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(s, S)-TYPE POLICY FOR A PRODUCTION INVENTORY PROBLEM WITH LIMITED BACKLOGGING AND WITH STOCKOUTS

Abstract. A production inventory problem with limited backlogging and with stockouts is described in a discrete time, stochastic optimal control framework with finite horizon. It is proved by dynamic programming methods that an optimal policy is of (s, S)-type. This means that in every period the policy is completely determined by two fixed levels of the stochastic inventory process considered.

1. Introduction. We discuss a dynamic, stochastic inventory model with limited backlogging and with stockouts when the inventory process crosses the backlogging limits. Usually in the literature both these situations are discussed in separate models ([6], [2], [1]). Our paper joins the models, as is done in Bylka [3] for the deterministic case. The negative levels of the inventory process are limited. In every period the limit results in that only a part of the excess of demand over supply is backlogged and satisfied when additional inventory becomes available. The remaining part cannot be satisfied and is completely lost. Both cases are considered in the shortage cost structure. The costs of shortage, storage and ordering (ordering being the sum of production and setup costs) are included in the objective functional.

2. Formulation of the problem. We describe the problem in the optimal control framework.

Let N be a positive integer and let $\{W_t : t = 0, 1, \ldots, N-1\}$ denote a sequence of independent, identically distributed, nonnegative random variables with finite expected value ($E(W_t) < \infty$). The variables represent the

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random demands which appear at the ends of periods $t = 0, 1, \ldots, N-1$. The inventory process $\{X_t : t = 0, 1, \ldots, N\}$ is given by the following real-valued decision process:

$$\text{(2.1)} \qquad \begin{aligned} X_0 &= x_0, \\ X_{t+1} &= \max(-\beta_t, X_t + u_t(X_t) - W_t), \qquad t = 0, 1, \ldots, N-1, \end{aligned}$$

where $\beta_t \ge 0$, $u_t : \mathbb{R} \to \mathbb{R}^+$ is a Borel-measurable decision (control) function and $x_0 \in \mathbb{R}$.

In this description x_0 is the initial level of the inventory process, β_t limits the backlogging and u_t denotes the order (or production) decision. We assume that the order decisions are made and the ordered goods delivered immediately at the beginning of each period. The demands are supposed to occur at the end of each period after the orders have entered the stock. The limit β_t makes the shortages of size up to β_t allowed and backordered. The shortages of size over β_t are completely lost (stockouts).

A sequence of decision functions

$$\pi = (u_0(\cdot), u_1(\cdot), \ldots, u_{N-1}(\cdot))$$

is called a *policy* (strategy). The model considers some costs connected with any strategy and corresponding process (2.1). The costs satisfy the following assumptions:

(2.2) Assumptions. (i) The ordering cost function at the period $t, t = 0, 1, \ldots, N-1$, is given by

$$c_t(u) = \begin{cases} K_t + c_t u & \text{if } u > 0, \\ 0 & \text{if } u = 0, \end{cases}$$

where $K_t \ge 0$, $c_t \ge 0$. K_t denotes the setup cost and c_t denotes the unit production cost.

(ii) $\{K_t\}$ is such that

$$K_t \ge K_{t+1} \qquad \text{for } t = 0, 1, \ldots, N-2.$$

(iii) For every period $t, t = 0, 1, \ldots, N-1$, the inventory cost function $h_t : \mathbb{R}^+ \to \mathbb{R}^+$ is convex and nondecreasing with $h_t(x) \le \bar{h}_t(1+x)$ for all $x \in \mathbb{R}^+$ and some $\bar{h}_t \in \mathbb{R}^+$.

(iv) For every period $t, t = 0, 1, \ldots, N-1$, the shortage cost function $p_t : \mathbb{R}^+ \to \mathbb{R}^+$ is convex and nondecreasing with $p_t(0) = 0$, $p_t(x) \le \bar{p}_t(1+x)$ for all $x \in \mathbb{R}^+$ and some $\bar{p}_t \in \mathbb{R}^+$. Additionally, $E(p_t'^{+}(W_t)) > c_t$ and $p_t'^{+}(\beta_t) - p_t'^{-}(\beta_t) > c_{t+1}$, where E denotes the expectation with respect to the probability distribution of W_t and p'^{+} and $p_t'^{-}$ denote the right- and left-hand derivatives. If $\beta_t = 0$ then we put $p_t'^{-}(0) = 0$.

The objective functional to be minimized is the expected value of all the cost incurred over the horizon

$$J(\pi) = \mathop{\mathbb{E}}_{\{W_t\}} \Big(\sum_{t=0}^{N-1} \{c_t(u_t(X_t)) + h_t(\max(0, X_t + u_t(X_t) - W_t)) + p_t(\max(0, W_t - X_t - u_t(X_t)))\} \Big). \tag{2.3}$$

We are looking for a strategy π^* (optimal) such that

$$J(\pi^*) = \min_{\pi} J(\pi). \tag{2.4}$$

Remarks on the model. Observe that for $\beta_t = \infty$, $t = 0, 1, \ldots, N-1$, all the negative levels of the inventory process (2.1) are admissible. This means that an unsatisfied demand is completely backlogged. If $\beta_t = 0$, $t = 0, 1, \ldots, N-1$, then only positive levels of the inventory process (2.1) are allowed. There is no backlogging and so unsatisfied demands are completely lost (stockouts). It is known ([6], [2], [1], [4], [5]) that in both these cases the optimal strategies are of (s, S)-type. This means that in every period t the decision function u_t^* is determined by two inventory levels s_t and S_t such that

$$u_t^*(x) = \begin{cases} S_t - x & \text{if } x \le s_t, \\ 0 & \text{if } x > s_t. \end{cases}$$

The aim of this paper is to generalize the result for the case $\beta_t \ge 0$, $\beta_t < \infty$, $t = 0, 1, \ldots, N-1$.

Remarks on the Assumptions (2.2). In the present model the shortages of size up to β_t and over β_t, $t = 0, 1, \ldots, N-1$, are differently treated. This fact is included into the structure of the shortage cost p_t. The discontinuity of p_t' at β_t assumed in (2.2)(iv) means that the rate of growth of the shortage cost above the limit β_t is substantially greater than below β_t. The size of the jump is greater than the next period marginal production cost c_{t+1}. The remaining assumptions in (2.2) are of the type similar to those which appeared in the discussion of the case $\beta_t = 0$, $t = 0, 1, \ldots, N-1$ ([1, §IV, Sec. IV]).

3. Auxiliary theorem and lemmas. Here and in the next sections we apply dynamic programming methods to determine an optimal strategy π^*.

THEOREM 3.1 [7, Theorem 1.1]. *Let $V_0, V_1, \ldots, V_N$ be nonnegative measurable functions defined on $\mathbb{R}$ such that*

$$\begin{aligned} V_N &\equiv 0, \\ V_t(x) &= \inf_{u \ge 0} \{c_t(u) + E(h_t(\max(0, u + x - W_t))) \\ &\quad + E(p_t(\max(0, W_t - x - u))) \\ &\quad + E(V_{t+1}(\max(-\beta_t, x + u - W_t)))\}, \qquad t = N-1, \ldots, 0. \end{aligned} \tag{3.1}$$

If $\pi^* = (u_0^*, u_1^*, \ldots, u_{N-1}^*)$ *is a strategy such that*

$$\begin{aligned} V_t(x) = c_t(u_t^*(x)) &+ E(h_t(\max(0, u_t^*(x) + x - W_t))) \\ &+ E(p_t(\max(0, W_t - u_t^*(x) - x))) \\ &+ E(V_{t+1}(\max(-\beta_t, u_t^*(x) + x - W_t))) \end{aligned}$$

then π^* *is optimal for the problem* (2.1)–(2.4).

It is convenient to rewrite the assumptions of Theorem 3.1 in a slightly different form. For this put $y = x + u$ and

$$\begin{aligned} G_t(y) = c_t y &+ E(h_t(\max(0, y - W_t))) + E(p_t(\max(0, W_t - y))) \\ &+ E(V_{t+1}(\max(-\beta_t, y - W_t))). \end{aligned}$$

Using this notation and the definition of $c_t(\cdot)$ the equations (3.1) may be written as

$$V_N(x) = 0, \qquad V_t(x) = \inf_{y \geq x}\{K_t \chi_{\{y>x\}} + G_t(y) - c_t x\}, \tag{3.2}$$

where

$$t = N-1, \ldots, 0, \qquad \chi_{\{y>x\}} = \begin{cases} 1 & \text{if } y > x, \\ 0 & \text{if } y = x. \end{cases}$$

Moreover, if $y_t^*(x)$ is such that

$$V_t(x) = \begin{cases} K_t + G_t(y_t^*(x)) - c_t x & \text{if } y_t^*(x) > x, \\ G_t(y_t^*(x)) - c_t x & \text{if } y_t^*(x) = x, \end{cases}$$

then

$$u_t^*(x) = y_t^*(x) - x. \tag{3.3}$$

Observe that the right-hand sides of (3.2) for $t = N-1, \ldots, 0$ are of the type

$$\inf_{y \geq x}\{K\chi_{\{y>x\}} + G(y) - cx\}, \qquad x \in \mathbb{R}, \tag{3.4}$$

where $K \geq 0$, $G : \mathbb{R} \to \mathbb{R}$ and $c > 0$. Therefore we start the discussion with the auxiliary problem (3.4). The problem means that for every $x \in \mathbb{R}$ we are looking for $y^*(x)$ such that

$$K\chi_{\{y^*(x)>x\}} + G(y^*(x)) - cx = \inf_{y \geq x}\{K\chi_{\{y>x\}} + G(y) - cx\}.$$

For this the notion of k-convexity will be useful.

DEFINITION 3.1 [6]. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be k-*convex*, $k \in \mathbb{R}^+$, if it satisfies

$$k + f(y+z) \geq f(y) + z\frac{f(y) - f(y-b)}{b} \qquad \text{for all } z \geq 0,\ b > 0 \text{ and } y \in \mathbb{R}.$$

Remark 3.1. It follows from the definition that a convex function is 0-convex; a k-convex function is also l-convex for any $l \geq k$; the sum of a

k-convex function and an l-convex function is $(k+l)$-convex; if f is k-convex and W is a random variable such that for every x, $E|f(x-W)| < \infty$ then $x \to E(f(x-W))$ is also k-convex.

LEMMA 3.1 [1, pp. 318–320]. *Consider* (3.4). *Let G be K-convex, continuous and such that $G(y) \to \infty$ as $|y| \to \infty$. Then there exist numbers s, S, $s \le S$, such that*

(i) $G(S) = \inf_{y\in\mathbb{R}} G(y)$,
(ii) $G(s) = K + G(S)$,
(iii) $G(x) \le K + G(y)$ *for all* x, y *with* $s \le x \le y$,
(iv) *the function*

$$y^*(x) = \begin{cases} S & \text{if } x \le s, \\ x & \text{if } x > s \end{cases}$$

is a solution of (3.4),
(v) *the function*

$$V(x) = \inf_{y\ge x}\{K\chi_{\{y>x\}} + G(y) - cx\} = \begin{cases} K + G(S) - cx & \text{if } x \le s, \\ G(x) - cx & \text{if } x > s \end{cases}$$

is continuous and K-convex.

Proof. The proof of (i)–(iii) is given by Bensoussan *et al.* [1, pp. 318–319]. The proof of (iv)–(v) follows from the proof of Proposition 8.2 in [1, pp. 319–320] and from the fact that the sum of a K-convex function and a linear function is K-convex.

Remark 3.2. Observe that if $G(x) \le \overline{G}(1+|x|)$ for some $\overline{G}$ then $V(x) \le \overline{V}(1+|x|)$ for some $\overline{V}$.

An optimal strategy for (2.1)–(2.4) is determined by Theorem 3.1 and remarks (3.3) and (3.2). So the second part of this section is connected with p_t, h_t describing G_t in (3.2).

Put

$$(lp_t)(z) = p_t'^{-}(\beta_t)(z-\beta_t) + p_t(\beta_t)$$

and

$$P_t(z) = \begin{cases} p_t(z) & \text{for } z \in [0,\beta_t), \\ (lp_t)(z) & \text{for } z \in [\beta_t,\infty). \end{cases} \tag{3.5}$$

Moreover, let

$$g_t(z) = \begin{cases} 0 & \text{for } z \in [0,\beta_t), \\ p_t(z) - (lp_t)(z) & \text{for } z \in [\beta_t,\infty). \end{cases} \tag{3.6}$$

Remark 3.3. Observe that the assumptions (2.2)(iv) imply that

(i) P_t and g_t are convex, nonnegative and nondecreasing on $\mathbb{R}^+$,
(ii) $P_t(z) \le p_t(z) \le \overline{\overline{p}}_t(1+z)$, for $z \in \mathbb{R}^+$ and some $\overline{\overline{p}}_t \in \mathbb{R}$, $\overline{\overline{p}}_t \le \overline{p}_t$,
(iii) $g_t(z) \le \overline{g}_t(1+z)$ with $\overline{g}_t \le \overline{p}_t$ and $z \in \mathbb{R}^+$.

The assumption $p_t'^{+}(\beta_t) - p_t'^{-}(\beta_t) > c_{t+1}$ gives

(iv) $g'^{+}(\beta_t) > c_{t+1}$.

Define

$$l_t(y) = c_t y + E(h_t(\max(0, y - W_t))) + E(P_t(\max(0, W_t - y))) \tag{3.7}$$

and let

$$\begin{aligned} L_t(y) &= l_t(y) + E(g_t(\max(0, W_t - y))) \\ &= c_t y + E(p_t(\max(0, W_t - y))) + E(h_t(\max(0, y - W_t))). \end{aligned} \tag{3.8}$$

Now we establish useful properties of l_t and L_t.

LEMMA 3.2. (i) *l_t and L_t are convex functions*,
(ii) *there exists $\overline{L}_t$ such that $L_t(y) \le \overline{L}_t(1 + |y|)$*,
(iii) *$L_t(y) \to \infty$ as $|y| \to \infty$.*

Proof. First observe that the functions defined by

$$\widetilde{P}_t(y) = P_t(\max(0, -y)),$$
$$\widetilde{h}_t(y) = h_t(\max(0, y)), \qquad \widetilde{g}_t(y) = g_t(\max(0, -y))$$

are convex, as superpositions of convex and convex nondecreasing functions. Secondly, for all y, $E|\widetilde{P}_t(y - W_t)| < \infty$, $E|\widetilde{h}_t(y - W_t)| < \infty$ and $E|\widetilde{g}_t(y - W_t)| < \infty$ because $E(W_t) < \infty$ and h_t, g_t, P_t are bounded by linear functions (assumptions (2.2)(ii), Remark 3.3(ii)–(iii)). So by the last part of Remark 3.1 the functions $E(\widetilde{P}_t(y - W_t))$, $E(\widetilde{h}_t(y - W_t))$ and $E(\widetilde{g}_t(y - W_t))$ are convex with respect to y, which, together with the definitions (3.7) and (3.8), gives (i).

The assertion (ii) follows from the assumption that p_t, h_t are bounded by linear functions.

For the proof of (iii) note that

(a) $E(p_t(\max(0, W_t - y)) \ge p_t(0)$ and $E(h_t(\max(0, y - W_t))) \ge h_t(0)$. Hence by (3.8), $L_t(y) \ge c_t y + p_t(0) + h_t(0) \to \infty$ as $y \to \infty$.

(b) Consider $y < 0$, $y \to -\infty$. So $p_t(\max(0, W_t - y)) = p_t(W_t - y) \ge p_t'^{+}(W_t)(-y) + p_t(W_t)$ because p_t is a convex function. Hence by (3.8),

$$L_t(y) \ge h_t(0) + p_t(0) + (E(p_t'^{+}(W_t)) - c_t)(-y) \to \infty$$

as $y \to -\infty$ because by assumption (2.2)(iv), $E(p_t'^{+}(W_t)) - c_t > 0$, which completes the proof.

In the next section we turn to the dynamic equations (3.2).

4. Optimal strategy. By induction we prove the following main result of the paper.

THEOREM 4.1. *Under the assumptions* (2.2) *there exists a sequence of pairs* (s_t, S_t) *with* $s_t \le S_t$, $t = 0, 1, \ldots, N-1$, *such that*

$$u_t^*(x) = \begin{cases} S_t - x & \text{for } x \le s_t, \\ 0 & \text{for } x > s_t, \end{cases}$$

is an optimal strategy for the problem (2.1)–(2.4).

Proof. *Step* 1. Consider $t = N-1$ and the dynamic equation of the form (3.2). Note that in this case $G_{N-1} = L_{N-1}$, where L_{N-1} is given by (3.8). Thus, by Lemma 3.2,

(a) G_{N-1} is a convex function and hence it is K_{N-1}-convex and continuous on $\mathbb{R}$,

(b) there exists $\overline{G}_{N-1}$ such that $G_{N-1}(y) \le \overline{G}_{N-1}(1+|y|)$,

(c) $G_{N-1}(y) \to \infty$ as $|y| \to \infty$.

Hence by Lemma 3.1(iv) there exist s_{N-1} and S_{N-1}, $s_{N-1} \le S_{N-1}$, such that

$$y_{N-1}^*(x) = \begin{cases} S_{N-1} & \text{if } x \le s_{N-1}, \\ x & \text{if } x > s_{N-1}, \end{cases}$$

which by (3.3) gives the assertion of the theorem for $t = N-1$.

Step 2. Let $t = N-k$, $N-1 \ge k \ge 1$. We specify the induction assumptions:

(i) G_t is K_t-convex and a continuous,

(ii) there exists $\overline{G}_t$ such that $G_t(y) \le \overline{G}_t(1+|y|)$,

(iii) $G_t(y) \to \infty$ as $|y| \to \infty$.

Note that by (3.2) and Lemma 3.1 the induction assumptions imply that there exists a pair (s_t, S_t) such that

$$y_t^*(x) = \begin{cases} S_t & \text{if } x \le s_t, \\ x & \text{if } x > s_t, \end{cases}$$

and moreover, the assertion (v) of Lemma 3.1 shows that

$$V_t(x) = -c_t x + \begin{cases} K_t + G_t(S_t) & \text{if } x \le s_t, \\ G_t(x) & \text{if } x > s_t, \end{cases} \tag{4.1}$$

is a continuous K_t-convex function. By the induction assumptions (i)–(iii) we have

$$\begin{aligned} &\text{(a) } G_t(S_t) \le G_t(y),\ y \in \mathbb{R};\ G_t(s_t) = K_t + G_t(S_t); \\ &\text{(b) } G_t(y) \le K_t + G_t(x) \text{ for all } x, y \text{ such that } s_t \le y \le x. \end{aligned} \tag{4.2}$$

Remark 3.2 says that there exists $\overline{V}_t$ such that

$$V_t(x) \le \overline{V}_t(1+|x|). \tag{4.3}$$

In order to prove the theorem it is sufficient to show that (i)–(iii) are satisfied for G_{t-1}. We recall the formulas for G_{t-1}:

$$\begin{aligned} G_{t-1}(y) &= c_{t-1}y + E(h_{t-1}(\max(0, y - W_{t-1}))) \\ &\quad + E(p_{t-1}(\max(0, W_{t-1} - y))) \\ &\quad + E(V_t(\max(-\beta_{t-1}, y - W_{t-1}))) \\ &= L_{t-1}(y) + E(V_t(\max(-\beta_{t-1}, y - W_{t-1}))) \\ &= l_{t-1}(y) + E(g_{t-1}(\max(0, W_{t-1} - y))) \\ &\quad + E(V_t(\max(-\beta_{t-1}, y - W_{t-1}))), \end{aligned}$$

where l_{t-1} and L_{t-1} are given by (3.7), (3.8).

Step 3. In this step we will show that

(a) $G_{t-1}(y) \to \infty$ as $|y| \to \infty$,
(b) G_{t-1} is a continuous function on $\mathbb{R}$,
(c) there exists $\overline{G}_{t-1}$ such that $G_{t-1}(y) \le \overline{G}_{t-1}(1 + |y|)$.

For the proof of (a) we recall that by the definition (3.1), $V_t \ge 0$. So together by Lemma 3.2(iii) we have $G_{t-1}(y) \ge L_{t-1}(y) \to \infty$ as $|y| \to \infty$.

For (b) observe that by Lemma 3.2(i), L_{t-1} is convex on $\mathbb{R}$ and so continuous. Thus it is sufficient to show that the map $y \to E(V_t(\max(-\beta_{t-1}, y - W_{t-1})))$ is continuous. Let $\{y_n\} \subset \mathbb{R}$ be a convergent sequence. Then (4.3) implies that

$$V_t(\max(-\beta_{t-1}, y_n - W_{t-1})) \le \overline{V}_t(1 + \sup |y_n| + W_{t-1}).$$

Observe that $E(\overline{V}_t(1 + \sup |y_n| + W_{t-1})) < \infty$ because $E(W_{t-1}) < \infty$. Thus the continuity follows from Lebesgue's theorem and the continuity of V_t.

The assertion (c) is a consequence of (4.3), Lemma 3.2(ii) and the assumption that $E(W_{t-1}) < \infty$.

For the proof of the theorem it remains to show that G_{t-1} is K_{t-1}-convex.

This will be done in the next step and in the Appendix.

Step 4. Now it is convenient to write G_{t-1} in the form

$$G_{t-1}(y) = l_{t-1}(y) + R(y),$$

where R is defined as

$$R(y) = E(g_{t-1}(\max(0, W_{t-1} - y))) + E(V_t(\max(-\beta_{t-1}, y - W_{t-1}))).$$

Lemma 3.2(i) states that l_{t-1} is convex and so 0-convex. Thus it is sufficient to prove that R is K_{t-1}-convex.

Let

$$r(y) = g_{t-1}(\max(0, -y)) + V_t(\max(-\beta_{t-1}, y)). \tag{4.4}$$

Hence, $R(y) = E(r(y - W_{t-1}))$ and by Remark 3.3(iii), (4.3) and the assumption that $E(W_{t-1}) < \infty$ we can assert that $E|r(y - W_{t-1})| < \infty$. So by the last part of Remark 3.1 it is sufficient to show that $r(\cdot)$ is a K_{t-1}-convex function. This will be done in the Appendix and will finish the proof of this theorem.

Remarks to the proof. For the case $\beta_t = 0, t = 0, 1, \ldots, N-1$, it was remarked by Bertsekas [2, p. 10] that K_t-convexity of G_t does not imply K_t-convexity of $E(V_t(\max(0, y-W_{t-1})))$. This explains the reason for which we introduce the function g_{t-1} and prove the K_{t-1}-convexity of R. The argument is similar to that given by Bertsekas [2, pp. 105–106] and Bensoussan *et al.* [1, pp. 345–348].

5. Appendix. **K_{t-1}-convexity of r.** We recall that by (3.6) and Remark 3.3(iv),

$$g_{t-1}(y) = \begin{cases} 0 & \text{for } y \in [0, \beta_{t-1}), \\ p_{t-1}(y) - (lp_{t-1})(y) & \text{for } y \in [\beta_{t-1}, \infty), \end{cases}$$

is a convex function with

$$g'^{+}_{t-1}(\beta_{t-1}) > c_t. \tag{5.1}$$

Thus the definition of g_{t-1} gives

$$g_{t-1}(\max(0, -y)) = \begin{cases} 0 & \text{for } y \ge -\beta_{t-1}, \\ g_{t-1}(-y) & \text{for } y < -\beta_{t-1}. \end{cases} \tag{5.2}$$

NOTATION. It is convenient in this section to drop some subscripts and put

$$g_{t-1} = g, \quad \beta_{t-1} = \beta \quad \text{and} \quad V_t = V, \quad G_t = G, \quad s_t = s, \quad S_t = S, \quad c_t = c.$$

Let us start with the following

LEMMA 5.1. *For $x \ge -\beta$ we have*

$$K_t + V(x) \ge V(-\beta) - g'^{+}(\beta)(x + \beta).$$

Proof. Consider three cases:

(i) $-\beta \le s < x$. By (4.1), (4.2)(a) and (5.1) we obtain

$$\begin{aligned} K_t + V(x) &= K_t - cx + G(x) \ge K_t - cx + G(S) \\ &= V(-\beta) + c(-\beta) - cx = V(-\beta) - c(x + \beta) \\ &\ge V(-\beta) - g'^{+}(\beta)(\beta + x). \end{aligned}$$

(ii) $-\beta \le x \le s$. Using (4.1) and (5.1) we get

$$K_t + V(x) = 2K_t - cx + G(S) \ge K_t - cx + G(S)$$

and the remaining part of the proof is as in (i).

(iii) $s < -\beta \le x$. By (4.1), (4.2)(b) and (5.1) we have

$$\begin{aligned} K_t + V(x) &= K_t - cx + G(x) \ge -cx + G(-\beta) \\ &= -cx + (V(-\beta) - c\beta) \ge V(-\beta) - g'^{+}(\beta)(x+\beta). \end{aligned}$$

Now we are in a position to check the K_{t-1}-convexity (Definition 3.1) of r.

PROPOSITION 5.1. *The function r given by* (4.4) *satisfies the inequality*

$$K_{t-1} + r(y+z) \ge r(y) + z\frac{r(y) - r(y-b)}{b} \tag{5.3}$$

for all $y \in \mathbb{R}$, $z \ge 0$, $b > 0$.

Proof. Let $y \in \mathbb{R}$, $z \ge 0$ and $b > 0$. Consider four cases.

(i) $-\beta \le y - b < y \le y + z$. By (4.4) and (5.2),

$$r(y+z) = 0 + V(y+z), \quad r(y) = 0 + V(y), \quad r(y-b) = 0 + V(y-b)$$

and (5.3) is a consequence of the K_t-convexity of V and the assumption (2.2)(ii) that $K_{t-1} \ge K_t$.

(ii) $y - b < y \le y + z \le -\beta$. Then (4.4) and (5.2) give $r(y+z) = g(-y-z) + V(-\beta)$, $r(y) = g(-y) + V(-\beta)$ and $r(y-b) = g(-(y-b)) + V(-\beta)$. The inequality (5.3) follows from the convexity of $g(\max(0,-y))$.

(iii) $y - b < y \le -\beta \le y + z$. In this case $r(y+z) = 0 + V(y+z)$, $r(y) = g(-y) + V(-\beta)$ and $r(y-b) = g(-(y-b)) + V(-\beta)$. So (5.3) and the assumptions $K_{t-1} \ge K_t$ and $g(\beta) = 0$ make it sufficient to show that

$$K_t + V(y+z) \ge g(-y) - g(\beta) + V(-\beta) + z\frac{g(-y) - g(-y+b)}{b}, \tag{5.4}$$

For this observe that from the convexity of g it follows that

$$g(\beta) - g(-y) = g(-y + (\beta + y)) - g(-y) \ge g'^{+}(-y)(\beta + y)$$

and

$$g(-y+b) - g(-y) \ge g'^{+}(-y)b.$$

Thus

$$\begin{aligned} g(-y) - g(\beta) + V(-\beta) + z\frac{g(-y) - g(-y+b)}{b} & \\ &\le -g'^{+}(-y)(\beta + y) + V(-\beta) + \frac{z}{b}(-g'^{+}(-y)b) \\ &= V(-\beta) - g'^{+}(-y)(\beta + y + z) \\ &\le V(-\beta) - g'^{+}(\beta)(\beta + y + z) \le K_t + V(y+z) \end{aligned}$$

because $g'^{+}(-y) \ge g'^{+}(\beta)$, $y + z \ge -\beta$ and we can use Lemma 5.1. So (5.4) is proved.

(iv) $y - b < -\beta < y \le y + z$. In this case we have $0 < y + \beta < b$ and $r(y+z) = V(y+z)$, $r(y) = V(y)$, $r(y-b) = V(-\beta) + g(-(y-b))$, which means that it is sufficient to check

$$K_t + V(y+z) \ge V(y) + z\frac{V(y) - V(-\beta) - g(b-y) + g(\beta)}{b}. \tag{5.5}$$

For the proof of (5.5) consider two opposite subcases:

$$\text{(a)} \qquad \frac{V(y) - V(-\beta)}{y+\beta} \ge \frac{V(y) - V(-\beta) + g(\beta) - g(b-y)}{b},$$

$$\text{(b)} \qquad \frac{V(y) - V(-\beta)}{y+\beta} < \frac{V(y) - V(-\beta) + g(\beta) - g(b-y)}{b}.$$

Case (a). V is K_t-convex. Thus

$$\begin{aligned} K_t + V(y+z) &\ge V(y) + z\frac{V(y) - V(-\beta)}{y+\beta} \\ &\ge V(y) + z\frac{V(y) - V(-\beta) + g(\beta) - g(b-y)}{b}. \end{aligned}$$

Case (b). The inequality (b) implies that

$$\frac{b(V(y) - V(-\beta)) - (y+\beta)(V(y) - V(-\beta))}{b(y+\beta)} < \frac{g(\beta) - g(b-y)}{b}$$

and so

$$V(y) < V(-\beta) + \frac{(y+\beta)(g(\beta) - g(b-y))}{b - y - \beta};$$

this implies that

$$\begin{aligned} V(y) &+ z\frac{V(y) - V(-\beta) + g(\beta) - g(b-y)}{b} \\ &\le V(-\beta) + \frac{(y+\beta)(g(\beta) - g(b-y))}{b-y-\beta} \\ &\quad + \frac{z}{b}\,\frac{(y+\beta)(g(\beta) - g(b-y)) + (b-y-\beta))(g(\beta) - g(b-y))}{b-y-\beta} \\ &= V(-\beta) + \frac{(y+\beta+z)(g(\beta) - g(b-y))}{b-y-\beta} \\ &\le V(-\beta) + \frac{(y+\beta+z)(-g'^{+}(\beta))(b-y-\beta)}{b-y-\beta} \\ &= V(-\beta) - g'^{+}(\beta)(y+\beta+z). \end{aligned}$$

Since $y + z \ge -\beta$, by Lemma 5.1 we have $V(-\beta) - g'^{+}(\beta)(y+\beta+z) \le K_t + V(y+z)$, which proves (5.5). This finishes the proof of Proposition 5.1 and so of Theorem 4.1.

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