Bin}\left(dk, \frac{k+2}{n}\right) \geq k/\sqrt{r}\right) \leq \exp\left(-\frac{1}{3}k/\sqrt{r}\right) < n^{-5}, \tag{12}$$

for any sufficiently large n, since $k^2/n = o(k/\sqrt{r})$.

We conclude by a union bound over all $r \leq k \leq 2q$,

$$\mathbb{P}(\{X_k(u) < k/\sqrt{r}, \text{ for all } \ r \leq k \leq 2q\}) \geq 1 - o(n^{-4}).$$

Define the event

$$R_u := \{X_r(u) \leq 1, \text{ and } \ X_k(u) < k/\sqrt{r}, \text{ for all } \ r < k \leq 2q\}, \tag{13}$$

such that $\mathbb{P}(R_u) \geq 1 - o(n^{-3/2})$ by above inequalities.

Thus defining $R := \bigcap_{u \in [n]} R_u$, we get by union bound

$$\mathbb{P}(R) \geq 1 - o(n^{-1/2}).$$

Consider now two uniformly chosen vertices u and v. We have

$$(|\pi(u, v)| \mid Q) \leq_{st} (H_q(u) + H_q(v) \mid Q)\,.$$

Furthermore, we have

$$(H_q(u) \mid R, Q) \leq_{st} \mathcal{H} := \sum_{i=1}^{q} I_i,$$

where $\{I_i\}_{i=1}^{\infty}$ are independent Bernoulli random variables with parameter

$$\mathbb{P}(I_i = 1) = \frac{d-1}{1 + (d-2)i},$$

for all $1 \leq i \leq r$, and

$$\mathbb{P}(I_i = 1) = \frac{d-1}{1 + (d-2)i - 2i/\sqrt{r}},$$

for all $r < i \leq q$.

We conclude

$$(|\pi(u, v)| \mid R, Q) \leq_{st} \mathcal{H}_1 + \mathcal{H}_2, \tag{14}$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are two independent copies of $\mathcal{H}$ defined above.

We have the following lemma.

Lemma 8. *We have for some constant C (depending only on d)*

$$\mathbb{P}(\mathcal{H}_1 + \mathcal{H}_2 > \alpha^*(\log n + \log\log n)) \le C(n\log n)^{-1},$$

and,

$$\mathbb{P}(\mathcal{H}_1 + \mathcal{H}_2 > \widehat{\alpha}(\log n + \log\log n)) \le C(n\log n)^{-2}.$$

We postpone the proof of this lemma to the end of this section.

We conclude by (14), Lemma 8 and union bound that

$$\mathbb{P}\left(\max_{j\in[n]} |\pi(1,j)| > \alpha^*(\log n + \log\log n)\right) \le \mathbb{P}(R^c) + \mathbb{P}(Q^c) + C/\log n,$$

and,

$$\mathbb{P}\left(\max_{i,j\in[n]} |\pi(i,j)| > \widehat{\alpha}(\log n + \log\log n)\right) \le \mathbb{P}(R^c) + \mathbb{P}(Q^c) + C/\log^2 n.$$

Since R and Q hold with high probability, we get (w.h.p.)

$$\max_{j\in[n]} |\pi(1,j)| \le \alpha^*(\log n + \log\log n),$$

and,

$$\max_{i,j\in[n]} |\pi(i,j)| \le \widehat{\alpha}(\log n + \log\log n).$$

This completes the proof of the upper bound for Theorem 1.

We end this section by presenting the proof of Lemma 7 and Lemma 8.

Proof of Lemma 7. Fix two vertices u and v. First consider the exploration process for $B_w(u,t)$ until reaching $t = T_q(u)$. We know that w.h.p. the event R holds. Conditioned on R we have

$$S_q(u) \ge (d-1-o(1))q.$$

Next, consider the exploration process started at v. Each matching adds a uniform half-edge to the neighborhood of v. Therefore, the probability that $B_w(v, T_q(v))$ does not intersect $B_w(u, T_q(u))$ is at most

$$\left(1 - \frac{(d-1-o(1))q}{dn}\right)^q \le \exp\left(4(d-1-o(1))\log n\right) < n^{-7},$$

for any large n. A union bound over u and v completes the proof.

□

Proof of Lemma 8. We have for $\lambda > 0$,

$$\mathbb{E}e^{\lambda \mathcal{H}_1} = \prod_{i=1}^{r} \left(1 + (e^\lambda - 1)\frac{d-1}{1+(d-2)i}\right) \prod_{i=r+1}^{q} \left(1 + (e^\lambda - 1)\frac{d-1}{1+(d-2)i - 2i/\sqrt{r}}\right).$$

Then using the fact that $\log(1+x) \le x$, we obtain

$$\begin{aligned}
\frac{1}{2}\log \mathbb{E}e^{\lambda(\mathcal{H}_1+\mathcal{H}_2)} &= \sum_{i=1}^{r} \log\left(1 + (e^\lambda - 1)\frac{d-1}{1+(d-2)i}\right) \\
&\quad + \sum_{i=r+1}^{q} \log\left(1 + (e^\lambda - 1)\frac{d-1}{1+(d-2)i-2i/\sqrt{r}}\right) \\
&\le (e^\lambda - 1)\left(\sum_{i=1}^{r} \frac{d-1}{1+(d-2)i} + \sum_{i=r+1}^{q} \frac{d-1}{1+(d-2)i-2i/\sqrt{r}}\right) \\
&\le (e^\lambda - 1)\left(\frac{d-1}{d-2}\sum_{i=1}^{r}\frac{1}{i} + \frac{d-1}{d-2}\frac{1}{1-2r^{-1/2}}\sum_{i=r+1}^{q}\frac{1}{i}\right) \\
&\le (e^\lambda - 1)\frac{d-1}{d-2}\left(1 + O(r^{-1/2})\right)(\log q + 2) \\
&\le (e^\lambda - 1)\frac{d-1}{d-2}(\log q + 3).
\end{aligned}$$

Recall that $q = \lfloor 2\sqrt{dn\log n}\rfloor$. Choosing $\lambda := \log\left(\frac{d-2}{d-1}\alpha^*\right)$, we get

$$\log \mathbb{E}e^{\lambda(\mathcal{H}_1+\mathcal{H}_2)} \le \left(\alpha^* - \frac{d-1}{d-2}\right)(\log n + \log\log n + \log d + 10).$$

By Markov's inequality we have

$$\begin{aligned}
\mathbb{P}(\mathcal{H}_1 + \mathcal{H}_2 > \alpha^*(\log n + \log\log n)) &\le \mathbb{E}e^{\lambda(\mathcal{H}_1+\mathcal{H}_2)}\exp(-\lambda\alpha^*(\log n + \log\log n)) \\
&\le \exp\left((\alpha^* - \frac{d-1}{d-2})(\log n + \log\log n + \log d + 10)\right) \\
&\qquad \exp\left(-\alpha^*\log(\frac{d-2}{d-1}\alpha^*)(\log n + \log\log n)\right) \\
&= C\exp\left((\alpha^* - \frac{d-1}{d-2} - \alpha^*\log\left(\frac{d-2}{d-1}\alpha^*\right))\right) \\
&\qquad \exp(\log n + \log\log n) \\
&= C\exp(-\log n - \log\log n) \\
&= C(n\log n)^{-1}.
\end{aligned}$$

Similarly, by taking $\lambda := \log\left(\frac{d-2}{d-1}\widehat{\alpha}\right)$ we get

$$\log \mathbb{E} e^{\lambda(\mathcal{H}_1+\mathcal{H}_2)} \leq \left(\widehat{\alpha} - \frac{d-1}{d-2}\right)(\log n + \log\log n + \log d + 10),$$

and by Markov's inequality we have

$$\begin{aligned} \mathbb{P}(\mathcal{H}_1 + \mathcal{H}_2 > \widehat{\alpha}(\log n + \log\log n)) &\leq \mathbb{E} e^{\lambda(\mathcal{H}_1+\mathcal{H}_2)} \exp(-\lambda\widehat{\alpha}(\log n + \log\log n)) \\ &\leq \exp\left((\widehat{\alpha} - \frac{d-1}{d-2})(\log n + \log\log n + \log d + 10)\right) \\ &\quad \exp\left(-\widehat{\alpha}\log(\frac{d-2}{d-1}\widehat{\alpha})(\log n + \log\log n)\right) \\ &= C\exp\left((\widehat{\alpha} - \frac{d-1}{d-2} - \widehat{\alpha}\log\left(\frac{d-2}{d-1}\widehat{\alpha}\right)\right) \\ &\quad \exp\left(\log n + \log\log n)\right) \\ &= C\exp\left(-2(\log n + \log\log n)\right) \\ &= C(n\log n)^{-2}, \end{aligned}$$

as required.

□

4 Proof of the lower bound

In this section we present the proof of the lower bound for Theorem 1.

For $\epsilon > 0$ (small enough) we define the function $f_\epsilon : \mathbb{R}^+ \to \mathbb{R}$ as follows

$$\begin{aligned} f_\epsilon(\alpha) &:= \alpha\log\left(\frac{d-2}{(d-1)(1-\epsilon)}\alpha\right) - \alpha(1-\epsilon) + \frac{1}{d-2} && (15) \\ &= f(\alpha) + \alpha(\epsilon - \log(1-\epsilon)). && (16) \end{aligned}$$

Let α^*_ϵ and $\widehat{\alpha}_\epsilon$ be respectively the unique solutions to $f_\epsilon(\alpha) = 0$ and $f_\epsilon(\alpha) = 1$ for $\alpha > \frac{d-1}{d-2}$. Note that $\alpha^*_\epsilon < \alpha^*, \widehat{\alpha}_\epsilon < \widehat{\alpha}$, and furthermore, $\alpha^*_\epsilon \to \alpha^*$ and $\widehat{\alpha}_\epsilon \to \widehat{\alpha}$ as $\epsilon \to 0$.

To prove the lower bound, it suffices to show that for all $\epsilon > 0$, there exist w.h.p. a vertex a such that

$$|\pi(1,a)| \geq \alpha^*_\epsilon \log n,$$

and there exists w.h.p. two vertices u and v such that

$$|\pi(u,v)| \geq \widehat{\alpha}_\epsilon \log n.$$

For a path $\gamma_l = v_0, e_1, v_1, \ldots, e_l, v_l$ where v_{i-1} and v_i are endpoints of e_i for all $i \in [l]$, let

$$w(\gamma_l) = \sum_{i=1}^{l} w(e_i).$$

We first show that given that a path $P(u, v)$ between u and v has small weight, it is very likely to be the shortest-weight path between its endpoints. More precisely, we have the following.

Lemma 9. *For all n sufficiently large, and any path $\gamma_k = v_0, e_1, v_1, \ldots, e_k, v_k$ with $k = O(\log n)$, we have (for all $\epsilon > 0$)*

$$\mathbb{P}(\gamma_k \neq \pi(v_0, v_k) \mid w(\gamma_k) \leq \frac{1-\epsilon}{d-2} \log n) = o(1).$$

For the sake of readability, we postpone the proof of the lemma to the end of this section. Consider a path $\gamma_l = v_0, e_1, v_1, \ldots, e_l, v_l$. It is easily seen that for $t > 0$, letting $\mathsf{Po}(t)$ denote a Poisson mean t random variable, we have

$$\mathbb{P}(w(\gamma_\ell) \leq t) = \mathbb{P}(\mathsf{Po}(t) \geq \ell) \geq \exp(-t)\frac{t^\ell}{\ell!} = \exp\left(-t + \ell \log t - \log \ell!\right). \tag{17}$$

In the following, we let $\ell = \ell_\epsilon$ be large enough such that (by Stirling formula)

$$\log \ell! \leq \ell \log \ell - \ell(1-\epsilon).$$

Thus we have for $\alpha > 0$,

$$\begin{aligned} \mathbb{P}\left(w(\gamma_\ell) \leq \frac{\ell(1-\epsilon)}{(d-2)\alpha}\right) &\geq \exp\left(-\frac{\ell(1-\epsilon)}{(d-2)\alpha} + \ell \log\left(\frac{\ell(1-\epsilon)}{(d-2)\alpha}\right) - \ell \log \ell + \ell(1-\epsilon)\right) \\ &= \exp\left(-\frac{\ell}{\alpha}\left(\frac{1-\epsilon}{d-2} + \alpha \log\left(\frac{(d-2)\alpha}{1-\epsilon}\right) - \alpha(1-\epsilon)\right)\right) \\ &= \exp\left(-\frac{\ell}{\alpha}\left(-\frac{\epsilon}{d-2} + f_\epsilon(\alpha) + \alpha \log(d-1)\right)\right). \end{aligned} \tag{18}$$

We get for $\alpha = \alpha_\epsilon^*$ in (18) (since $f_\epsilon(\alpha_\epsilon^*) = 0$)

$$\mathbb{P}\left(w(\gamma_\ell) \leq \frac{\ell(1-\epsilon)}{(d-2)\alpha_\epsilon^*}\right) \geq (d-1)^{-\ell} \exp\left(\frac{\epsilon \ell}{(d-2)\alpha_\epsilon^*}\right), \tag{19}$$

and for $\alpha = \widehat{\alpha}_\epsilon$ in (18) (since $f_\epsilon(\widehat{\alpha}_\epsilon) = 1$)

$$\mathbb{P}\left(w(\gamma_\ell) \leq \frac{\ell(1-\epsilon)}{(d-2)\widehat{\alpha}_\epsilon}\right) \geq (d-1)^{-\ell} \exp\left(\left(-1 + \frac{\epsilon}{d-2}\right)\frac{\ell}{\widehat{\alpha}_\epsilon}\right). \tag{20}$$

Lemma 10. *Assume $t_n = \lfloor c \log n \rfloor$ for some positive constant c. For any function $\omega(n)$ tending to ∞ with n, w.h.p., there exists $v \in \partial B(1, t_n)$ such that*

$$w(\gamma_1(v)) \leq \frac{t_n(1-\epsilon)}{(d-2)\alpha^*_\epsilon} + \omega(n),$$

where $\gamma_1(v)$ denote the path from 1 to v in $B(1, t_n)$.

Proof. We first prove the lemma for the case $c < \frac{1}{5\log(d-1)}$.

Consider now $B(1, \lfloor c \log n \rfloor)$ for $c < \frac{1}{5\log(d-1)}$. By Lemma 4 w.h.p. $\text{tx}(B(1, \lfloor c \log n \rfloor) \leq 1$, and then by removing at most one of the children of 1 (and its descendants) we have the tree structure and then, $|\partial B(1, \lfloor c \log n \rfloor)| \geq (d-1)^{\lfloor c \log n \rfloor}$. In the following we assume that one of the children of node 1 is removed (even if $\text{tx}(B(1, \lfloor c \log n \rfloor) = 0)$ such that $|\partial B(1, \lfloor c \log n \rfloor)| = (d-1)^{\lfloor c \log n \rfloor}$.

Let $t_0 = \log_{d-1} \log \omega(n)$. Note that for any path γ_{t_0} of length t_0, by Markov inequality

$$\mathbb{P}(w(\gamma_{t_0}) \geq \epsilon\omega(n))) \leq \frac{t_0}{\epsilon\omega(n)}.$$

Thus, by union bound, the probability that this would be true for one of the nodes at level t_0 of node 1 (i.e., in $\partial B(1, t_0)$) is smaller than

$$(d-1)^{t_0} \frac{t_o}{\epsilon\omega(n)} = \frac{\log \omega(n) \log_{d-1} \log \omega(n)}{\epsilon\omega(n)},$$

which goes to zero as n goes to ∞. Then w.h.p. the path from the root (1) to all nodes at level t_0 has weight smaller that $\epsilon\omega(n)$.

We assume $\ell = \ell_\epsilon$ is large enough such that $\exp\left(\frac{\epsilon\ell}{(d-2)\alpha^*_\epsilon}\right) > 1$. Now consider the following branching process starting from a node r at level t_0, i.e., $r \in \partial B(1, t_0)$.

We call a vertex v *good* if either v is the root ($v = r$), or if v lies ℓ levels below a good vertex u and $w(\gamma_1(u, v)) \leq \frac{\ell(1-\epsilon)}{(d-2)\alpha^*_\epsilon}$, where $\gamma_1(u, v)$ denote the path from u to v (in $B(1, \lfloor c \log n \rfloor)$).

The collection of good nodes form a Galton-Watson tree. Let Z denote the progeny distribution of this process. Without need to calculate its distribution, from (19) we know that

$$\mathbb{E}Z = (d-1)^\ell \mathbb{P}\left(w(\gamma_\ell) \leq \frac{\ell(1-\epsilon)}{(d-2)\alpha^*_\epsilon}\right) \geq \exp\left(\frac{\epsilon\ell}{(d-2)\alpha^*_\epsilon}\right) > 1.$$

Hence, with some positive probability q_ϵ this process survives. We conclude with probability at least q_ϵ we have a good node at level $\lfloor c \log n \rfloor$ from the root r at level t_0. Considering the same process for all nodes at level t_0, we conclude that there exists a good vertex at level $\lfloor c \log n \rfloor$, with probability at least (by independence of these processes)

$$1 - (1-q_\epsilon)^{(d-1)^{t_0}} = 1 - (1-q_\epsilon)^{\log \omega(n)} \to 1,$$

as $n \to \infty$. Then w.h.p. we have a node v at level t_n, such that

$$w(\gamma_1(v)) \leq \frac{t_n(1-\epsilon)}{(d-2)\alpha_\epsilon^*} + \epsilon\omega(n).$$

This completes the proof of lemma for the case $c < \lfloor \frac{1}{5\log(d-1)} \rfloor$.

Now consider the case $c \geq \frac{1}{5\log(d-1)}$, and let K be an integer such that $c' := c/K < \frac{1}{5\log(d-1)}$. By previous argument, we know that w.h.p. there exists a node v_1 at level $\lfloor c' \log n \rfloor$ such that $w(\gamma_1(v_1)) \leq \frac{\lfloor c' \log n \rfloor (1-\epsilon)}{(d-2)\alpha_\epsilon^*} + \epsilon\omega(n)$. We know repeat the same argument to find a node v_2 at level $\lfloor c' \log n \rfloor$ below of node v_1 such that $w(\gamma_1(v_1, v_2)) \leq \frac{\lfloor c' \log n \rfloor (1-\epsilon)}{(d-2)\alpha_\epsilon^*} + \epsilon\omega(n)$, where $\gamma_1(u, v)$ denote the path from u to v on $B(1, t)$. Note that the tree excess is again at most one, and the number of nodes at level t_0 of node v_1 is at least $(d-2)(d-1)^{t_0 - 1}$ which goes to infinity as $n \to \infty$, and we have the similar arguments. Now repeating this process $K - 1$ times completes the proof. $\square$

Thus, by above lemma, there exists w.h.p. a node a at level $\alpha_\epsilon^* \log n$ such that $w(\gamma_1(a)) \leq \frac{1-\epsilon}{d-2} \log n$. By Lemma 9, this path is optimal. We conclude w.h.p. there exists a node a such that $\pi(1, a) \geq \alpha_\epsilon^* \log n$.

We now prove that there exists w.h.p. two vertices u and v such that

$$|\pi(u, v)| \geq \widehat{\alpha}_\epsilon \log n.$$

Indeed, we prove that there exists w.h.p. a path γ of length $\widehat{\alpha}_\epsilon \log n$ such that $w(\gamma) \leq \frac{1-\epsilon}{d-2} \log n$. Then again using Lemma 9, we conclude the proof.

Consider the following exploration process starting from a node a. We call a vertex v, a-good if either v is the root ($v = a$), or if v lies ℓ levels below a good vertex u and $w(\gamma_a(u, v)) \leq \frac{\ell(1-\epsilon)}{(d-2)\widehat{\alpha}_\epsilon}$, where $\gamma_a(u, v)$ denote the path from u to v in $B(a, .)$.

To find the nodes which are a-good, we first explore the nodes in $B(a, \ell)$, and we find the set of nodes at this level which are a-good. Then, for each of these (a-good) nodes, we explore again ℓ level behind and we continue the exploration until finding all of the a-good nodes. Let us denote by $G_\ell(a)$ the explored graph (starting from a) to find the set of all a-good nodes.

The following lemma bounds from above the size of $G_\ell(a)$.

Lemma 11. *Let $G \sim G(n, d)$ for some fixed $d \geq 3$. Then there exists a constant A such that w.h.p., $|G_\ell(u)| \leq A \log n$ for all $u \in V(G)$.*

The proof of this lemma is given at the end of this section. Hence, we can assume $G_\ell(u) \leq A \log n$ for all $u \in V(G)$ in the rest of the proof.

We now call a vertex u *nice* if $G_\ell(u)$ is a tree and the height of $G_\ell(u)$, denoted by $D_\ell(u)$, is at least $\widehat{\alpha}_\epsilon \log n$, i.e., $D_\ell(u) \geq \widehat{\alpha}_\epsilon \log n$.

Note that when u is nice, then there exists a node v at level $\widehat{\alpha}_\epsilon \log n$ behind u such that $w(\gamma_u(v)) \leq \frac{1-\epsilon}{d-2} \log n$, where $\gamma_u(v)$ denote the path from u to v in $B(u, .)$. Using the second moment method, we now prove that there exists at least one nice vertex.

Let N_a denote the event that node a is nice, and $X = \sum_{a \in [n]} \mathbf{1}(N_a)$ be the total number of nice vertices. We now show that $X \geq 1$ w.h.p., which concludes the proof.

Let Z be the distribution of the number of a-good nodes at level ℓ in $(d-1)$-array tree having a as a root. Conditioning on the tree structure of $G_\ell(a)$ and by removing one of the children of a (and all its descendants), the set of a-good nodes are distributed as a branching process with distribution Z. Note that by (20), we have

$$\mathbb{E}Z = (d-1)^\ell \mathbb{P}\left(w(\gamma_\ell) \leq \frac{\ell(1-\epsilon)}{(d-2)\widehat{\alpha}_\epsilon} \right) \geq \exp\left(\left(-1 + \frac{\epsilon}{d-2} \right) \frac{\ell}{\widehat{\alpha}_\epsilon} \right). \tag{21}$$

Let $P_k(a)$ be the probability that this branching process survives for at least k generations. By basic recurrent argument, we have

$$P_{k+1}(a) = 1 - \Phi_Z(1 - P_k(a)),$$

where $\Phi_Z(s) = \mathbb{E}s^Z$ denote the generation function of Z.

Note that $\mathbb{E}Z < 1$ (for ϵ small enough) and the branching process is subcritical. Hence, $P_k(a) \to 0$ as $k \to \infty$. Using $1 - \Phi_Z(1-x) = \Phi'_Z(1)x + O(x^2)$, and $\Phi'_Z(1) = \mathbb{E}Z$, it follows easily that

$$P_k(a) = (\mathbb{E}Z + o(1))^k, \text{ as } k \to \infty. \tag{22}$$

Thus, conditioning on the tree-structure of $G_\ell(a)$ (and by choosing $k = \frac{\widehat{\alpha}_\epsilon}{\ell} \log n$), we get

$$\mathbb{P}(D_\ell(a) \geq \widehat{\alpha}_\epsilon \log n) \geq (1 \pm o(1)) n^{-1+\frac{\epsilon}{d-2}}.$$

Since the size of $G_\ell(a)$ is (w.h.p.) smaller that $A \log n$ (by Lemma 11), with probability at least $1 - O(\log n / n)$, $G_\ell(a)$ is a tree.

Putting all these together, we have

$$\mathbb{E}X = \sum_a \mathbb{P}(N_a) \geq \frac{2}{3} n^{\frac{\epsilon}{d-2}}.$$

And,

$$\begin{aligned}
\mathbb{E}X^2 &= \mathbb{E}(\sum_a \mathbf{1}(N_a))^2 = \mathbb{E} \sum_{a,b} \mathbf{1}(N_a)\mathbf{1}(N_b) \\
&= \mathbb{E}\left[\sum_a \mathbf{1}(N_a) \sum_{b:\ G_\ell(a) \cap G_\ell(b) \neq \emptyset} \mathbf{1}(N_b) + \sum_{a,b:\ G_\ell(a) \cap G_\ell(b) = \emptyset} \mathbf{1}(N_a)\mathbf{1}(N_b) \right] \\
&\leq (A \log n)^2 \mathbb{E}X + (\mathbb{E}X)^2,
\end{aligned}$$

where the last inequality follows by Lemma 11. We conclude that

$$Var[X] = \mathbb{E}X^2 - (\mathbb{E}X)^2 \leq (A\log n)^2 \mathbb{E}X.$$

Then, by Chebysev's inequality w.h.p. $X \geq \frac{1}{2} n^{\frac{\epsilon}{d-2}}$.

This completes the proof of the lower bound.

We end this section by presenting the proof of Lemma 9 and Lemma 11.

Proof of Lemma 9. We condition on the path γ_k between v_0 and v_k. We now remove the path γ_k and consider the exploration process defined is Section 2.4 starting from v_0. (The proof is similar to [13, Lemma 3.5].)
Let τ_i denote the time of the i'th exploration step (for $i \geq 0$, $\tau_0 = 0$). Note that $\tau_{i+1} - \tau_i \geq_{st} Y_i$, where Y_i are independent exponential random variables with

$$\mathbb{E}[Y_i] = (1 + (d-2)(i+1))^{-1}.$$

Note that this is true since the worst case is when $X_i(a) = 0$, i.e., the explored set forms a tree.

We let $z = \lfloor \sqrt{n/\log n} \rfloor$. We will show later that the growing balls in the exploration process starting from v_0 and v_k will not intersect w.h.p. provided that they are of size less than z. We now prove that $\tau_z > \frac{1-\epsilon}{2(d-2)} \log n$ with high probability.

We have

$$\begin{aligned}
\mathbb{P}(\tau_z \leq t) &\leq \int_{\sum_{i=1}^z x_i \leq t} \prod_{i=1}^{z} [1 + (d-2)i] e^{-\sum_{i=1}^z (1+(d-2)i)x_i} dx_1 \ldots dx_z \\
&= \int_{0 \leq y_1 \leq \cdots \leq y_z \leq t} \prod_{i=1}^{z} [1 + (d-2)i] e^{-y_z} e^{-(d-2)\sum_{i=1}^z y_i} dy_1 \ldots dy_z,
\end{aligned}$$

where $y_k = \sum_{i=0}^{k-1} x_{z-i}$. Letting $y = y_z$ and accounting for all permutations over $y_1, \ldots, y_{z-1}$ (by giving to these variables the range $[0, y]$), we obtain

$$\begin{aligned}
\mathbb{P}(\tau_z \leq t) &\leq \int_0^t e^{-(d-1)y} \frac{\prod_{i=1}^z (i + \frac{1}{d-2})}{(z-1)!} \\
&\quad \cdot \left(\int_{[0,y]^{z-1}} (d-2)^z e^{-(d-2)\sum_{i=1}^{z-1} y_i} dy_1 \ldots dy_{z-1} \right) dy \\
&\leq \int_0^t e^{-(d-1)y} \frac{\prod_{i=1}^z (i + \frac{1}{d-2})}{(z-1)!} \cdot \left(\prod_{i=1}^{z-1} \int_o^y (d-2) e^{-(d-2)y_i} dy_i \right) dy \\
&\leq C(d-2) z^{\frac{d-1}{d-2}} \int_0^t e^{-(d-1)y} \left(1 - e^{-(d-2)y}\right)^{z-1} dy,
\end{aligned}$$

where $C > 0$ is an absolute constant. Now using the fact that $\left(1 - e^{-(d-2)y}\right)^{z-1} \leq e^{-n^\alpha}$, for some $\alpha > 0$ and for all $0 \leq y \leq \frac{1-\epsilon}{2(d-2)} \log n =: t_0$, we obtain

$$\mathbb{P}\left(\tau_z \leq \frac{1-\epsilon}{2(d-2)} \log n\right) \leq C(d-2) z^{\frac{d-1}{d-2}} \int_0^{t_0} e^{-n^\alpha} dy = o(n^{-4}).$$

Similarly considering the exploration process for v_k, again after time t_0, we obtain w.h.p. a set of size at most z. Now remark that, because each matching is uniform among the remaining half-edges, the probability of hitting the ball of size t_0 around v_0 is at most z/n. Altogether,

$$\mathbb{P}(\gamma_k \neq \pi(v_0, v_k) \mid w(\gamma_k) \leq \frac{1-\epsilon}{d-2} \log n) \leq \frac{z^2}{n} + o(1) = o(1),$$

as desired. □

Proof of Lemma 11. Let $Z_\ell(a)$ denote the number of a-good nodes in $B_\ell(a)$ (i.e., the nodes in generation ℓ behind a with (weighted) distance smaller than $\frac{\ell(1-\epsilon)}{(d-2)\widehat{\alpha}_\epsilon}$ from a). By Markov inequality and from (20), we obtain

$$\begin{aligned}
\mathbb{P}(Z_\ell(a) \geq 1) &\leq \mathbb{E}Z_\ell(a) \\
&\leq d(d-1)^{\ell-1}(d-1)^{-\ell} \exp\left(\left(-1 + \frac{\epsilon}{d-2}\right)\frac{\ell}{\widehat{\alpha}_\epsilon}\right) \\
&= \frac{d}{d-1} \exp\left(\left(-1 + \frac{\epsilon}{d-2}\right)\frac{\ell}{\widehat{\alpha}_\epsilon}\right) =: \beta_\epsilon
\end{aligned}$$

(this follows from the fact that the worst case is when $B_\ell(a)$ forms a tree).

Thus, for $\ell = \ell_\epsilon$ large enough, we have $\mathbb{P}(Z_\ell(a) \geq 1) \leq \beta_\epsilon < 1$.

We conclude (for any integer K)

$$\mathbb{P}\left(G_\ell(a) \leq Kd(d-1)^{\ell-1}\right) \leq \beta_\epsilon^K.$$

Now by choosing $K = 2\log n / |\log \beta_\epsilon|$, we get

$$\mathbb{P}\left(G_\ell(a) \leq 2d(d-1)^{\ell-1} \log n / |\log \beta_\epsilon|\right) \leq n^{-2}.$$

Taking a union bound over all a finishes the proof.

□

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