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Rates of approximation for general sampling-type operators in the setting of filter convergence ☆

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ABSTRACT

We investigate the order of approximation of a real-valued function f by means of suitable families of sampling type operators, which include both discrete and integral ones. We give a unified approach, by means of which it is possible to consider several kinds of classical operators, for instance Urysohn integral operators, in particular Mellin-type convolution integrals, and generalized sampling series.

We deal with filter convergence, obtaining proper extensions of classical results.

1. Introduction

We investigate, in the context of modular spaces, the rates of approximation of a real-valued function f by means of a family of operators of type

$$(T_w f)(s) = \int_{H_w} K_w(s,t,f(t))\, d\mu_w(t), \quad w \in W,\ s \in G,$$

where $W \subset \mathbb{R}$ is a suitable directed set, $(G,+)$ is a locally compact abelian Hausdorff topological group endowed with its Borel σ-algebra $\mathcal{B}$, $(H_w)_w$ is a net of nonempty closed sets of $\mathcal{B}$ with $G = \overline{\bigcup_{w \in W} H_w}$, μ_w is a regular measure defined on the Borel σ-algebra $\mathcal{B}_w$ of H_w and f belongs to the domain of the operators T_w for each $w \in W$.

In this paper we continue the study in [4,20,21] and consider a unified approach which includes a wide class of nonlinear integral operators of sampling type, among which both discrete and Mellin convolution operators.

We follow the approach given in [4,9], by examining particular choices of the subspaces H_w and the measures μ_w.

Here, we deal with the rates of approximation of $T_w f$ in the setting of modular spaces, which contain as particular cases L^p, Orlicz and Musielak-Orlicz spaces (see also [2,8,18,19,28,33]). We extend some results obtained in [2,18] both to wider classes of operators and with respect to modular filter convergence. In the context of filter convergence, some computations of rates of approximation were given in [26] for Korovkin-type theorems. A particular case of filter convergence is the statistical convergence, introduced in [27,37].

As applications of our results, we consider both integral operators of Urysohn type (for instance Mellin convolution operators), and discrete generalized sampling operators. These topics are useful in the reconstructions of signals, images

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and videos (see also [3,11–17,31,39,18,38] and their bibliography) and have also several applications in Computational Analysis (see for instance [1,32,34]).

In the first case, for every $w \in W$ we choose $H_w = G$ and $\mu_w = \mu$, where μ is a suitable regular measure defined on all the Borel subsets of G. In the second case we take $G = \mathbb{R}$ or $\mathbb{R}^N$ with the (N-dimensional) Lebesgue measure μ, and $H_w = \frac{1}{w}\mathbb{Z}$ or $H_w = \frac{1}{w}\mathbb{Z}^N$ endowed with the counting measure μ_w.

The paper is structured as follows. In Sections 2 and 3 we present filter convergence, structural assumptions, the modular spaces and some related basic concepts. In Section 4 we prove our main result, while in Section 5 we give some applications both to Urysohn-type integral operators and to generalized sampling series.

2. Preliminaries

Let $G = (G, +)$ be a locally compact abelian Hausdorff topological group with neutral element θ. Let $\mathcal{B}$ be the σ-algebra of all Borel subsets of $G, \mu : \mathcal{B} \to \mathbb{R}$ be a positive σ-finite regular measure, and $\mathcal{U}$ be a base of μ-measurable symmetric neighborhoods of θ. Let us denote by $L^0(G, \mathcal{B}, \mu) = L^0(G)$ the space of all real-valued μ-measurable functions with identification up to sets of measure μ zero.

Let W be any abstract infinite set. A nonempty class $\mathcal{F}$ of subsets of W is called a *filter* of W iff $\emptyset \neg\in \mathcal{F}, A \cap B \in \mathcal{F}$ whenever $A, B \in \mathcal{F}$ and for each $A \in \mathcal{F}$ and $B \supset A$ we get $B \in \mathcal{F}$.

As classical examples of filters we recall the filter $\mathcal{F}_{\text{cofin}}$ of all subsets of W whose complement is finite and, for $W = \mathbb{N}$, the filter $\mathcal{F}_s$ associated with the statistical convergence, namely the class of all subsets of $\mathbb{N}$ whose asymptotic density is 1. The asymptotic density d of a set $A \subset \mathbb{N}$ is defined as

$$d(A) := \lim_n \frac{\sharp(A \cap \{1, \dots, n\})}{n},$$

where the symbol $\sharp$ denotes the cardinality of the set in brackets (see also [27,37]).

A filter $\mathcal{F}$ of W is said to be *free* iff it contains $\mathcal{F}_{\text{cofin}}$. Observe that $\mathcal{F}_s$ is a free filter of $\mathbb{N}$.

A net $x_w, w \in W$, in $\mathbb{R}$ is *$\mathcal{F}$-convergent* to $x \in \mathbb{R}$ (and we write $x = (\mathcal{F})\lim_{w \in W} x_w$) iff

$$\{w \in W : |x_w - x| \leqslant \varepsilon\} \in \mathcal{F} \quad \text{for every } \varepsilon > 0.$$

Given two functions $f_1, f_2 : W \to \mathbb{R}$ and a filter $\mathcal{F}$ of W, we say that $f_1(w) = O(f_2(w))$ *with respect to* $\mathcal{F}$ iff there exists a $D > 0$ with

$$\{w \in W : |f_1(w)| \leqslant D|f_2(w)|\} \in \mathcal{F}.$$

From now on we suppose that $W = (W, \succeq)$ is a directed set, and $\mathcal{F}$ is a free filter of W. Some examples used frequently in the literature are $(W, \succeq) = (\mathbb{N}, \geqslant)$, or $W \subset [a, w_0[\subset \mathbb{R}$ endowed with the usual order, where $w_0 \in \mathbb{R} \cup \{+\infty\}$ is a limit point of W (see for instance [18, Section 3.2]). We also will consider the above set G endowed with the filter $\mathcal{H}_\theta$ of all neighborhoods of its neutral element θ. For each $w \in W$, let H_w be a nonempty closed set of $\mathcal{B}$, with $\overline{\bigcup_{w \in W} H_w} = G$, and μ_w be a regular measure defined on the Borel σ-algebra $\mathcal{B}_w$ generated by the family $\{A \cap H_w : A \text{ is an open subset of } G\}$. For every $w \in W$ let $\mathcal{L}_w$ be the set of all measurable non-negative functions $L_w : G \times G \to \mathbb{R}$, and suppose that L_w is *$\mathcal{F}$-homogeneous uniformly with respect to* $w \in W$, namely there is a set $F^* \in \mathcal{F}$ with

$$L_w(\sigma + s, u + s) = L_w(\sigma, u) \quad \text{for every } \sigma, s, u \in G \text{ and } w \in F^* \tag{1}$$

(see also [18, Section 4.1]).

Let $\mathbb{R}_0^+$ be the set of all non-negative real numbers and Ψ be the class of all functions $\psi : G \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that $\psi(t, \cdot)$ is continuous, nondecreasing, $\psi(t, 0) = 0$ and $\psi(t, u) > 0$, for every $t \in G$ and $u > 0$. We consider a family $(\psi_w)_w \subset \Psi$, with the property that there exist two constants $E_1, E_2 \geqslant 1$ and measurable functions $\phi_w : G \times G \to \mathbb{R}_0^+, w \in W$, with

$$\psi_w(t, u) \leqslant E_1 \psi_w(t - s, E_2 u) + \phi_w(t, s - t) \quad \text{for all } u \in \mathbb{R}_0^+,\ s, t \in G,\ w \in F^*. \tag{2}$$

We now give the fundamental properties of the types of kernels, which we will use throughout the paper.

Let $\mathcal{K}$ be the class of all families of functions $K_w : G \times H_w \times \mathbb{R} \to \mathbb{R}, w \in W$, satisfying the following conditions:

- $K_w(\cdot, \cdot, u)$ is measurable on $G \times H_w$ for each $w \in W$ and $u \in \mathbb{R}$;
- $K_w(s, t, 0) = 0$ for every $w \in W,\ s \in G$ and $t \in H_w$;
- for each $w \in W$ there are $L_w \in \mathcal{L}_w$ and $\psi_w \in \Psi$, with

$$|K_w(s, t, u) - K_w(s, t, v)| \leqslant L_w(s, t)\psi_w(t, |u - v|) \tag{3}$$

for all $s \in G,\ t \in H_w$ and $u, v \in \mathbb{R}$.

Let $\mathbb{K} = (K_w)_w \in \mathcal{K}$ and $\mathbf{T} = (T_w)_{w \in W}$ be a net of operators defined by

$$(T_w f)(s) = \int_{H_w} K_w(s, t, f(t))\, d\mu_w(t), \quad s \in G, \tag{4}$$

where $f \in \mathrm{Dom}\,\mathbf{T} = \bigcap_{w \in W} \mathrm{Dom}\,T_w$, and for every $w \in W$, $\mathrm{Dom}\,T_w$ is the subset of $L^0(G)$ on which $T_w f$ is well-defined as a μ-measurable function of $s \in G$.

For $s \in G, w \in W$ and $L_w \in \mathcal{L}_w$, set $l_w(s) := L_w(\theta, s)$, and suppose that

- l_w is a μ-measurable function with $l_w(\cdot - s) \in L^1(H_w)$ for every $s \in G$;
- there are $D^* > 0$ and $\overline{F} \in \mathcal{F}$ with

$$\int_{H_w} l_w(t-s)\, d\mu_w(t) \leqslant D^* \tag{5}$$

for each $s \in G$ and $w \in \overline{F}$.

Let now Ξ be the class of all functions $\xi : W \to \mathbb{R}_0^+$ such that $(\mathcal{F})\ \lim_{w\in W} \xi(w) = 0$.

Definition 2.1. Let $\xi \in \Xi, \mathbb{K} \in \mathcal{K}, l_w$ be as before and $\pi_w : G \to \mathbb{R}_0^+, w \in W$, be μ-measurable functions. We say that $\mathbb{K}$ is $(\mathcal{F}, \xi)$-*singular with respect to* l_w *and* π_w iff

(2.1.1) for each $U \in \mathcal{U}$,

$$\int_{G \setminus U} l_w(s)\,(\pi_w(s) + 1) d\mu(s) = O(\xi(w)) \quad \text{with respect to } \mathcal{F};$$

(2.1.2) If $r^w(s) := \sup_{u \in \mathbb{R}\setminus\{0\}} \left| \frac{1}{u} \int_{H_w} K_w(s,t,u)\, d\mu_w(t) - 1 \right|,\ s \in G$, then $\sup_{s\in G} r^w(s) = O(\xi(w))$ with respect to $\mathcal{F}$;

(2.1.3) there exist $F^* \in \mathcal{F}$ and $D' > 0$ such that for every $s \in G$ and $w \in F^*$ we get $r^w(s) \leqslant D'$ and

$$\int_G l_w(s)\, d\mu(s) \leqslant D'. \tag{6}$$

We now give the concept of regularity for families of measures with respect to a filter (for similar notions existing in the literature, see also [2,4,9]).

A family $m_w : G \times \mathcal{B}_w \to \mathbb{R}_0^+$, $w \in W$, is said to be $\mathcal{F}$-*regular* iff it is of the type

$$m_w(s, A) = \int_A \gamma_w(s,t)\, d\mu_w(t), \quad s \in G, w \in W, A \in \mathcal{B}_w,$$

where $\gamma_w : G \times G \to \mathbb{R}$ is measurable and the following properties are fulfilled:

- there is a constant $D_1 > 0$ such that, if $b_w^*(s) := m_w(s, H_w)$ for any $w \in W$ and $s \in G$, then

$$\left\{ w \in W : 0 < b_w^*(s) \leqslant D_1 \text{ for all } s \in G \right\} \in \mathcal{F};$$

- putting

$$\omega_w^t(A) := \int_A \gamma_w(t, s+t)\, d\mu(s), \quad w \in W,\ t \in H_w,\ A \in \mathcal{B}_w,$$

there is a family of measures $\omega_w, w \in W$, such that

$$\{ w \in W : \omega_w^t(A) \leqslant \omega_w(A) \text{ for all } t \in H_w \text{ and } A \in \mathcal{B}(G)\} \in \mathcal{F}. \tag{7}$$

Remark 2.2. (a) For some examples of regular families in the classical context see also [2,9].(b) Analogously as pointed out in [2, Section 3], observe that, by virtue of our assumptions and taking into account (5), the family $l_w, w \in W$, generates a family of $\mathcal{F}$-regular measures m_w. Indeed, it is enough to set

$$\begin{aligned} &\gamma_w(s,t) = l_w(t-s), \\ &m_w(s,A) = \int_A l_w(t-s)\, d\mu_w(t), \quad s \in G,\ w \in W,\ A \in \mathcal{B}_w, \\ &\omega_w^t(A) = \omega_w(A) = \int_A l_w(s)\, d\mu(s), \quad w \in W,\ t \in H_w,\ A \in \mathcal{B}. \end{aligned} \tag{8}$$

3. The modular spaces

We now give the basic properties of modular spaces (see also [18,28,33]).

A *modular* is a functional $\rho : L^0(G) \to \widetilde{\mathbb{R}_0^+}$, satisfying the following properties:

- $\rho(f) = 0 \iff f = 0$ μ-almost everywhere on G;
- $\rho(-f) = \rho(f)$ for every $f \in L^0(G)$;
- $\rho(\alpha_1 f + \alpha_2 g) \leqslant \rho(f) + \rho(g)$ whenever $f, g \in L^0(G)$ and $\alpha_1 \geqslant 0,\ \alpha_2 \geqslant 0$ with $\alpha_1 + \alpha_2 = 1$;
- $\rho(F(t, \cdot))$ is a measurable function of $t \in G$ for every measurable function $F : G \times G \to \mathbb{R}_0^+$.

A modular ρ is said to be *monotone* iff $\rho(f) \leqslant \rho(g)$ for each $f, g \in L^0(G)$ with $|f| \leqslant |g|$.

A modular ρ is *convex* iff $\rho(\alpha_1 f + \alpha_2 g) \leqslant \alpha_1\rho(f) + \alpha_2\rho(g)$ for all $f, g \in L^0(G)$ and whenever $\alpha_1, \alpha_2 \geqslant 0$ with $\alpha_1 + \alpha_2 = 1$.

A modular ρ is said to be *quasi-convex* iff there is an $M > 0$ with

$$\rho(\alpha_1 f + \alpha_2 g) \leqslant M(\alpha_1\rho(Mf) + \alpha_2\rho(Mg)) \tag{9}$$

for every choice of $f, g \in L^0(G)$ and $\alpha_1, \alpha_2 \geqslant 0$ with $\alpha_1 + \alpha_2 = 1$.

The space

$$L^\rho(G) := \left\{ f \in L^0(G) : \lim_{\lambda \to 0^+} \rho(\lambda f) = 0 \right\}$$

is the *modular space* associated with ρ, and the subspace

$$E^\rho(G) := \{ f \in L^\rho(G) : \rho(\lambda f) < +\infty \text{ for all } \lambda > 0 \}$$

is the *space of the finite elements of* $L^\rho(G)$.

Note that, if ρ is a quasi-convex modular, then

$$L^\rho(G) = \{ f \in L^0(G) : \text{ there is } \lambda > 0 \text{ with } \rho(\lambda f) < +\infty \}$$

(see also [18]).

We now present a fundamental example of modular space. Let Φ (resp. $\widetilde{\Phi}$) be the set of all continuous non-decreasing (resp. convex) functions $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $\varphi(0) = 0, \varphi(u) > 0$ for any $u > 0$ and $\lim_{u \to +\infty} \varphi(u) = +\infty$.

For every $\varphi \in \Phi$ (resp. $\widetilde{\Phi}$), put

$$\rho^\varphi(f) = \int_G \varphi(|f(s)|)\, d\mu(s), \quad f \in L^0(G). \tag{10}$$

The functional ρ^φ is a (resp. convex) modular on $L^0(G)$, satisfying the given properties of the modulars, and the subspace

$$L^\varphi(G) = \{ f \in L^0(G) : \rho^\varphi(\lambda f) < +\infty \text{ for some } \lambda > 0 \}$$

is the Orlicz space generated by the function φ (see also [33,35]).

A family $(f_w)_w$ of functions in $L^\rho(G)$ is said to be $\mathcal{F}$*-modularly convergent* to $f \in L^\rho(G)$ iff there exists $\lambda > 0$ with

$$(\mathcal{F}) \lim_{w \in W} \rho[\lambda(f_w - f)] = 0.$$

Observe that the $\mathcal{F}_{\text{cofin}}$-modular convergence is equivalent to usual modular convergence (see also [18]).

For $w \in W$, let ρ_w, η_w be modulars on $L^0(H_w, \mathcal{B}_w, \mu_w) = L^0(H_w)$. We denote by $L^{\rho_w}(H_w), L^{\eta_w}(H_w)$ the spaces of all functions $f \in L^0(G)$, whose restriction $f_{|H_w}$ belongs to the modular spaces generated by ρ_w, η_w respectively.

An $\mathcal{F}$-regular family $(m_w)_w$ is $\mathcal{F}$*-compatible* with the pair (ρ, ρ_w) with respect to a net $(b_w)_w$ in $\mathbb{R}$ iff there are two positive real numbers N, Q and a set $F_1 \in \mathcal{F}$ with

$$\rho\left(\int_{H_w} g(t, \cdot)\, dm_w^{(\cdot)}(t) \right) \leqslant Q \int_G \rho_w(Ng(\cdot, s + \cdot))\, d\omega_w(s) + b_w \tag{11}$$

for every measurable function $g : G \times G \to \mathbb{R}_0^+$ and for each $w \in F_1$.

Some examples of compatibility, in the classical setting, can be found in [2,9].

Let $\Gamma = (\psi_w)_w \subset \Psi$ be as in (2). The triple $(\rho_w, \psi_w, \eta_w), w \in W$, is said to be $\mathcal{F}$*-properly directed* with respect to a net $(c_w)_w$ in $\mathbb{R}$, iff for every $\lambda \in (0, 1)$ there are $C_\lambda \in (0, 1)$ and $F_2 \in \mathcal{F}$ with

$$\rho_w(C_\lambda \psi_w(s, g(\cdot))) \leqslant \eta_w(\lambda g(\cdot)) + c_w \quad \text{whenever } w \in W,\ s \in G,\ 0 \leqslant g \in L^0(G). \tag{12}$$

Let $\mathcal{T}$ be the class of all measurable functions $\tau : G \to \mathbb{R}_0^+$, continuous at θ, with $\tau(\theta) = 0$ and $\tau(t) > 0$ for all $t \neq \theta$. For a fixed $\tau \in \mathcal{T}$, let $Lip(\tau)$ be the class of all functions $f \in L^0(G)$ such that there are $\lambda > 0$ and $\widetilde{F} \in \mathcal{F}$ with

$$\sup_{w \in \widetilde{F}} [\eta_w(\lambda |f(\cdot) - f(\cdot + t)|)] = O(\tau(t)) \tag{13}$$

with respect to the filter $\mathcal{H}_\theta$ of all neighborhoods of θ.

A family of modulars $\eta_w, w \in W$, is $\mathcal{F}$*-subbounded* iff there are $C \geqslant 1, \pi_w : G \to \mathbb{R}_0^+$, $\widehat{F} \in \mathcal{F}$ and a non-trivial linear subspace Y_η of $L^0(G)$, with

$$\eta_w(f(s + \cdot)) \leqslant \eta_w(Cf) + \pi_w(s) \quad \text{for all } f \in Y_\eta,\ s \in G \text{ and } w \in \widehat{F}. \tag{14}$$

We say that $f \in L^{\eta_w}(H_w)$ *$\mathcal{F}$-uniformly with respect to $w \in W$* iff there are $R^* > 0$ and $v > 0$ with

$$\{w \in W : \eta_w(vf) \leqslant R^*\} \in \mathcal{F}. \tag{15}$$

Let now $\phi_w, w \in W$, be as in (2) and $\tau \in \mathcal{T}$. We say that $(\phi_w)_w$ *satisfies property* $(*)$ iff there exist $E_3 > 0$, $\lambda' > 0$ and $\underline{F} \in \mathcal{F}$ with

$$\rho_w(\lambda' \phi_w(\cdot, s)) \leqslant E_3 \text{ for each } s \in G \text{ and } \sup_{w \in \underline{F}} \rho_w(\lambda' \phi_w(\cdot, s)) = O(\tau(s)) \tag{16}$$

with respect to the filter $\mathcal{H}_\theta$ of all neighborhoods of θ.

4. The main result

We prove our main theorem about rates of approximation with respect to filter convergence for $T_w f - f$, where $T_w, w \in W$, are defined in (4), and $f \in Lip(\tau)$ for a fixed $\tau \in \mathcal{T}$.

Theorem 4.1. *Let ρ be a quasi-convex and monotone modular on $L^0(G)$, $\rho_w, \eta_w, w \in W$, be monotone modulars on $L^0(H_w)$, such that the triple (ρ_w, ψ_w, η_w) is $\mathcal{F}$-properly directed with respect to a net $(c_w)_w$ in $\mathbb{R}$, where $c_w = O(\xi(w))$ with respect to $\mathcal{F}$.*

Let K_w, L_w, l_w satisfy the assumptions in Section 2. *Let $\xi \in \Xi$ and $\tau \in \mathcal{T}$ be fixed.*

Assume that $\mathbb{K}$ is $(\mathcal{F}, \xi)$-singular with respect to l_w and π_w, η_w is $\mathcal{F}$-subbounded, $f \in L^\rho(G) \cap Lip(\tau) \cap Y_\eta$, and $f \in L^{\eta_w}(H_w)$ $\mathcal{F}$-uniformly with respect to $w \in W$.

Suppose that the family of measures $(m_w)_w$ defined in (8) is $\mathcal{F}$-compatible with the pair (ρ, ρ_w) with respect to a net $(b_w)_w$, with $b_w = O(\xi(w))$ with respect to $\mathcal{F}$, and let $(\phi_w)_w$ satisfy property $(*)$ as in (16).

Finally, assume that there is a neighborhood U of θ with $U \in \mathcal{U}$ and

$$\int_U l_w(s)\tau(s)\, d\mu(s) = O(\xi(w)) \text{ with respect to } \mathcal{F}. \tag{17}$$

Then there is a constant $c > 0$ with

$$\rho(c(T_w f - f)) = O(\xi(w)) \text{ with respect to } \mathcal{F}.$$

Proof. We first estimate the quantity $|T_w f - f|$. Let E_1, E_2 be as in (2). We have:

$$|(T_w f)(s) - f(s)| \leqslant \int_{H_w} |K_w(s,t,f(t)) - K_w(s,t,f(s))|\, d\mu_w(t) + \left| \int_{H_w} K_w(s,t,f(s))\, d\mu_w(t) - f(s) \right| = I_1 + I_2. \tag{18}$$

We now estimate the term I_1. From (3), (1) and (2) with $u = f(t)$, $v = f(s)$, $\sigma = \theta$, $u = t - s$ and $u = |f(t) - f(s)|$ respectively, we get the existence of a set $F_0 \in \mathcal{F}$ such that

$$\begin{aligned} I_1 &\leqslant \int_{H_w} L_w(s,t)\psi_w(t, |f(t) - f(s)|)\, d\mu_w(t) = \int_{H_w} L_w(\theta, t - s)\, \psi_w(t, |f(t) - f(s)|)\, d\mu_w(t) \\ &= \int_{H_w} l_w(t - s)\, \psi_w(t, |f(t) - f(s)|)\, d\mu_w(t) \\ &\leqslant E_1 \int_{H_w} l_w(t - s)\psi_w(t - s, E_2\, |f(t) - f(s)|)\, d\mu_w(t) + \int_{H_w} l_w(t - s)\phi_w(t, s - t)\, d\mu_w(t) \quad \text{for each } s \in G \text{ and } w \in F_0. \end{aligned} \tag{19}$$

From (18) and (19), by applying the modular ρ, for every $c > 0$ and $w \in F_0$ we have

$$\begin{aligned} \rho(c(T_w f - f)) &\leqslant \frac{1}{3}\rho\left(3cE_1 \int_{H_w} l_w(t - \cdot)\psi_w(t - \cdot, E_2\, |f(t) - f(\cdot)|)\, d\mu_w(t)\right) + \frac{1}{3}\rho\left(3c \int_{H_w} l_w(t - \cdot)\, \phi_w(t, \cdot - t)\, d\mu_w(t)\right) \\ &\quad + \frac{1}{3}\rho\left(3c\left|\int_{H_w} K_w(\cdot, t, f(\cdot))\, d\mu_w(t) - f(\cdot)\right|\right) = \frac{1}{3}(J_1 + J_2 + J_3). \end{aligned} \tag{20}$$

We first estimate J_1. If we choose

$$g(t,s) = \psi_w(t - s, E_2\, |f(t) - f(s)|), \quad t, s \in G, \ w \in W,$$

then, by the condition (11) of $\mathcal{F}$-compatibility, there are two positive real constants N, Q and an element $F_1 \in \mathcal{F}$ with

$$J_1 = \rho\left(3cE_1 \int_{H_w} l_w(t - \cdot)\, \psi_w(t - \cdot, E_2\, |f(t) - f(\cdot)|)\, d\mu_w(t)\right) \leqslant Q \int_G l_w(s)\rho_w(3cE_1 N \psi_w(-s, E_2\, |f(\cdot) - f(s + \cdot)|))\, d\mu(s) + b_w \tag{21}$$

whenever $w \in F_0 \cap F_1$, where $(b_w)_w$ is as in (11).

Take λ as in (13) and put $\lambda_0 := \frac{\lambda}{E_2}$. Without loss of generality, we can suppose that $\lambda_0 \in (0,1)$. Since the triple (ρ_w, ψ_w, η_w) is $\mathcal{F}$-properly directed, in correspondence with λ_0 there exist $C_{\lambda_0} \in (0,1)$, $(c_w)_w$ in $\mathbb{R}$ and $F_2 \in \mathcal{F}$, satisfying (12).

Choose now $c > 0$ small enough, so that $3cE_1N \leqslant C_{\lambda_0}$. From (21) and monotonicity of ρ_w it follows that

$$J_1 \leqslant Q\int_G l_w(s)\rho_w(C_{\lambda_0}\psi_w(-s, E_2|f(\cdot) - f(s+\cdot)|))\,d\mu(s) + b_w \tag{22}$$

for each $w \in F_0 \cap F_1 \cap F_2$. From (22), (12) and (6), we obtain the existence of a set $F_3 \in \mathcal{F}$, with

$$\begin{aligned} J_1 &\leqslant Q\int_G l_w(s)\,\eta_w(\lambda_0 E_2|f(\cdot) - f(s+\cdot)|)d\mu(s) + b_w + c_w\int_G l_w(s)\,d\mu(s) \\ &\leqslant Q\int_G l_w(s)\,\eta_w(\lambda|f(\cdot) - f(s+\cdot)|)d\mu(s) + b_w + D'c_w \end{aligned} \tag{23}$$

for every $w \in \bigcap_{j=0}^{3} F_j$, where D' is as (6). Since $f \in Lip(\tau)$, there are $D > 0$, $U \in \mathcal{U}$ and $F_4 \in \mathcal{F}$ with

$$\eta_w(\lambda|f(\cdot) - f(s+\cdot)|) \leqslant D\,\tau(s) \tag{24}$$

for each $s \in U$ and $w \in F_4$. Without loss of generality, we can choose U small enough, so that the condition (17) is satisfied. Thus from (23) we get:

$$J_1 \leqslant Q\int_U l_w(s)\eta_w(\lambda|f(\cdot) - f(s+\cdot)|)d\mu(s) + Q\int_{G\setminus U} l_w(s)\eta_w(\lambda|f(\cdot) - f(s+\cdot)|)d\mu(s) + b_w + D'c_w = Q(J_1^1 + J_1^2) + b_w + D'c_w \tag{25}$$

for every $w \in \bigcap_{j=0}^{4} F_j$. From (17) and (24) it follows that there are $A^* > 0$ and $F_5 \in \mathcal{F}$ with

$$J_1^1 = \int_U l_w(s)\eta_w(\lambda|f(\cdot) - f(s+\cdot)|)\,d\mu(s) \leqslant D\int_U l_w(s)\tau(s)\,d\mu(s) \leqslant DA^*\,\xi(w) \tag{26}$$

whenever $w \in \bigcap_{j=0}^{5} F_j$.

We now estimate the term J_1^2. Proceeding analogously as in [2, p. 865], by monotonicity and $\mathcal{F}$-subboundedness of η_w, and since $f \in Y_\eta$, there exists $F_6 \in \mathcal{F}$ such that

$$\begin{aligned} \eta_w(\lambda|f(\cdot) - f(s+\cdot)|) &\leqslant \frac{1}{2}\eta_w(2\lambda f) + \frac{1}{2}\eta_w(2\lambda(f(s+\cdot))) \leqslant \frac{1}{2}\eta_w(2\lambda f) + \frac{1}{2}\eta_w(2\lambda Cf) + \frac{1}{2}\pi_w(s) \\ &\leqslant \eta_w(2\lambda Cf) + \pi_w(s) \quad \text{for all } s \in G \text{ and } w \in F_6, \end{aligned} \tag{27}$$

where $C \geqslant 1$ and π_w is as in (14). Hence, from (27) and 2.1.1) we obtain the existence of a constant $B > 0$ and a set $F_7 \in \mathcal{F}$ with

$$\begin{aligned} J_1^2 &= \int_{G\setminus U} l_w(s)\eta_w(\lambda(f(\cdot) - f(s+\cdot)))\,d\mu(s) \leqslant \eta_w(2\lambda Cf)\int_{G\setminus U} l_w(s)\,d\mu(s) + \int_{G\setminus U} l_w(s)\,\pi_w(s)\,d\mu(s) \\ &\leqslant B\,(\eta_w(2\lambda Cf) + 1)\,\xi(w) \end{aligned} \tag{28}$$

for all $w \in \bigcap_{j=0}^{7} F_j$.

We now estimate the term J_2. Again thanks to $\mathcal{F}$-compatibility, we get

$$J_2 = \rho\left(3c\int_{H_w} l_w(t-\cdot)\,\phi_w(t,\cdot - t)\,d\mu_w(t)\right) \leqslant Q\int_G l_w(s)\rho_w(3cN\phi_w(\cdot,s))\,d\mu(s) + b_w \tag{29}$$

for each $w \in F_0 \cap F_1$, where $N, Q, (b_w)_w, F_0$ and F_1 are as in (22), and in (29) we have chosen

$$g(t,s) = \phi_w(t, s-t), \quad s,t \in G,\ w \in W.$$

We choose $c > 0$ small enough, so that $3cN \leqslant \lambda'$, and use monotonicity of ρ_w. From (29), condition 2.1.1) of $(\mathcal{F}, \xi)$-singularity and (17) we get

$$J_2 \leqslant Q\int_G l_w(s)\rho_w(\lambda'\phi_w(\cdot,s))\,d\mu(s) + b_w \leqslant QE_3\int_{G\setminus U} l_w(s)\,d\mu(s) + QB'\int_U l_w(s)\,\tau(s)\,d\mu(s) = O(\xi(w)). \tag{30}$$

We now estimate the term J_3. First of all, set

$$\Omega^w(s) := \int_{H_w} K_w(s,t,f(s))\,d\mu_w(t) - f(s), \quad s \in G,\ w \in W.$$

For every $w \in W$, if $f(s) \neq 0$, then

$$\Omega^w(s) = f(s)\left(\frac{1}{f(s)}\int_{H_w} K_w(s,t,f(s))\,d\mu_w(t) - 1\right),$$

while, if $f(s) = 0$, then $\Omega^w(s) = 0$, since $K(\cdot,\cdot,0) = 0$. Hence, for each $s \in G$ and $w \in W$, we get

$$|\Omega^w(s)| \leqslant r^w(s)\,|f(s)|, \tag{31}$$

where $r^w(s)$ is as in (2.1.2). From (31) it follows that

$$J_3 = \rho\left(3c\left|\int_{H_w} K_w(\cdot,t,f(\cdot))\,d\mu_w(t) - f(\cdot)\right|\right) \leqslant \rho(3c\,r^w f), \quad w \in W.$$

As $\sup_{s\in G} r^w(s) = O(\xi(w))$ with respect to $\mathcal{F}$, there are $P > 0$ and $F_8 \in \mathcal{F}$ with

$$J_3 \leqslant \rho(3cPf\,\xi(w)) \tag{32}$$

for every $w \in F_8$. By quasi-convexity of ρ, from (32) we get

$$J_3 \leqslant M\rho(3cMPf)\,\xi(w), \quad w \in F_8, \tag{33}$$

where M is as in (9), namely a constant related to quasi-convexity of ρ.

Finally, from (20), (25), (26), (28), (30) and (33) it follows that

$$\rho(c(T_w f - f)) \leqslant \frac{1}{3}\{(DA^*Q + BQ\,\eta_w(2\lambda Cf) + BQ + M\rho(3cMPf))\xi(w) + b_w + D' c_w\} + O(\xi(w)) \tag{34}$$

for every $w \in \bigcap_{j=0}^{8} F_j$. Observe that, since $f \in L^\rho(G)$, we get that $\rho(3cMPf) < +\infty$ for $c > 0$ sufficiently small. Moreover, as $f \in L^{\eta_w}(H_w)$ $\mathcal{F}$-uniformly with respect to $w \in W$, and since η_w is monotone and $\lambda \leqslant \frac{v}{2C}$, where v is as in (15), we obtain the existence of a set $F_9 \in W$ with

$$\eta_w(2\lambda Cf) \leqslant \eta_w(vf) \leqslant R^* \tag{35}$$

for all $w \in F_9$, where R^* is as in (15). Thus, from (34) and (35), we deduce the existence of a $c > 0$ sufficiently small, so that

$$\rho(c(T_w f - f)) = O(\xi(w)) \quad \text{with respect to } \mathcal{F}.$$

This concludes the proof. □

Remark 4.2.

(a) Observe that, in general, the hypothesis that $\overline{\bigcup_{w\in W} H_w} = G$ is essential (see also [7, Remark 4]).

(b) If $G = \mathbb{R}$ or $G = H_w$ for every $w \in W$, $\tau(t) = |t|^\alpha$ with $\alpha > 0$, then $Lip(\tau)$ is the classical discrete Zygmund class (see also [2,23]). If further $W = [1, +\infty[$ and $\xi(w) = w^{-\alpha}$, then we observe that condition (17) is linked to the existence of suitable moments of order α (see for instance [18, Example 3.9]).

5. Applications

5.1. Urysohn-type operators

As a first application, we deal with Mellin-type convolution operators (see also [12,13,20,21,30]).

Let $(G,+)$ be the multiplicative group $(\mathbb{R}^+,\cdot)$, $W \subset \mathbb{R}^+$ such that $+\infty$ is a limit point for W, $\mathcal{M}^1$ be the class of all measurable subsets of $\mathbb{R}^+$, and set

$$\mu(A) = \mu_w(A) = \int_A \frac{dt}{t}, \quad A \in \mathcal{M}^1,\ w \in W. \tag{36}$$

Let $\widetilde{\mathcal{L}}$ be the set of all families of measurable functions $\widetilde{L_w} : \mathbb{R}^+ \to \mathbb{R}_0^+$, $w \in W$, such that $\widetilde{L_w} \in L^1(\mu)$.

Let $(\psi_w)_w \subset \Psi$ be as in Section 2, and denote by $\widetilde{\mathcal{K}}$ the set of all families of functions $\widetilde{K_w} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $w \in W$, such that:

(i) $\widetilde{K_w}(\cdot,u)$ is measurable for all $u \in \mathbb{R}$ and $w \in W$, and $\widetilde{K_w}(t,0) = 0$ for every $w \in W$ and $t \in \mathbb{R}^+$;

(ii) for each $w \in W$ there are $\widetilde{L_w} \subset \widetilde{\mathcal{L}}$ and $\psi_w \subset \Psi$, with

$$|\widetilde{K_w}(t,u) - \widetilde{K_w}(t,v)| \leqslant \widetilde{L_w}(t)\psi_w(|u - v|) \tag{37}$$

whenever $w \in W$, $t \in \mathbb{R}^+$ and $u, v \in \mathbb{R}$.

Let $\widetilde{\mathbb{K}} = (\widetilde{K_w})_w \in \widetilde{\mathcal{K}}$ and $\widetilde{T_w}$, $w \in W$, be a nonlinear Mellin-type operator defined as

$$(\widetilde{T_w} f)(s) = \int_0^{+\infty} \widetilde{K_w}\left(\frac{t}{s}, f(t)\right)\frac{dt}{t}, \quad s \in \mathbb{R}^+, \tag{38}$$

where $f \in$Dom $\widetilde{\mathbf{T}} = \bigcap_{w\in W}$ Dom $\widetilde{T_w}$, and Dom $\widetilde{T_w}$ is the subset of $L^0(\mathbb{R}^+,\mathcal{B},\mu)$ on which $\widetilde{T_w} f$ is well-defined. Set

$$L_w(s,t) = \widetilde{L_w}\left(\frac{t}{s}\right), \quad K_w(s,t,u) = \widetilde{K_w}\left(\frac{t}{s}, u\right), \quad s,t \in \mathbb{R}^+,\ u \in \mathbb{R},\ w \in W.$$

If $\widetilde{K_w}, \widetilde{L_w}, w \in W$ fulfil the above assumptions (i) and (ii), then $K_w, L_w, w \in W$, satisfy the assumptions in Section 2.

Note that the $\widetilde{L_w}$'s are homogeneous and $\widetilde{L_w}(s) = L_w(1,s)$ for every $w \in W$ and $s \in \mathbb{R}^+$. In this setting, after a change of variables, condition (5) reduces to

$$\int_0^{+\infty} \widetilde{L_w}\left(\frac{t}{s}\right)\frac{dt}{t} = \int_0^{+\infty} \widetilde{L_w}(t)\,\frac{dt}{t} \leqslant D^*, \quad s \in \mathbb{R}^+,\ w \in \overline{F}, \tag{39}$$

for a suitable element $\overline{F} \in \mathcal{F}$ and a constant $D^* > 0$, while, for $\xi \in \Xi, (\mathcal{F},\xi)$-singularity of $\widetilde{\mathbb{K}} = (\widetilde{K_w})_w$ (with respect to $\widetilde{L_w}$ and π_w) is formulated as follows:

(i) for each $\delta > 1$, putting $U_\delta := \left[\frac{1}{\delta}, \delta\right]$, we have

$$\int_{\mathbb{R}^+ \setminus U_\delta} \widetilde{L_w}(s)\,(\pi_w(s)+1)\frac{ds}{s} = O(\xi(w)) \quad \text{with respect to } \mathcal{F};$$

(ii) if $\widetilde{r^w}(s) := \sup_{u \in \mathbb{R}\setminus\{0\}}\left|\frac{1}{u}\int_0^{+\infty}\widetilde{K_w}\left(\frac{t}{s},u\right)\frac{dt}{t} - 1\right|, s \in \mathbb{R}^+$, then $\sup_{s\in\mathbb{R}^+}\widetilde{r^w}(s) = O(\xi(w))$ with respect to $\mathcal{F}$;

(iii) there are $F^* \in \mathcal{F}$ and $D' > 0$ with $\widetilde{r^w}(s) \leqslant D'$ and

$$\int_0^{+\infty} \widetilde{L_w}(t)\,\frac{dt}{t} \leqslant D' \tag{40}$$

whenever $s \in \mathbb{R}^+$ and $w \in F^*$.

In particular, if $\widetilde{K_w}(t,u) = \widetilde{L_w}(t)\cdot u$, $t \in \mathbb{R}^+, u \in \mathbb{R}$, then we have

$$\widetilde{r^w}(s) = \left|\int_0^{+\infty}\widetilde{L_w}(t)\frac{dt}{t} - 1\right|, \quad s \in \mathbb{R}^+. \tag{41}$$

Observe that, in this setting, $Y_\eta = L^\eta(\mathbb{R}^+)$. Thus it is possible to give a version of Theorem 4.1 in the context of Mellin operators.

Theorem 5.1. *Let ρ,η be monotone modulars on $L^0(\mathbb{R}^+)$, assume that ρ is quasi-convex and that the triple (ρ,ψ_w,η) is $\mathcal{F}$-properly directed with respect to a net $(c_w)_w$ in $\mathbb{R}$, where $c_w = O(\xi(w))$ with respect to $\mathcal{F}$. Let $\widetilde{K_w}, \widetilde{L_w}$ be as above, and $\xi \in \Xi, \tau \in \mathcal{T}$ be fixed. Suppose that $\widetilde{\mathbb{K}} = (\widetilde{K_w})_w$ is $(\mathcal{F},\xi)$-singular with respect to $\widetilde{L_w}$ and π_w, η_w is $\mathcal{F}$-subbounded, and $f \in L^{\rho+\eta}(\mathbb{R}^+) \cap Lip(\tau)$. Assume that the family $(m_w)_w$, defined by*

$$m_w(s,A) = \int_A \widetilde{L_w}\left(\frac{t}{s}\right)d\mu(t), \quad s \in \mathbb{R}^+,\ A \in \mathcal{M}^1,\ w \in W,$$

is $\mathcal{F}$-compatible with the modular ρ with respect to the identically zero net. Moreover, suppose that

$$\int_a^b \widetilde{L_w}\left(\frac{t}{\cdot}\right)\frac{dt}{t} \in E^\rho(\mathbb{R}^+)$$

for every compact interval $[a,b] \subset \mathbb{R}^+$, and let $(\phi_w)_w$ satisfy property $(*)$ as in (16).

Let there exist $\delta_0 > 1$ with

$$\int_{\mathbb{R}^+\setminus U_{\delta_0}} \widetilde{L_w}(s)\,\tau(s)\,d\mu(s) = O(\xi(w)) \quad \text{with respect to } \mathcal{F}. \tag{42}$$

Then there is a constant $c > 0$ with

$$\rho(c(\widetilde{T_w}f - f)) = O(\xi(w)) \quad \text{with respect to } \mathcal{F}.$$

Observe that our theory, according to [2], includes also the case of multidimensional Mellin convolution operators, by setting $G = H_w = ((\mathbb{R}^+)^N, \cdot)$, where the operation $\cdot$ is defined as

$$s\cdot t = (s_1 t_1, \ldots, s_N t_N), \quad s = (s_1,\ldots,s_N),\ t = (t_1,\ldots,t_N) \in (\mathbb{R}^+)^N,$$

the neutral element of G is $\theta = (1,\ldots,1)$, the inverse element of $(t_1,\ldots,t_N)$ is $\left(\frac{1}{t_1},\ldots,\frac{1}{t_N}\right)$, and

$$\mu(A) = \mu_w(A) = \int_A \frac{(dt)^N}{\prod_{j=1}^N t_j}, \quad A \in \mathcal{M}^N,\ w \in W,$$

where $\mathcal{M}^N$ denotes the class of all measurable subsets of $(\mathbb{R}^+)^N$.

A particular Mellin-type kernel is the *moment kernel*, defined by

$$M_w(t) = w\,t^w \chi_{(0,1)}(t), \quad t \in \mathbb{R}^+,\ w \in W,$$

where $\chi_{(0,1)}$ is the characteristic function associated with $(0,1)$. For each $w \in W, t \in \mathbb{R}^+, u \in \mathbb{R}$, put

$$\widetilde{L_w}(t) = M_w(t), \quad \widetilde{K_w}(t,u) = \widetilde{L_w}(t) \cdot u. \tag{43}$$

Observe that for every $\delta > 1$ and $w \in W$ we get

$$\int_0^{+\infty} \widetilde{L_w}(t) \frac{dt}{t} = w \int_0^1 t^{w-1} dt = 1; \quad \int_{\mathbb{R}^+ \setminus U_\delta} \widetilde{L_w}(t) \frac{dt}{t} = w \int_0^{1/\delta} t^{w-1}\, dt = \frac{1}{\delta^w}. \tag{44}$$

Now, fix arbitrarily $0 < \alpha \leqslant 1$, and choose $\tau(t) = |\log t|^\alpha$, $\xi(w) = w^{-\alpha}$. Then, $\tau \in \mathcal{T}$ and $\xi \in \Xi$.
For every $\delta > 1$, putting $v = -w \log t$, we get:

$$0 \leqslant \int_{1/\delta}^{\delta} \widetilde{L_w}(t) \tau(t) \frac{dt}{t} = w \int_{1/\delta}^{\delta} t^{w-1} |\log t|^\alpha \chi_{(0,1)}(t)\, dt = w \int_{1/\delta}^{1} t^{w-1} (-\log t)^\alpha\, dt = w^{-\alpha} \int_0^{w \log \delta} e^{-v}\, v^\alpha\, dv$$
$$\leqslant w^{-\alpha} \int_0^{+\infty} e^{-v}\, v^\alpha\, dv. \tag{45}$$

From (44) and (45) we obtain

$$\int_{\mathbb{R}^+ \setminus U_\delta} \widetilde{L_w}(t) \frac{dt}{t} = O(w^{-\alpha}) \text{and} \int_{1/\delta}^{\delta} \widetilde{L_w}(t)\, \tau(t) \frac{dt}{t} = O(w^{-\alpha}) \text{ for every } \delta > 1. \tag{46}$$

Another Mellin-type kernel is the *Mellin–Poisson–Cauchy kernel*, defined by

$$\widetilde{L_w}(t) = \frac{2^{p-1}(p-1)!}{\pi(2p-3)!!} \frac{w}{(1 + w^2 \log^2 t)^p}, \quad \widetilde{K_w}(t,u) = \widetilde{L_w}(t) \cdot u, \tag{47}$$

$w \in W, p \geqslant 2, t \in \mathbb{R}^+, u \in \mathbb{R}$, where $(-1)!! = 1!! = 1$ and $(2q+1)!! = 1 \cdot 3 \cdot \dots \cdot (2q+1)$ for all $q \in \mathbb{N}$. It is known that

$$\int_0^{+\infty} \widetilde{L_w}(t) \frac{dt}{t} = 1 \quad \text{for every } w \in W \tag{48}$$

(see also [21]). Moreover, for each $p \geqslant 2, \delta > 1$ and for w large enough (depending on δ) we have:

$$w \int_{\mathbb{R}^+ \setminus U_\delta} \frac{1}{(1 + w^2 \log^2 t)^p} \frac{dt}{t} = 2 \int_{w \log \delta}^{+\infty} \frac{1}{(1+y^2)^p}\, dy \leqslant 2 \int_{w \log \delta}^{+\infty} \frac{y}{(1+y^2)^2}\, dy = 2 \int_{w^2 \log^2 \delta}^{+\infty} \frac{1}{(1+z)^2}\, dz = \frac{1}{1 + w^2 \log^2 \delta}. \tag{49}$$

Moreover, for every $\delta > 1, w \in W$ and p we get:

$$0 \leqslant w \int_{1/\delta}^{\delta} \frac{1}{(1 + w^2 \log^2 t)^p} |\log t|^\alpha \frac{dt}{t} = w \int_{1/\delta}^{1} \frac{1}{(1 + w^2 \log^2 t)^p} (-\log t)^\alpha \frac{dt}{t} + w \int_1^{\delta} \frac{1}{(1 + w^2 \log^2 t)^p} (\log t)^\alpha \frac{dt}{t}$$
$$= 2 w^{-\alpha} \int_0^{w \log \delta} \frac{1}{(1+v^2)^p}\, v^\alpha\, dv \leqslant 2 w^{-\alpha} \int_0^{+\infty} \frac{1}{(1+v^2)^p}\, v^\alpha\, dv. \tag{50}$$

Since $0 < \alpha \leqslant 1$ and $p \geqslant 2$, from (49) and (50) it follows that

$$\int_{\mathbb{R}^+ \setminus U_\delta} \widetilde{L_w}(t) \frac{dt}{t} = O(w^{-\alpha}) \text{ and } w \int_{1/\delta}^{\delta} \frac{1}{(1 + w^2 \log^2 t)^p} |\log t|^\alpha \frac{dt}{t} = O(w^{-\alpha}) \text{ for any } \delta > 1. \tag{51}$$

The *Mellin–Gauss–Weierstrass kernel* is defined by setting

$$\widetilde{L_w}(t) = \frac{w}{2\sqrt{\pi}} e^{-\frac{w^2}{4} \log^2 t}, \quad \widetilde{K_w}(t,u) = \widetilde{L_w}(t) \cdot u, \quad w \in W,\ t \in \mathbb{R}^+,\ u \in \mathbb{R}. \tag{52}$$

We get

$$\int_0^{+\infty} \widetilde{L_w}(t) \frac{dt}{t} = 1 \quad \text{for all } w \in W \tag{53}$$

(see also [20,21]). Moreover, for each $\delta > 1$ and for w large enough (depending on δ), we have

$$\int_{\mathbb{R}^+ \setminus U_\delta} \widetilde{L_w}(t) \frac{dt}{t} = \frac{2}{\sqrt{\pi}} \int_{\frac{w \log \delta}{2}}^{+\infty} e^{-v^2} dv \leqslant \frac{2}{\sqrt{\pi}} \int_{\frac{w \log \delta}{2}}^{+\infty} e^{-v} dv = \frac{2}{\sqrt{\pi}} e^{-\frac{w \log \delta}{2}}. \tag{54}$$

Furthermore, for every $\delta > 1$ we get:

$$0 \leqslant w \int_{1/\delta}^{\delta} e^{-\frac{w^2}{4} \log^2 t} |\log t|^\alpha \frac{dt}{t} = w \int_{1/\delta}^{1} e^{-\frac{w^2}{4} \log^2 t} (-\log t)^\alpha \frac{dt}{t} + w \int_1^{\delta} e^{-\frac{w^2}{4} \log^2 t} (\log t)^\alpha \frac{dt}{t} = 2^{\alpha+2} w^{-\alpha} \int_0^{\frac{w \log \delta}{2}} e^{-v^2}\, v^\alpha\, dv$$
$$\leqslant 2^{\alpha+2} w^{-\alpha} \int_0^{+\infty} e^{-v^2}\, v^\alpha\, dv. \tag{55}$$

From (54) and (55) we obtain

$$\int_{\mathbb{R}^+\setminus U_\delta} \widetilde{L_w}(t)\frac{dt}{t} = O(w^{-\alpha}) \text{ and } w\int_{1/\delta}^{\delta} e^{-\frac{w^2}{4}\log^2 t}|\log t|^\alpha \frac{dt}{t} = O(w^{-\alpha}) \quad \text{for any } \delta > 1. \tag{56}$$

For other examples of kernels existing in the literature (for instance Féjer, Picard, box-spline, Abel-Poisson and Féjer-Korovkin-type kernels), see also [2, Section 6].

Note that for every compact interval $[a,b] \subset \mathbb{R}^+$ the function

$$s \mapsto \int_a^b \widetilde{L_w}\left(\frac{t}{s}\right)\frac{dt}{t}$$

belongs to $L^q(\mathbb{R}^+,\mu)$, where μ is as in (36), for every $1 \leqslant q < +\infty$ and w large enough (depending on q), where $\widetilde{L_w}, w \in W$, is as in (43), (47) or (52) (see also [21]).

Now, proceeding analogously as in [21, Example 2], it is possible to check that in general our results are proper extensions of the corresponding classical ones.

More precisely, for a sake of simplicity let $W = \mathbb{N}$, and take any free filter $\mathcal{F} \neq \mathcal{F}_{\text{cofin}}$ of $\mathbb{N}$ and any infinite set H, with $\mathbb{N}\setminus H \in \mathcal{F}$. Note that such a set H does exist, since $\mathbb{N}\setminus H \in \mathcal{F}$. As examples, let $\mathcal{F} := \mathcal{F}_s$ be the filter of all subsets of $\mathbb{N}$ having asymptotic density 1 and let H be the set of prime numbers, the Fibonacci set or the set of all perfect squares. Observe that these three sets have asymptotic density zero (see also [25]).

For each $t > 0$ and $n \in \mathbb{N}$, set

$$L_n^*(t) = \begin{cases} \widetilde{L_n}(t) & \text{if } n \in \mathbb{N}\setminus H, \\ e^{3n^2}\widetilde{L_n}(t) & \text{if } n \in H, \end{cases} \tag{57}$$

where $\widetilde{L_n}(t)$ is defined analogously as in (43), (47) or (52).

From (44) and (46) we get that the moment kernel satisfies (42) and all the $(\mathcal{F},\xi)$-singularity conditions, with $\pi_w = 0, \tau(t) = |\log t|^\alpha$ and $\xi(w) = w^{-\alpha}$, where $0 < \alpha \leqslant 1$.

These properties are fulfilled also by the Mellin–Poisson–Cauchy kernel, thanks to (48) and (51), and by the Mellin–Gauss–Weierstrass kernel, by virtue of (53) and (56).

Furthermore (see also [29, Proposition 1]) observe that the family of functions $m_n : \mathbb{R}^+ \times \mathcal{M}^1 \to \mathbb{R}_0^+$ defined by

$$m_n(s,A) = \int_A \widetilde{L_n}\left(\frac{t}{s}\right)\frac{dt}{t}, \quad s \in \mathbb{R}^+,\ A \in \mathcal{M}^1,\ n \in \mathbb{N},$$

is $\mathcal{F}$-regular and $\mathcal{F}$-compatible with the modular ρ^φ defined as in (10) for every $\varphi \in \widetilde{\Phi}$, where $\widetilde{\Phi}$ is as in Section 2.

Set now $\varphi(u) = \eta(u) = u^q$, where $u \in \mathbb{R}_0^+$ and $q \geqslant 1$ is taken arbitrarily. For each $n \in \mathbb{N}$, define $\psi_n : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ by setting $\psi_n(u) = u, u \in \mathbb{R}_0^+$. It is easy to check that the triple $(\rho^\varphi, \psi_n, \rho^\eta)$ is $\mathcal{F}$-properly directed with $c_n = 0$ for all $n \in \mathbb{N}$. So the hypotheses of Theorem 5.1 are fulfilled, and hence the kernels L_n^* satisfy our main results with respect to $\mathcal{F}$-convergence.

We now see that the kernels $L_n^*, n \in \mathbb{N}$, do not fulfil the classical versions of theorems analogous to Theorem 5.1. Proceeding as in [21], let $S = [e^{-1/4}, e^{1/4}]$, f be a positive continuous functions defined on $\mathbb{R}^+$, with $f(t) = 1$ for every $t \in [e^{-3}, e^{-2}]$ and whose support is contained in $[e^{-4}, e^{-1}]$. For every $n \in \mathbb{N}$ and $s > 0$, put

$$(T_n^* f)(s) = \int_0^{+\infty} L_n^*\left(\frac{t}{s}\right) f(t)\frac{dt}{t}.$$

In [21] it is proved that

$$\lim_{n\in H}(T_n^* f)(s) = +\infty \quad \text{for every } s \in S. \tag{58}$$

From (58) we deduce that $\lim_{n\in H}\rho^\varphi[\lambda(T_n^* f - f)\chi_S] = +\infty$. Thus, we get that our results are proper extensions of the corresponding classical ones.

5.2. Discrete operators

We now consider the case of generalized sampling-type operators, following the approach of [2,4,9]. Let $W \subset \mathbb{R}^+$ be such that $+\infty$ is a limit point for W. In the literature one takes often $W = \mathbb{R}^+$ or $W = \mathbb{N}$ (see also [4,9,18]). We consider nonlinear discrete operators of type

$$(T_w f)(s) = \sum_{k=-\infty}^{+\infty} K_w\left(s, \frac{k}{w}, f\left(\frac{k}{w}\right)\right), \quad w \in W,\ s \in \mathbb{R}, \tag{59}$$

where $f \in \text{Dom}\,\mathbf{T}$. In this setting, let $G = \mathbb{R}$ endowed with the Lebesgue measure μ, and for $w \in W$ let $H_w = \frac{1}{w}\mathbb{Z}$, $\mathcal{B}_w$ be the set of all subsets of H_w, μ_w be the counting measure.

We assume that $\mathbb{K} = (K_w)_w$ belongs to $\mathcal{K}$, for a fixed family $(\psi_w)_w \subset \Psi$, and identify $L^1(H_w)$ with l^1.

For each $w \in W$, $t = \frac{k}{w} \in \frac{1}{w}\mathbb{Z}$ and $s \in \mathbb{R}$ we get:

$$\int_{\mathbb{R}} L_w(s,t)\,d\mu(s) = \int_{\mathbb{R}} L_w\left(s,\frac{k}{w}\right)ds, \quad \int_{\frac{1}{w}\mathbb{Z}} L_w(s,t)\,d\mu_w(t) = \sum_{k=-\infty}^{+\infty} L_w\left(s,\frac{k}{w}\right).$$

For each $s \in \mathbb{R}$ and $w \in W$, set $l_w(s) := L_w(s,0)$. The condition in (5) and those involving $(\mathcal{F},\xi)$-singularity, for $\xi \in \Xi$, are expressed as follows.

(i) There exist $D^* > 0$ and $\overline{F} \in \mathcal{F}$ such that

$$\sum_{k=-\infty}^{+\infty} l_w\left(\frac{k}{w} - s\right) \leqslant D^* \text{ for every } s \in \mathbb{R} \text{ and } w \in \overline{F}.$$

(ii) For every $\delta > 0$,

$$\int_{|s|\geqslant\delta} l_w(s)(\pi_w(s)+1)\,ds = O(\xi(w)) \text{ with respect to } \mathcal{F};$$

(iii)
$$\sup_{s\in\mathbb{R},u\in\mathbb{R}\setminus\{0\}} \left| \frac{1}{u}\sum_{k=-\infty}^{+\infty} K_w(s,\tfrac{k}{w},u) - 1 \right| = O(\xi(w)) \text{ with respect to } \mathcal{F};$$

(iv) there exist $F^* \in \mathcal{F}$ and $D' > 0$ such that

$$r^w(s) \leqslant D' \text{and} \int_{\mathbb{R}} L_w\left(s,\frac{k}{w}\right)ds \leqslant D' \text{ for each } s \in \mathbb{R} \text{ and } w \in F^*.$$

Let us take modulars $\eta = \eta_{\mathbb{R}}$ on $L^0(\mathbb{R})$ and $\eta_w = \eta_{\frac{1}{w}\mathbb{Z}}$ on $L^0(\frac{1}{w}\mathbb{Z})$. We consider the case when η and η_w, $w \in W$, generate an Orlicz space, that is

$$\eta_{\mathbb{R}}(f) = \int_{-\infty}^{+\infty} \varphi(|f(s)|)\,ds, \quad \eta_w = \eta_{\frac{1}{w}\mathbb{Z}} = \sum_{k=-\infty}^{+\infty} \varphi\left(\left|f\left(\frac{k}{w}\right)\right|\right),$$

where $\varphi \in \Phi$. Note that, if φ is convex, then it is possible to see that the space Y_η in (14) contains the set of bounded real-valued functions with compact support on $\mathbb{R}$ (see also [2]). So the theory in Sections 3 and 4 can be applied to the operators in (59), dealing with the modulars $\rho_w := \rho_{\frac{1}{w}\mathbb{Z}}, \eta_w := \eta_{\frac{1}{w}\mathbb{Z}}$.

A particular case of operators (59) is given by the *linear generalized sampling series,* defined by

$$(\widetilde{T_w} f)(s) := \sum_{k=-\infty}^{+\infty} f\left(\frac{k}{w}\right)\kappa(ws-k), \quad w \in W,\ s \in \mathbb{R}, \tag{60}$$

where $\kappa \in L^1(\mathbb{R})$ is a kernel function, and $f : \mathbb{R} \to \mathbb{R}$. These kinds of operators arise from the problem of reconstructing a real-valued function f (signal) on the whole real line from its sampled values $f(\frac{k}{w})$ computed at the nodes $\frac{k}{w}$, where k varies in $\mathbb{Z}$ and $w > 0$ is the rate of the sampling. Among the related literature, we quote for instance [18,22,38] and the bibliography therein, and in particular [2,4,6,16,23,24]. In this setting we put

$$K_w\left(s,\frac{k}{w},f\left(\frac{k}{w}\right)\right) = \kappa(ws-k)\cdot f\left(\frac{k}{w}\right), \quad L_w\left(s,\frac{k}{w}\right) = |\kappa(ws-k)|. \tag{61}$$

Note that $l_w(s) = |\kappa(ws)|$ for each $s \in \mathbb{R}$ and $w \in W$. Let us assume the classical hypotheses that

$$\sum_{k=-\infty}^{+\infty} \kappa(u-k) = 1 \text{ uniformly with respect to } u \in \mathbb{R},$$

$$\sup_{u\in\mathbb{R}} \sum_{k=-\infty}^{+\infty} |\kappa(u-k)| < +\infty,$$

$$M_\alpha(\kappa) := \int_{-\infty}^{+\infty} |\kappa(s)|\,|s|^\alpha\,ds < +\infty,$$

where $M_\alpha(\kappa)$ is the *moment of order* α of κ, and $0 < \alpha \leqslant 1$ (see also [23]). Under these assumptions it is possible to prove that the condition in (17) and those of $(\mathcal{F},\xi)$-singularity are satisfied with respect to $\mathcal{F} = \mathcal{F}_{\text{cofin}}, \xi(w) = w^{-\alpha}$ and $\tau(t) = |t|^\alpha$ (see also [2, Proposition 3], [5, Lemma 3.1], [18, Theorem 8.1], [36, Lemma 1]). Indeed, for every $\delta > 0$ and $w \in W$, taking $s = wt$, we get

$$\begin{aligned} w\int_{-\delta}^{\delta}|\kappa(wt)|\,|t|^{\alpha}dt &= w^{-\alpha}\int_{-w\delta}^{w\delta}|\kappa(s)|\,|s|^{\alpha}ds \leqslant w^{-\alpha}\int_{-\infty}^{+\infty}|\kappa(s)|\,|s|^{\alpha}ds = O(w^{-\alpha}), \quad w\int_{\delta}^{+\infty}|\kappa(wt)|\,dt \\ &= w\delta^{-\alpha}\int_{\delta}^{+\infty}|\kappa(wt)|\,\delta^{\alpha}dt \leqslant w\delta^{-\alpha}\int_{\delta}^{+\infty}|\kappa(wt)|\,t^{\alpha}dt = w^{-\alpha}\delta^{-\alpha}\int_{w\delta}^{+\infty}|\kappa(s)|\,|s|^{\alpha}ds \\ &\leqslant w^{-\alpha}\delta^{-\alpha}\int_{-\infty}^{+\infty}|\kappa(s)|\,|s|^{\alpha}ds = O(w^{-\alpha}), \quad w\int_{-\infty}^{-\delta}|\kappa(wt)|\,dt = w\delta^{-\alpha}\int_{-\infty}^{-\delta}|\kappa(wt)|\,\delta^{\alpha}dt \\ &\leqslant w\delta^{-\alpha}\int_{-\infty}^{-\delta}|\kappa(wt)|\,(-t)^{\alpha}dt = w\delta^{-\alpha}\int_{\delta}^{+\infty}|\kappa(-wy)|\,y^{\alpha}dy = O(w^{-\alpha}). \end{aligned} \tag{62}$$

We now show that our results are proper extensions of the corresponding classical ones (see also [4, Section 6] and [9, Corollary 4]).

Take $W = \mathbb{N}$, and pick the modulars $\rho, \eta, \eta_n, n \in \mathbb{N}$, generating the Lebesgue spaces L^p, with $p \geqslant 1$. Let $C_c^{\infty}(\mathbb{R})$ be the space of all real-valued functions, defined on the whole real line, having compact support and admitting derivatives of any order on $\mathbb{R}$. Pick $f \in C_c^{\infty}(\mathbb{R})$ such that, for r large enough, $f(t) \geqslant |t|^{-r}$ for $|t| \geqslant 1$, and $f(t) = 0$ whenever $|t| < 1$. Let $\mathcal{F}$ and H be as in Section 5.1.

Analogously as in [4], let $(v_n)_n$ be a sequence of positive real numbers with $\lim_n v_n = 0$ and B^* be a set of positive Lebesgue measure, independent of n, with $\sum_{k=n}^{2n} L_n(s, \frac{k}{n}) \geqslant v_n$ for any $s \in B^*$. Let $(y_n)_n$ be a sequence in $\mathbb{R}^+$, with $\lim_n v_n y_n = +\infty$. For each $n \in \mathbb{N}$, $s \in \mathbb{R}$ and $t \in H_n$, put

$$L_n^*(s,t) = \begin{cases} L_n(s,t), & \text{if } n \in \mathbb{N} \setminus H, \\ y_n L_n(s,t), & \text{if } n \in H, \end{cases} \tag{63}$$

where L_n is analogously as in (61), and set

$$K_n^*\left(s, \frac{k}{n}, f\left(\frac{k}{n}\right)\right) = L_n^*\left(s, \frac{k}{n}\right) \cdot f\left(\frac{k}{n}\right). \tag{64}$$

In [4] it is shown that $\lim_n (T_n f)(s) = +\infty$ for every $s \in B^*$. Thus in this setting, taking into account (62), it is possible to see that the hypotheses of Theorem 4.1 are satisfied, but the corresponding classical result, that is when $\mathcal{F} = \mathcal{F}_{\text{cofin}}$ (see also [9, Corollary 4]), does not hold.

Our general approach on discrete operators includes also the nonlinear multivariate sampling series, which have applications, for instance, in the reconstruction of images and videos (see also [2,10]).

In this case, $G = (\mathbb{R}^N, +)$, μ is the N-dimensional Lebesgue measure, $H_w = \frac{1}{w}\mathbb{Z}^N$, $\mathcal{B}_w$ is the set of all subsets of $\mathbb{Z}^N$ and μ_w is the counting measure. In this setting the operators (59) and (60) are expressed by

$$(V_w f)(\mathbf{s}) = \sum_{\mathbf{k} \in \mathbb{Z}^N} K_w\left(\mathbf{s}, \frac{\mathbf{k}}{w}, f\left(\frac{\mathbf{k}}{w}\right)\right), \quad w \in W, \ \mathbf{s} \in \mathbb{R}^N,$$

$$(\widetilde{V_w} f)(\mathbf{s}) = \sum_{\mathbf{k} \in \mathbb{Z}^N} f\left(\frac{\mathbf{k}}{w}\right) \kappa(w\mathbf{s} - \mathbf{k}), \quad w \in W, \ \mathbf{s} \in \mathbb{R}^N$$

respectively, where $\kappa \in L^1(\mathbb{R}^N)$, $\mathbf{s} = (s_1, \ldots, s_N)$, $\mathbf{k} = (k_1, \ldots, k_N)$, $w\mathbf{s} = (ws_1, \ldots, ws_N)$ and $\frac{\mathbf{k}}{w} = \left(\frac{k_1}{w}, \ldots, \frac{k_N}{w}\right)$.

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References

[1] Z.A. Anastassi, T.E. Simos, Numerical methods for the efficient solution of problems in quantum mechanics, Phys. Rep. 482–483 (2009) 1–240.
[2] L. Angeloni, G. Vinti, Rate of approximation for nonlinear integral operators with application to signal processing, Differ. Integral Equ. 18 (8) (2005) 855–890.
[3] C. Bardaro, A. Boccuto, X. Dimitriou, I. Mantellini, Abstract Korovkin-type theorems in modular spaces and applications, Cent. Eur. J. Math. 11 (10) (2013) 1774–1784, http://dx.doi.org/10.2478/s11533-013-0288-7.
[4] C. Bardaro, A. Boccuto, X. Dimitriou, I. Mantellini, Modular filter convergence theorems for abstract sampling-type operators, Appl. Anal. 92 (11) (2013) 2404–2423, http://dx.doi.org/10.1080/00036811.2012.738480.
[5] C. Bardaro, P.L. Butzer, R.L. Stens, G. Vinti, Kantorovich-type generalized sampling series in the setting of Orlicz spaces, Sampling Theor. Sign. Image Process. 6 (1) (2007) 29–52.
[6] C. Bardaro, P.L. Butzer, R.L. Stens, G. Vinti, Prediction by samples from the past with error estimates covering discontinuous signals, IEEE Trans. Inf. Theory 56 (1) (2010) 614–633.
[7] C. Bardaro, I. Mantellini, On some estimates for general sampling operators and approximation properties, Int. J. Math. Sci. 2 (2) (2003) 289–326.
[8] C. Bardaro, I. Mantellini, Uniform modular integrability and convergence properties for a class of Urysohn integral operators in function spaces, Math. Slovaca 56 (4) (2006) 465–482.
[9] C. Bardaro, I. Mantellini, Approximation properties in abstract modular spaces for a class of general sampling type operators, Appl. Anal. 85 (4) (2006) 383–413.

[10] C. Bardaro, I. Mantellini, Generalized sampling approximation of bivariate signals: rate of pointwise convergence, Numer. Funct. Anal. Optim. 31 (2) (2010) 131–154.
[11] C. Bardaro, I. Mantellini, A note on the Voronovskaja theorem for Mellin convolution operators, Appl. Math. Lett. 24 (2011) 2064–2067.
[12] C. Bardaro, I. Mantellini, On Voronovskaja formula for linear combinations of Mellin–Gauss–Weierstrass operators, Appl. Math. Comput. 218 (2012) 10171–10179.
[13] C. Bardaro, I. Mantellini, On the moments of the bivariate Mellin–Picard type kernels and applications, Integral Transforms Spec. Funct. 23 (2) (2012) 135–148.
[14] C. Bardaro, I. Mantellini, On the iterates of Mellin–Féjer convolution operators, Acta Applicandae Mathematicae 121 (2) (2012) 213–229.
[15] C. Bardaro, I. Mantellini, On linear combination of multivariate generalized sampling type series, Mediterranean J. Math. 10 (4) (2013) 1831–1850.
[16] C. Bardaro, I. Mantellini, Asymptotic formulae for linear combinations of generalized sampling operators, Zeitschrift Anal. Anw. 32 (3) (2013) 279–298.
[17] C. Bardaro, I. Mantellini, On Mellin convolution operators: a direct approach to the asymptotic formulae, Integral Transforms Spec. Funct. (2013), to appear.
[18] C. Bardaro, J. Musielak, G. Vinti, Nonlinear Integral Operators and Applications, de Gruyter, Berlin, 2003.
[19] C. Bardaro, G. Vinti, Uniform convergence and rate of approximation for a nonlinear version of the generalized sampling operator, Result. Math. 34 (1998) 224–240.
[20] A. Boccuto, X. Dimitriou, Modular filter convergence theorems for Urysohn integral operators and applications, Acta Math. Sin. Engl. Ser. 29 (6) (2013) 1055–1066, http://dx.doi.org/10.1007/s10114-013-1443-6.
[21] A. Boccuto, X. Dimitriou, Modular convergence theorems for integral operators in the context of filter exhaustiveness and applications, Mediterranean J. Math. 10 (2) (2013) 823–842, http://dx.doi.org/10.1007/s00009-012-0199-z.
[22] P.L. Butzer, A survey on the Whittaker–Shannon sampling theorem and some of its extension, J. Math. Res. Exposition 3 (1983) 185–212.
[23] P.L. Butzer, R.J. Nessel, Fourier Analysis and Approximation I, Academic Press, New York-London, 1971.
[24] P.L. Butzer, R.L. Stens, Reconstruction of signals in $L^p(\mathbb{R})$-spaces by generalized sampling series based on linear combinations of B-splines, Integral Transforms Spec. Funct. 19 (1–2) (2008) 35–58.
[25] R.J. Cintra, L.C. Rêgo, H.M. de Oliveira, R.M. Campello De Souza, On a density for sets of integers, Anais do VII ERMAC Regional, 3 (2007), Recife, available on http://www2.ee.ufpe.br/codec/ER3MAC.pdf.
[26] O. Duman, M.A. Őrzaslan, E. Erkuş-Duman, Rates of ideal convergence for approximation operators, Mediterranean J. Math. 7 (1) (2010) 111–121, http://dx.doi.org/10.1007/s00009-010-0031-6.
[27] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951) 241–244.
[28] W.M. Kozlowski, Modular function spaces, Pure and Applied Mathematics, Marcel Dekker, New York, 1988.
[29] I. Mantellini, Generalized sampling operators in modular spaces, Comment. Math. 38 (1998) 77–92.
[30] I. Mantellini, On the asymptotic behaviour of linear combinations of Mellin–Picard type operators, Math. Nachr. 286 (17–18) (2013) 1820–1832, http://dx.doi.org/10.1002/mana201100248.
[31] I. Mantellini, G. Vinti, Modular estimates for nonlinear integral operators and applications in fractional calculus, Numer. Funct. Anal. Optim. 17 (1–2) (1996) 143–165.
[32] G.V. Milovanović, D.R. Djordjević, Numerical methods in computational engineering, University of Niš, Faculty of Civil Engineering and Architecture, Niš, 2007.
[33] J. Musielak, Orlicz spaces and modular spaces, Lecture Notes in Mathematics, 1034, Springer-Verlag, New York, 1983.
[34] B. Neta, M. Scott, On a family of Halley-like methods to find simple roots of nonlinear equations, Appl. Math. Comput. 219 (15) (2013) 7940–7944.
[35] M.M. Rao, Z.D. Ren, Theory of Orlicz Spaces, Marcel Dekker, New York, 1991.
[36] S. Ries, R.L. Stens, Approximation by generalized sampling series, in: B.l. Sendov, P. Petrushev, R. Maleev, S. Tashev (Eds.), Constructive Theory of Functions, Proc. Conf. Varna, Bulgaria, 1984, Publishing House of the Bulgarian Academy of Sciences, Sofia, 1984, pp. 746–756.
[37] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math. 2 (1951) 73–74.
[38] C. Vinti, A survey on recent results of the Mathematical Seminar in Perugia, inspired by the work of Professor P.L. Butzer, Results Math. 34 (1998) 32–55.
[39] G. Vinti, A general approximation result for nonlinear integral operators and applications to signal processing, Appl. Anal. 79 (2001) 217–238.