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Lagrange multipliers theorem and saddle point optimality criteria in mathematical programming

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Abstract

We prove a version of Lagrange multipliers theorem for nonsmooth functionals defined on normed spaces. Applying these results, we extend some results about saddle point optimality criteria in mathematical programming.

Keywords: Lagrange multipliers theorem; Saddle point; Mathematical programming

1. Introduction

Let U be an open neighborhood of a vector x in a normed space E, $g_1, \ldots, g_N$ and f be real functions on U. We consider the following optimization problem with inequality constraints:

$$(PI) \quad \begin{array}{ll} \min f(y) & \\ \text{s.t.} \quad g_i(y) \leqslant 0 & \forall i = 1, \ldots, N. \end{array}$$

If E is a Banach space, x is a solution of (PI), $g_1, \ldots, g_N$ and f are Fréchet differentiable at x and $D((g_1, \ldots, g_N))(x)(E) = \mathbb{R}^N$, Ioffe and Tihomirov in [7] proved that there are a real number a_0 and nonpositive real numbers $a_1, \ldots, a_N$ such that $(a_0, \ldots, a_N) \neq 0 \in \mathbb{R}^{N+1}$ and

$$a_0 Df(x) = a_1 Dg_1(x) + \cdots + a_N Dg_N(x).$$

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In [4], Halkin extended this result to the following optimization problem:

$$(PIE)\quad \begin{aligned} &\min f(y) \\ &\text{s.t.}\quad g_i(y) \leqslant 0 \quad \forall i = 1, \dots, n, \\ &g_{n+j}(y) = 0 \quad \forall j = 1, \dots, N - n. \end{aligned}$$

Halkin also proved that $a_1, \dots, a_n$ are nonpositive. Furthermore, if $g_1, \dots, g_N$ are not only Fréchet differentiable but also C^1 at x, that is the Fréchet derivatives of $g_1, \dots, g_N$ exist on a neighborhood of x and are continuous at x, Ioffe and Tihomirov in [7, p. 73] proved that a_0 in their cited result is not equal to 0.

Using the notation of generalized gradients, Clarke, Ioffe, Michel and Penot, Mordukhovich, Rockafellar and Treiman have extended the above results for Lipschitz constraint functions in [3,5,6,10,11,17,19,20]. In [21,22], Ye considered the problem (*PIE*) with mixed assumptions of Gâteaux, Fréchet differentiability and Lipschitz continuity of constraint functions. In [8,9,12–16], using the extremal principle, Kruger, Mordukhovich and Wang consider the problem (*PIE*) for locally Lipschitzian constraint functions on subsets in Asplund spaces.

If $N = 1$, we reduced the Lagrange multipliers rule to a two-dimensional problem in [1] and obtained a version of Lagrange multipliers theorem, in which we only required the smoothness of the restrictions of f and g on $F \cap U$, where F is any two-dimensional vector subspace containing x of E. This smoothness is very weak and may not imply the continuity of the functions.

In the next section of the present paper, we prove a discrete implicit mapping theorem (see Lemma 2.1) and apply it to extend the results in [1] to the case $N > 1$ for f and g, whose restrictions on $F \cap U$ are smooth for any $(N + 1)$-dimensional vector subspace F containing x of E. We note that our results can be applied to functions which are not C^1-Fréchet differentiable neither Lipschitz continuous, even they are not continuous at x (see Remark 2.3). Applying these results, we extend some results of Bector et al. [2] in the last section.

2. Lagrange multipliers rule

Let U be a nonempty open subset of a normed space $(E, \|\cdot\|_E)$, X be a linear subspace of E, Z be a finite-dimensional linear subspace of X and J be a mapping from U into a normed space Y. We consider Z as a normed subspace of X. Let v be a vector in X and x be in U. Denote by $Z(v)$ and $Z(x, v)$ the vector subspaces of E generated by $Z \cup \{v\}$ and $Z \cup \{x, v\}$, respectively. We put

$$U_{x,Z} = \{y \in Z\colon\ x + y \in U\},$$
$$J_{x,Z}(y) = J(x + y) \quad \forall y \in U_{x,Z}.$$

Then $U_{x,Z}$ is an open subset of Z. We say

(i) J is (X, Z)-continuous at x on U if and only if for every v in X, there is a positive real number η_v such that $J_{u,Z}$ is continuous at 0 for any u in $Z(x, v) \cap B_E(x, \eta_v)$;
(ii) J is X-differentiable at x if and only if there exists a linear mapping $DJ(x)$ from X into Y such that

$$\lim_{t\to 0} \frac{J(x + th) - J(x)}{t} = DJ(x)(h) \quad \forall h \in X;$$

(iii) J is (X, Z)-differentiable at x if and only if J is X-differentiable at x and for any v in X, if the sequence $\{(h_m, t_m)\}_{m\in\mathbb{N}} \subset Z(v) \times \mathbb{R}$ converges to $(h, 0)$ in $Z(v) \times \mathbb{R}$ then

$$\lim_{m\to\infty} \frac{J(x + t_m h_m) - J(x)}{t_m} = DJ(x)(h).$$

Remark 2.1. Let J be a linear mapping from E into $\mathbb{R}^n$, X be a linear subspace of E and Z be a finite-dimensional vector subspace of X. It is clear that J is (X, Z)-continuous at any x in E and (X, Z)-differentiable at any x in E although it may not be continuous on E.

Remark 2.2. If J is (X, F)-differentiable at x then there are a positive real number ϵ_v and a mapping ϕ_v from $B_E(0, \epsilon_v) \cap Z(v)$ into Y such that $B_E(x, \epsilon_v) \subset U$, $\lim_{z\to 0} \phi_v(z) = 0$ and

$$J(x + z) = J(x) + DJ(x)(z) + \|z\|_E \phi_v(z) \quad \forall z \in B_E(0, \epsilon_v) \cap Z(v).$$

Indeed, it is sufficient to prove that

$$\lim_{z\in Z(v),\, z\to 0} \frac{J(x + z) - J(x) - DJ(x)(z)}{\|z\|_E} = 0.$$

We assume by contradiction that there exist a sequence $\{z_m\}_{m\in\mathbb{N}}$ in $Z(v)$ and a positive real number ϵ such that $0 < \|z_m\|_E < m^{-1}$ and

$$\left\| \frac{J(x + z_m) - J(x) - DJ(x)(z_m)}{|z_m|_E} \right\|_Y > \epsilon \quad \forall n \in \mathbb{N}. \tag{$*$}$$

We put $s_m = \|z_m\|_E^{-1} z_m \in Z(v)$. Because $Z(v)$ is a finite-dimensional subspace of X, there exists a subsequence $\{s_{m_k}\}_{k\in\mathbb{N}}$ of $\{s_m\}_{m\in\mathbb{N}}$ such that $\lim_{k\to\infty} s_{m_k} = s$ in $Z(v)$. Because J is (X, F)-differentiable at x then

$$\lim_{k\to\infty} \frac{J(x + z_{m_k}) - J(x)}{\|z_{m_k}\|_E} = \lim_{k\to\infty} \frac{J(x + \|z_{m_k}\|_E s_{m_k}) - J(x)}{\|z_{m_k}\|_E} = DJ(x)(s).$$

Since $Z(v)$ is finite-dimensional, $Df(x)$ is continuous on $Z(v)$ and

$$\lim_{k\to\infty} DJ(x)\left(\frac{z_{m_k}}{\|z_{m_k}\|_E}\right) = \lim_{k\to\infty} DJ(x)(s_{m_k}) = DJ(x)(s).$$

Then we have

$$\lim_{k\to\infty} \frac{J(x + z_{m_k}) - J(x) - DJ(x)(z_{m_k})}{\|z_{m_k}\|_E} = 0,$$

we get the contradiction with $(*)$ and we get the result.

We have the following result:

Lemma 2.1 *(Discrete Implicit Mapping Theorem). Let U be an open neighborhood of a vector x in a normed linear space E, X be a vector subspace of E, F be a n-dimensional vector subspace of X and g be a mapping from U into $\mathbb{R}^n$ with $g = (g_1, \dots, g_n)$. Assume that g is (X, F)-continuous and (X, F)-differentiable at x.*

Put $M = \{y \in U\colon\ g(y) = g(x)\}$ and $e_i = (\delta_i^1, \dots, \delta_i^n)$ for any i in $\{1, \dots, n\}$, where δ_i^j is the Kronecker number.

Let v be in X and $h_1, \dots, h_n$ be n vectors in F such that $Dg(x)(v) = 0$ and $Dg(x)(h_i) = e_i$ for any i in $\{1, \dots, n\}$.

Then there exist a sequence $\{s_m = (s_m^1, \dots, s_m^n)\}_{m\in\mathbb{N}}$ *converging to* 0 *in* $\mathbb{R}^n$ *and a sequence of positive real numbers* $\{\alpha_m\}_{m\in\mathbb{N}}$ *converging to* 0 *in* $\mathbb{R}$ *such that*

$$u_m \equiv \alpha_m\big(s_m^1 h_1 + \cdots + s_m^n h_n\big) + \alpha_m v + x \in M \quad \forall m \in \mathbb{N}.$$

Proof. We can assume without loss of generality that $x = 0$ and $g(x) = 0$. Fix a vector v in X such that $Dg(0)(v) = 0$. By Remark 2.1 about the (X, F)-continuity and (X, F)-differentiability of g at 0, there is a positive real number ϵ_v and a mapping ϕ_v from $B_{F(v)}(0, \epsilon_v) \equiv F(v) \cap B_E(0, \epsilon_v)$ to $\mathbb{R}^n$ such that: $B_E(x, \epsilon_v) \subset U$, $g_{y,F}$ is continuous at 0 for every $y \in B_{F(v)}(0, \epsilon_v)$, $\lim_{y\to 0}\phi_v(y) = 0$ and

$$g(z) = Dg(0)(z) + \|z\|_E \phi_v(z) \quad \forall z \in B_{F(v)}(0, \epsilon_v).$$

There is a real positive number r such that $\alpha(s^1 h_1 + \cdots + s^n h_n + v)$ belongs to $B_{F(v)}(0, \epsilon_v)$ for any $(\alpha, s^1, \dots, s^n) \in (0, r) \times B_n(0, r)$, where $B_k(0, p) = \{t = (t^1, \dots, t^k) \in \mathbb{R}^k\colon\ \|t\|_{\mathbb{R}^k} \equiv \sqrt{(t^1)^2 + \cdots + (t^k)^2} < p\}$.

We put

$$\eta(s) = s^1 h_1 + \cdots + s^n h_n \quad \forall s = \big(s^1, \dots, s^n\big) \in B_n(0, r),$$
$$G_\alpha(s) = \alpha^{-1} g\big(\alpha\big(\eta(s) + v\big)\big) \quad \forall (\alpha, s) \in (0, r) \times B_n(0, r).$$

Because $g_{y,F}$ is continuous at 0 for every y in $B_{F(v)}(0, \epsilon_v)$, we see that G_α is continuous on $B_n(0, r)$ for any fixed α in $(0, r)$ and

$$\begin{aligned} G_\alpha(s) &= \alpha^{-1} g\big(\alpha\big(s^1 h_1 + \cdots + s^n h_n + v\big)\big) = \alpha^{-1} g\big(\alpha\big(\eta(s) + v\big)\big) \\ &= \alpha^{-1} Dg(0)\big(\alpha\big(\eta(s) + v\big)\big) + \alpha^{-1}\big\|\alpha\big(\eta(s) + v\big)\big\|_E \phi_v\big(\alpha\big(\eta(s) + v\big)\big) \\ &= Dg(0)\big(\eta(s)\big) + \big\|\eta(s) + v\big\|_E \phi_v\big(\alpha\big(\eta(s) + v\big)\big) \\ &= \big(s^1, \dots, s^n\big) + \big\|\eta(s) + v\big\|_E \phi_v\big(\alpha\big(\eta(s) + v\big)\big) \quad \forall s = \big(s^1, \dots, s^n\big) \in B_n(0, r). \end{aligned}$$

Note that there is a positive real number M such that $\|\eta(s) + v\|_E \leqslant M$ for any $s = (s^1, \dots, s^n)$ in $B_n(0, r)$. Thus

$$\lim_{\alpha\to 0}\big[\sup\big\{\big\|\phi_v\big(\alpha\big(\eta(s) + v\big)\big)\big\|_E\colon\ \big(s^1, \dots, s^n\big) \in B_n(0, r)\big\}\big] = 0$$

and

$$\begin{aligned} \big\langle G_\alpha(s), s\big\rangle &= \langle s, s\rangle + \big\|\eta(s) + v\big\|_E \big\langle \phi_v\big(\alpha\big(\eta(s) + v\big)\big), s\big\rangle \\ &\geqslant m^{-2}\big\{1 - mM\big\|\phi_v\big(\alpha\big(\eta(s) + v\big)\big)\big\|_E\big\} \quad \forall s \in \partial B_n\big(0, m^{-1}\big), \end{aligned}$$

where m is an integer greater than r^{-1}.

Thus there is a real number α_m in $(0, m^{-1})$ such that

$$\big\langle G_{\alpha_m}(s), s\big\rangle > 0 \quad \forall s \in \partial B_n\big(0, m^{-1}\big).$$

By Lemma 4.1 in [18, p. 14], we have a solution s_m in $B_n(0, m^{-1})$ to the equation $G_{\alpha_m}(s) = 0$ for any integer m greater than r^{-1}, which yields the lemma. □

Remark 2.3. If E is a Banach space, U is an open subset of E, g is Fréchet differentiable on U and Dg is continuous at x, then by the Ljusternik theorem (see [7, p. 41]), M is a manifold and the tangent space $TM_x = Dg(x)^{-1}(\{0\})$. Moreover, M is a C^1-manifold if g is C^1-Fréchet differentiable on U (see [23, Theorem 43.C]). The smoothness of g in Lemma 2.1 is very weak so that M may not be a smooth manifold and we cannot define the tangent space of M at x. Note that the sequence $\{\alpha_m^{-1}(u_m - x)\}_{m\in\mathbb{N}}$ converges to v in E. Therefore, we can admit v as a "generalized" tangent vector of M at x for any v in $Dg(x)^{-1}(\{0\})$, if we consider $\{u_m\}_{m\in\mathbb{N}}$ as a discrete curve passing through x. This idea is illustrated by the following example.

Let $E = X = U = \mathbb{R}^3$ and $Z = \mathbb{R} \times \{0\} \times \{0\}$. We define

$$\begin{aligned}
&P_1 = \bigl\{(s,t) \in \mathbb{R}^2\colon\ \exists(\alpha,\beta) \in \mathbb{Q}\times\mathbb{Q},\ \alpha s + \beta t = 0\bigr\},\\
&P_2 = \bigl\{(s,t) \in \mathbb{R}^2\colon\ \sqrt{s^2+t^2} \in \mathbb{Q}\bigr\},\\
&P = [P_1 \cap P_2] \cup \bigl[\bigl(\mathbb{R}^2 \setminus P_1\bigr) \cap \bigl(\mathbb{R}^2 \setminus P_2\bigr)\bigr],\\
&A = \bigl\{(r,s,t) \in \mathbb{R}^3\colon\ (s,t) \in P\bigr\},\\
&B = \bigl\{(r,s,t) \in \mathbb{R}^3\colon\ r = 0,\ s > 0,\ t = s^2\bigr\},\\
&g(r,s,t) = r + \bigl(s^2+t^2\bigr)\chi_A(r,s,t) + \chi_B(r,s,t) \quad \forall (r,s,t) \in \mathbb{R}^3,
\end{aligned}$$

where χ_C is the characteristic function of the set C.

Let $v = (0,s,t)$ in $\mathbb{R}^3$ such that $st \neq 0$. Put $\delta_v = \frac{1}{2}\sqrt{s^{-2}t^2 + s^{-4}t^4}$, we have $B_E((0,0,0),\delta_v) \cap Z(v) \cap B = \emptyset$. Therefore, for any v in $\{0\}\times\mathbb{R}^2$, there is a positive real number δ_v such that $B_E((0,0,0),\delta_v) \cap Z(v) \cap B = \emptyset$. Thus we see that g is (X,Z)-continuous at $(0,0,0)$ and (X,Z)-differentiable at $(0,0,0)$. But g is not Fréchet differentiable at $(0,0,0)$ because g is not continuous at $(0,0,0)$. Put $M = \{(r,s,t) \in \mathbb{R}^3\colon\ g(r,s,t) = g(0,0,0)\}$, we have

$$\begin{aligned}
&Dg\bigl((0,0,0)\bigr)(Z) = \mathbb{R},\\
&M = \bigl\{(0,s,t)\colon\ (s,t) \notin (P \cup B)\bigr\} \cup \bigl\{(r,s,t)\colon\ (s,t) \in (P \setminus B),\ r = -s^2 - t^2\bigr\}.
\end{aligned}$$

Note that $(0,s,t)$ is a "generalized" tangent vector of M at $(0,0,0)$ for any (s,t) in $\mathbb{R}^2$.

The results in [3–7,17,19–22] cannot be applied to this case. It is easy to derive this example to the case of vector functions.

The idea of "generalized" tangent vectors is essential to get the following generalized Lagrange multipliers theorem.

Theorem 2.1. *Let U be an open subset of normed vector space E, X be a vector subspace of E, F be a n-dimensional vector subspace of X, u be in U, r be in $\mathbb{R}^n$, f be a mapping from U into $\mathbb{R}$, $g = (g_1,\ldots,g_n)$ be a mapping from U into $\mathbb{R}^n$, $M = \{x \in U\colon\ g(x) = r\}$ and $u \in M$. Assume that*

(i) *$f(u)$ is the minimum (or maximum) of $f(M)$,*
(ii) *f is (X,F)-differentiable at u,*
(iii) *g is (X,F)-continuous at u and (X,F)-differentiable at u,*
(iv) *$Dg(u)(F) = \mathbb{R}^n$.*

Then there exists a unique mapping $\Lambda \in L(\mathbb{R}^n,\mathbb{R})$ such that

$$Df(u)(k) = \Lambda\bigl(Dg(u)(k)\bigr) \quad \forall k \in X.$$

Proof. Assume $f(u)$ is the minimum of $f(M)$. Choose n vectors $h_1, \dots, h_n$ in F such that $Dg(u)(h_i) = e_i \equiv (\delta_i^1, \dots, \delta_i^n)$ for any i in $\{1, \dots, n\}$. We define a real linear mapping Λ on $\mathbb{R}^n$ as follows:

$$\Lambda(e_i) = Df(u)(h_i) \quad \forall i = 1, \dots, n.$$

Now fix a vector k in X. Put

$$v = k - \sum_{i=1}^{n} Dg_i(u)(k)h_i \in X.$$

Then $Dg(u)(v) = 0$. By Lemma 2.1, there are a sequence $\{\alpha_m\}_{m\in\mathbb{N}}$ of positive real numbers and a sequence $\{s_m = (s_m^1, \dots, s_m^n)\}_{m\in\mathbb{N}}$ in $\mathbb{R}^n$ such that $\{\alpha_m\}_{m\in\mathbb{N}}$ and $\{s_m\}_{m\in\mathbb{N}}$ converge to 0 in $\mathbb{R}$ and $\mathbb{R}^n$ respectively, $u_m \in U$, and $g(u_m) = g(u)$, where

$$u_m = u + \alpha_m\big(s_m^1 h_1 + \cdots + s_m^n h_n\big) + \alpha_m v \quad \forall m \in \mathbb{N},$$

or $u_m \in M$ for every $m \in \mathbb{N}$.

Since $f(u)$ is the minimum of $f(M)$ and f is (X, F)-differentiable at u, we have

$$\begin{aligned}
&Df(u)(k) - \Lambda\big(Dg(u)(k)\big)\\
&\quad = Df(u)(k) - \Lambda\left(\sum_{i=1}^{n} Dg_i(u)(k)e_i\right)\\
&\quad = Df(u)(k) - \sum_{i=1}^{n} D_i(u)(k)\Lambda(e_i) = Df(u)(k) - \sum_{i=1}^{n} Dg_i(u)(k)Df(u)(h_i)\\
&\quad = Df(u)(k) - Df(u)\left(\sum_{i=1}^{n} Dg_i(u)(k)h_i\right) = Df(u)\left(k - \sum_{i=1}^{n} Dg_i(u)(k)h_i\right)\\
&\quad = Df(u)(v) = \lim_{m\to\infty} \frac{[f(u + \alpha_m(s_m^1 h_1 + \cdots + s_m^n h_n + v)) - f(u)]}{\alpha_m} \geqslant 0.
\end{aligned}$$

Therefore, $Df(u)(k) \geqslant \Lambda(Dg(u)(k))$ for any k in X. Replacing k in the above inequality by $-k$, we get the theorem.

Since $Dg(u)(F) = \mathbb{R}^n$, we can get the uniqueness of Λ.

The proof for the case $f(u) = \max f(M)$ is similar and omitted. $\square$

Remark 2.4. If E is a Banach space, g is Fréchet differentiable on U and Dg is continuous at u, then Theorem 2.1 has been proved in [7, p. 73]. Here we only need the differentiability of f and g at u. If $n = 1$, Theorem 2.1 has been proved in [1].

Theorem 2.2. *Let U be an open subset of normed vector space E, X be a vector subspace of E, F be a $(n+m)$-dimensional vector subspace of X, u be in U, f be a mapping from U into $\mathbb{R}$, $g = (g_1, \dots, g_{n+m})$ be a mapping from U into $\mathbb{R}^{n+m}$ and*

$$M = \big\{x \in U\colon\ g_i(x) \leqslant 0, g_{n+j}(x) = 0\ \forall i \in \{1, \dots, n\},\ j \in \{1, \dots, m\}\big\}.$$

Assume that

(i) *$u \in M$ and $f(u)$ is the minimum of $f(M)$,*

(ii) *f is (X,F)-differentiable at u,*
(iii) *g is (X,F)-continuous at u and (X,F)-differentiable at u,*
(iv) *$Dg(u)(F)=\mathbb{R}^{n+m}$.*

Then $g(u)=0$ and there exists a unique $(a_1,\dots,a_{n+m})$ in $\mathbb{R}^{n+m}$ such that $a_1,\dots,a_n$ are negative and

$$Df(u)(k)=\sum_{i=1}^{n+m}a_i Dg_i(u)(k)\quad \forall k\in X.$$

Proof. We prove the theorem by the following steps.

Step 1. Put $\alpha=(\alpha_1,\dots,\alpha_{n+m})=g(u)$ and $S=\{x\in U\colon\ g(x)=\alpha\}$ we see that $S\subset M$. Since $f(u)=\min f(S)$, by Theorem 2.1, there exists a unique $(a_1,\dots,a_{n+m})$ in $\mathbb{R}^{n+m}$ such that

$$Df(u)(k)=\sum_{i=1}^{n+m}a_i Dg_i(u)(k)\quad \forall k\in X.$$

Step 2. We prove that $a_i<0$ for every $i\in\{1,\dots,n\}$. First, we prove that $a_1<0$. Since $Dg(u)(F)=\mathbb{R}^{n+m}$, there is k in F such that $Dg_1(u)(k)=-1$ and $Dg_2(u)(k)=\cdots=Dg_n(u)(k)=-\varepsilon<0$ and $Dg_{n+j}(u)(k)=0$ for any j in $\{1,\dots,m\}$.

Let $\{h_i\}_{i=1,\dots,n+m}$ be in F such that $Dg(u)(h_i)=(\delta_i^1,\dots,\delta_i^{n+m})$ for any $i\in\{1,\dots,n+m\}$, we have

$$D(g_{n+1},\dots,g_{n+m})(u)(h_i)=\left(\delta_i^{n+1},\dots,\delta_i^{n+m}\right)\quad \forall i\in\{n+1,\dots,n+m\}.$$

By Lemma 2.1, there are a sequence $\{s_l=(s_l^{n+1},\dots,s_l^{n+m})\}_{l\in\mathbb{N}}$ converging to 0 in $\mathbb{R}^m$ and a sequence of positive real numbers $\{\alpha_l\}_{l\in\mathbb{N}}$ converging to 0 in $\mathbb{R}$ such that

$$\begin{aligned}&u_l\equiv\alpha_l\left(s_l^{n+1}h_{n+1}+\cdots+s_l^{n+m}h_{n+m}\right)+\alpha_l k+u,\\&(g_{n+1},\dots,g_{n+m})(u_l)=(g_{n+1},\dots,g_{n+m})(u)=0\quad \forall l\in\mathbb{N}.\end{aligned}$$

Note that

$$\begin{aligned}\lim_{l\to\infty}\frac{g_i(u_l)-g_i(u)}{\alpha_l}&=\lim_{l\to\infty}\frac{g_i(\alpha_l(s_l^{n+1}h_{n+1}+\cdots+s_l^{n+m}h_{n+m})+\alpha_l k+u)-g_i(u)}{\alpha_l}\\&=Dg_i(u)(k)<0\quad \forall i\in\{1,\dots,n\}.\end{aligned}$$

Thus there exists $l_0\in\mathbb{N}$ such that

$$g_i(u_l)<g_i(u)\quad \forall i\in\{1,\dots,n\},\ \forall l\geqslant l_0,$$

which implies that u_l is in M for any $l\geqslant l_0$ and

$$-a_1-\varepsilon\sum_{i=2}^{n}a_i=Df(u)(k)=\lim_{l\to\infty}\frac{f(u_l)-f(u)}{\alpha_l}\geqslant 0\quad \forall\varepsilon>0.$$

Let ε tend to 0, we have $-a_1\geqslant 0$ or $a_1\leqslant 0$. Since $Dg(u)(F)=\mathbb{R}^{n+m}$, we have $a_1<0$. Similarly, we have

$$a_i<0\quad \forall i=1,\dots,n.$$

Step 3. We shall prove $g(u)=0$. Put

$$I=\left\{i\in\{1,\dots,n\}\colon\ g_i(u)<0\right\}\quad\text{and}\quad J=\left\{j\in\{1,\dots,n\}\colon\ g_j(u)=0\right\}.$$

We assume by contradiction that I is not empty. In this case, we see that

$$\sum_{i\in I}a_i<0. \tag{1}$$

On the other hand, there is k in F such that

$$\begin{aligned}&Dg_i(u)(k)=1\quad\forall i\in I,\\&Dg_j(u)(k)=-\varepsilon\quad\forall j\in J,\\&Dg_{n+l}(u)(k)=0\quad\forall l=1,\dots,m.\end{aligned}$$

By Lemma 2.1, there are a sequence $\{s_l=(s_l^{n+1},\dots,s_l^{n+m})\}_{l\in\mathbb{N}}$ converging to 0 in $\mathbb{R}^m$ and a sequence of positive real numbers $\{\alpha_l\}_{l\in\mathbb{N}}$ converging to 0 in $\mathbb{R}$ such that

$$\begin{aligned}&u_l\equiv\alpha_l\big(s_l^{n+1}h_{n+1}+\cdots+s_l^{n+m}h_{n+m}\big)+\alpha_l k+u\quad\text{and}\\&0=(g_{n+1},\dots,g_{n+m})(u_l)=(g_{n+1},\dots,g_{n+m})(u)\quad\forall l\in\mathbb{N}.\end{aligned}$$

We have

$$\begin{aligned}\lim_{l\to\infty}\frac{g_i(u_l)-g_i(u)}{\alpha_l}&=\lim_{l\to\infty}\frac{g_i(\alpha_l(s_l^{n+1}h_{n+1}+\cdots+s_l^{n+m}h_{n+m})+\alpha_l k+u)-g_i(u)}{\alpha_l}\\&=Dg_i(u)(k)=1\quad\forall i\in I.\end{aligned}$$

Thus $\lim_{l\to\infty}g_i(u_l)=g_i(u)<0$. Then there exists an integer l_1 such that

$$g_i(u_l)<0\quad\forall i\in I,\ \forall l\geqslant l_1.$$

We have

$$\begin{aligned}\lim_{l\to\infty}\frac{g_j(u_l)-g_j(u)}{\alpha_l}&=\lim_{l\to\infty}\frac{g_j(\alpha_l(s_l^{n+1}h_{n+1}+\cdots+s_l^{n+m}h_{n+m})+\alpha_l k+u)-g_j(u)}{\alpha_l}\\&=Dg_j(u)(k)=-\varepsilon<0\quad\forall j\in J.\end{aligned}$$

Thus there exists $l_2\in\mathbb{N}$ such that

$$g_j(u_l)<g_j(u)=0\quad\forall j\in J,\ \forall l\geqslant l_2.$$

Therefore u_l is in M for any $l\geqslant\max\{l_1,l_2\}$ and

$$\sum_{i\in I}a_i-\varepsilon\sum_{j\in J}a_j=Df(u)(k)=\lim_{l\to\infty}\frac{f(u_l)-f(u)}{\alpha_l}\geqslant 0\quad\forall\varepsilon>0.$$

It implies that $\sum_{i\in I}a_i\geqslant 0$, which contradicts to (1) and I should be empty and we get the result. $\square$

3. Applications in programming problems

Let $X \equiv \mathbb{R}^n$ denote the n-dimensional Euclidean space and let $\mathbb{R}^n_+$ be its nonnegative orthant. Let $f, h_1, \ldots, h_m$ be real functions on an open subset S of $\mathbb{R}^n$. We consider the following nonlinear programming problem

$$(P) \quad \begin{array}{l} \min f(x) \\ \text{subject to} \quad h_j(x) \leqslant 0 \quad \forall j \in \{1, 2, \ldots, m\}. \end{array}$$

Put $J = \{1, \ldots, r\}$ and $K = \{r+1, \ldots, m\}$, $h_J(x) = (h_1(x), \ldots, h_r(x))$ and $h_K(x) = (h_{r+1}(x), \ldots, h_m(x))$ for any x in S. Let $h(x)$ denote the column vector $(h_1(x), h_2(x), \ldots, h_m(x))^T$ and be partitioned as $h(x) = (h_J(x), h_K(x))^T$. We denote

$$S_h = \{x \in S\colon\ h_j(x) \leqslant 0,\ j \in \{1, \ldots, m\}\},$$
$$S_K = \{x \in S\colon\ h_k(x) \leqslant 0,\ k \in K\}.$$

Let $a = (a_1, \ldots, a_N) \in \mathbb{R}^N$ with $N \in \mathbb{N}$. We say $a \geqslant 0$ (or $\leqslant 0$) if and only if $a_i \geqslant 0$ (or $\leqslant 0$) for every $i \in \{1, \ldots, N\}$.

Definition 3.1. Let S be an open subset of $\mathbb{R}^n$, u be in S, g be a real function on S and Gâteaux differentiable at u, and η is a function from $S \times S$ to $\mathbb{R}^n$. We say

(i) the function g is said to be invex at u with respect to the function η, if for every $x \in S$, we have

$$g(x) - g(u) \geqslant \eta(x, u)^T \nabla g(u);$$

(ii) the function g is said to be pseudo-invex at u with respect to the function η, if for every $x \in S$, we have

$$\eta(x, u)^T \nabla g(u) \geqslant 0 \quad \Rightarrow \quad g(x) \geqslant g(u);$$

(iii) the function g is said to be quasi-invex at u with respect to the function η, if for every $x \in S$, we have

$$g(x) \leqslant g(u) \quad \Rightarrow \quad \eta(x, u)^T \nabla g(u) \leqslant 0.$$

Applying results in the first section, we study the following programming problems.

Problem 1: Nonlinear programming problem

Let $f, g, h_1, \ldots, h_m$ be real functions on an open subset S of $\mathbb{R}^n$. Let J, K, h_J and h_K be as in the beginning of this section.

Put $L(x, \lambda_J) = f(x) + (\lambda_J)^T h_J(x)$ for any (x, λ_J) in $S_K \times \mathbb{R}^{|J|}_+$. The map L is called the incomplete Lagrange function of the problem (P).

Definition 3.2. A point $(\bar{x}, \bar{\lambda}_J) \in S_K \times \mathbb{R}^{|J|}_+$ is called a saddle point of the incomplete Lagrange function L if

$$L(\bar{x}, \lambda_J) \leqslant L(\bar{x}, \bar{\lambda}_J) \leqslant L(x, \bar{\lambda}_J) \quad \forall (x, \lambda_J) \in S_K \times \mathbb{R}^{|J|}_+.$$

Theorem 3.1. *Let S be an open subset of $X \equiv \mathbb{R}^n$, f and $h_1, \dots, h_m$ be real functions on S and Gâteaux differentiable at $\bar{x}$ with $\bar{x} \in S$. F be a m-dimensional vector subspace of X and $\bar{x}$ be optimal for (P). Assume that*

(i) *$L(\cdot, \lambda_J)$ is pseudo invex and $(\lambda_K)^T h_K(\cdot)$ is quasi invex at $\bar{x}$ with respect to a function η for any (λ_J, λ_K) in $\mathbb{R}^m_+$;*
(ii) *f is (X, F)- differentiable at $\bar{x}$;*
(iii) *h is (X, F)-differentiable at $\bar{x}$ and (X, F)-continuous at $\bar{x}$;*
(iv) *$Dh(\bar{x})(F) = \mathbb{R}^m$.*

Then there exists $\bar{\lambda}_J \in \mathbb{R}^{|J|}_+$ such that $(\bar{x}, \bar{\lambda}_J)$ is a saddle point of the incomplete Lagrange function L.

Proof. Applying Theorem 2.2 in case $E = X = \mathbb{R}^n$, $U = S$ for f and $\{h_j\}_{j=\overline{1,m}}$, there exists $\bar{\lambda} = (\bar{\lambda}_J, \bar{\lambda}_K) \in \mathbb{R}^m$, $\bar{\lambda}_J \in \mathbb{R}^{|J|}_+$, $\bar{\lambda}_K \in \mathbb{R}^{|K|}_+$, such that

$$\begin{aligned}
&\nabla\big[f(\bar{x}) + (\lambda_J)^T h_J(\bar{x}) + (\lambda_K)^T h_K(\bar{x})\big] = 0,\\
&(\lambda_J)^T h_J(\bar{x}) = 0, \qquad (\lambda_K)^T h_K(\bar{x}) = 0,\\
&\bar{\lambda} = (\bar{\lambda}_J, \bar{\lambda}_K) \geqslant 0.
\end{aligned}$$

Now arguing as in the proof of Theorem 2.1 in [2], we get the theorem. □

We consider the following propositions.

(A1) $\bar{x}$ be optimal for (P).
(A2) There are positive real numbers $\lambda_1, \dots, \lambda_m$ such that $h_i(\bar{x}) = 0$ for any i in $\{1, \dots, m\}$ and $Df(\bar{x}) + \lambda_1 Dh_1(\bar{x}) + \cdots + \lambda_m Dh_m(\bar{x}) = 0$.

We have the following result.

Theorem 3.2. *Let S be an open subset of $X \equiv \mathbb{R}^n$, f and $h_1, \dots, h_m$ be real functions on S. Let $\bar{x}$ be in S. Then*

(i) *if f, $h_1, \dots, h_m$ satisfy the conditions* (ii), (iii) *and* (iv) *of Theorem* 3.1*, then* (A1) *implies* (A2);
(ii) *if $f(\cdot) + \sum_{i=1}^m \lambda_i h_i(\cdot)$ is Gâteaux differentiable at $\bar{x}$ and invex with respect to the function η at $\bar{x}$ then* (A2) *implies* (A1).

Proof. Applying Theorem 2.2 in case $E = X = \mathbb{R}^n$, $U = S$ for f and $\{h_j\}_{j=\overline{1,m}}$, we get (i).

Consider (ii). By the invexity property of $f(\cdot) + \sum_{i=1}^m \lambda_i h_i(\cdot)$ with respect to the function η at $\bar{x}$, we have

$$\begin{aligned}
&f(x) + \sum_{i=1}^m \lambda_i h_i(x) - \left[f(\bar{x}) + \sum_{i=1}^m \lambda_i h_i(\bar{x})\right]\\
&\quad\geqslant \eta(x, \bar{x})^T \nabla\left[f + \sum_{i=1}^m \lambda_i h_i\right](\bar{x}) = 0 \quad \forall x \in S_h.
\end{aligned}$$

It follows that

$$f(x) \geqslant f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) \geqslant f(\bar{x}) + \sum_{i=1}^{m} \lambda_i h_i(\bar{x}) = f(\bar{x}) \quad \forall x \in S_h,$$

and we get (ii). □

Problem 2: Fractional programming problem

Let $f, g, h_1, \dots, h_m$ be real functions on an open subset S of $\mathbb{R}^n$. Let J, K, h_J and h_K be as in the beginning of this section. Put

$$S_h = \{x \in S\colon\ h_j(x) \leqslant 0,\ j \in \{1, \dots, m\}\},$$
$$S_K = \{x \in S\colon\ h_k(x) \leqslant 0,\ k \in K\}.$$

Assume that $g(x) \neq 0$ for any x in S and $g(x) > 0$ for any x in S_K. We now consider the fractional programming problem

$$(FP) \quad \begin{aligned} &\min \frac{f(x)}{g(x)} \\ &\text{subject to} \quad h_j(x) \leqslant 0 \quad \forall j \in \{1, 2, \dots, m\}. \end{aligned}$$

We note that if $\bar{x}$ is optimal for the problem (FP) then $\bar{x}$ is also optimal for the following problem

$$(FP1) \quad \begin{aligned} &\min \frac{f(x)}{g(x)} \\ &\text{subject to} \quad \frac{h_j(x)}{g(x)} \leqslant 0 \quad \forall j \in J \text{ and } h_k(x) \leqslant 0\ \forall k \in K. \end{aligned}$$

This form of $(FP1)$ suggests the choice of the incomplete Lagrange function $L_F : S_K \times \mathbb{R}_+^{|J|} \to \mathbb{R}$ as follows:

$$L_F(x, \lambda_J) = \frac{f(x) + (\lambda_J)^T h_J(x)}{g(x)}.$$

Definition 3.3. A point $(\bar{x}, \bar{\lambda}_J) \in S_K \times \mathbb{R}_+^{|J|}$ is called a saddle point of the incomplete Lagrange function L_F if

$$L_F(\bar{x}, \lambda_J) \leqslant L_F(\bar{x}, \bar{\lambda}_J) \leqslant L_F(x, \bar{\lambda}_J) \quad \forall (x, \lambda_J) \in S_K \times \mathbb{R}_+^{|J|}.$$

Theorem 3.3. *Let S be an open subset of $X \equiv \mathbb{R}^n$, f and $h_1, \dots, h_m$ be real functions on S and Gâteaux differentiable at $\bar{x}$ with $\bar{x} \in S$. F be a m-dimensional vector subspace of X and $\bar{x}$ be optimal for (FP). Assume that*

(i) *$L_F(\cdot, \lambda_J)$ is pseudo invex and $(\lambda_K)^T h_K(\cdot)$ is quasi invex at $\bar{x}$ with respect to a function η for any (λ_J, λ_K) in $\mathbb{R}_+^m$;*
(ii) *$\frac{f}{g}$ is (X, F)-differentiable at $\bar{x}$;*
(iii) *$\frac{h_1}{g}, \dots, \frac{h_r}{g}, h_{r+1}, \dots, h_m$ are (X, F)-differentiable at $\bar{x}$ and (X, F)-continuous at $\bar{x}$;*
(iv) *$D((\frac{h_1}{g}, \dots, \frac{h_r}{g}, h_{r+1}, \dots, h_m))(\bar{x})(F) = \mathbb{R}^m$.*

Then there exists $\bar{\lambda}_J \in \mathbb{R}_+^{|J|}$ such that $(\bar{x}, \bar{\lambda}_J)$ is a saddle point of the incomplete Lagrange function L_F.

Proof. Since $\bar{x}$ is optimal for (*FP*) (and hence for (*FP*1)), applying Theorem 2.2 in case $E = X = \mathbb{R}^n$, $U = S$ for $\frac{f}{g}$ and $\frac{h_1}{g}, \dots, \frac{h_r}{g}, h_{r+1}, \dots, h_m$, we can find $\bar{\lambda} = (\bar{\lambda}_J, \bar{\lambda}_K)$ in $\mathbb{R}_+^{|J|} \times \mathbb{R}_+^{|K|}$, such that

$$\nabla\left[\frac{f(\bar{x})}{g(\bar{x})} + (\bar{\lambda}_J)^T \frac{h_J(\bar{x})}{g(\bar{x})} + (\bar{\lambda}_K)^T h_K(\bar{x})\right] = 0,$$
$$(\lambda_J)^T h_J(\bar{x}) = 0 \quad \text{and} \quad (\lambda_K)^T h_K(\bar{x}) = 0.$$

Fix x in S_K, we have $(\bar{\lambda}_K)^T h_K(x) \leqslant 0 = (\bar{\lambda}_K)^T h_K(\bar{x})$. By the quasi-invexity of $(\bar{\lambda}_K)^T h_K(\cdot)$ with respect to the function η at $\bar{x}$, we obtain

$$\eta(x, \bar{x})^T \nabla\big[(\bar{\lambda}_K)^T h_K(\bar{x})\big] \leqslant 0.$$

Thus

$$\eta(x, \bar{x})^T \nabla\left[\frac{f(\bar{x})}{g(\bar{x})} + (\bar{\lambda}_J)^T \frac{h_J(\bar{x})}{g(\bar{x})}\right] \geqslant 0.$$

By the pseudo-invexity of $\frac{f(\cdot)+(\bar{\lambda}_J)^T h_J(\cdot)}{g(\cdot)}$ with respect to the function η at $\bar{x}$, we see that

$$\frac{f(x)}{g(x)} + (\bar{\lambda}_J)^T \frac{h_J(x)}{g(x)} \geqslant \frac{f(\bar{x})}{g(\bar{x})} + (\bar{\lambda}_J)^T \frac{h_J(\bar{x})}{g(\bar{x})}.$$

And we get

$$L_F(x, \bar{\lambda}_J) \geqslant L_F(\bar{x}, \bar{\lambda}_J).$$

If λ_J is in $\mathbb{R}_+^{|J|}$, we have

$$L_F(\bar{x}, \bar{\lambda}_J) = \frac{f(\bar{x})}{g(\bar{x})} + (\bar{\lambda}_J)^T \frac{h_J(\bar{x})}{g(\bar{x})} = \frac{f(\bar{x})}{g(\bar{x})} \geqslant \frac{f(\bar{x})}{g(\bar{x})} + (\lambda_J)^T \frac{h_J(\bar{x})}{g(\bar{x})} = L_F(\bar{x}, \lambda_J).$$

Thus we get the theorem. □

Problem 3: Generalized fractional programming problem

Let $f_1, \dots, f_p, g_1, \dots, g_p, h_1, \dots, h_m$ be real functions on an open subset S of $\mathbb{R}^n$. Put

$$S_h = \big\{x \in S\colon\ h_j(x) \leqslant 0,\ j \in \{1, \dots, m\}\big\},$$
$$S_K = \big\{x \in S\colon\ h_k(x) \leqslant 0,\ k \in K\big\}.$$

Assume that $g_1(x), \dots, g_p(x) \neq 0$ for any x in S and $g_1(x), \dots, g_p(x)$ are positive for every x in S_h. We consider the generalized fractional programming problem

$$(GFP) \qquad \min_{x \in S} \max_{1 \leqslant i \leqslant p} \frac{f_i(x)}{g_i(x)}$$
$$\text{subject to} \quad h_j(x) \leqslant 0 \quad \forall j \in \{1, 2, \dots, m\}.$$

Put $Y = \{y \in \mathbb{R}_+^p, \sum_{i=1}^p y_i = 1\}$. The incomplete Lagrange function $L_G : S_K \times Y \times \mathbb{R}_+^{|J|} \to \mathbb{R}$ for the problem (*GFP*) can be chosen as

$$L_G(x, y, \lambda_J) = \frac{y^T f(x) + (\lambda_J)^T h_J(x)}{y^T g(x)}.$$

Definition 3.4. A point $(\bar{x}, \bar{y}, \bar{\lambda}_J) \in S_K \times Y \times \mathbb{R}_+^{|J|}$ is called a saddle point of the incomplete Lagrange function L_G if

$$L_G(\bar{x}, y, \lambda_J) \leqslant L_G(\bar{x}, \bar{y}, \bar{\lambda}_J) \leqslant L_G(x, \bar{y}, \bar{\lambda}_J) \quad \forall (x, y, \lambda_J) \in S_K \times Y \times \mathbb{R}_+^{|J|}.$$

We consider the problem

$$(EGFP)_v \quad \begin{array}{l} \min q \\ \text{subject to} \quad f_i(x) - v g_i(x) \leqslant q \quad \text{and} \\ h_j(x) \leqslant 0 \quad \forall (i, j) \in \{1, \ldots, p\} \times \{1, \ldots, m\}. \end{array}$$

We have the following lemma (see [2, Lemma 2.3, p. 9]).

Lemma 3.1. *The point x^* is (GFP) optimal with corresponding optimal value of the (GFP) objective equal to v^* if and only if (x^*, q^*) is $(EGFP)_{v^*}$ optimal with corresponding optimal value of the $(EGFP)_{v^*}$ objective equal to 0, i.e., $q^* = 0$.*

Now we define

$$H^{v^*}(x, q) = q \quad \forall (x, q) \in S \times \mathbb{R},$$
$$G_{1i}^{v^*}(x, q) = f_i(x) - v^* g_i(x) - q \quad \forall (x, q, i) \in S \times \mathbb{R} \times \{1, \ldots, p\},$$
$$G_{2j}^{v^*}(x, q) = h_j(x) \quad \forall (x, q, j) \in S \times \mathbb{R} \times \{1, \ldots, m\}.$$

Theorem 3.4. *Let S be an open subset of $X \equiv \mathbb{R}^n$, $f_1, \ldots, f_p, g_1, \ldots, g_p, h_1, \ldots, h_m$ be real functions on S and Gâteaux differentiable at $\bar{x}$ with $\bar{x} \in S$. F be a $(p+m)$-dimensional vector subspace of $X \times \mathbb{R} \equiv \mathbb{R}^n \times \mathbb{R}$ and $\bar{x}$ be optimal for (GFP) with corresponding optimal value of the (GFP) objective equal to v^*. Assume that*

(i) *$L_G(\cdot, y, \lambda_J)$ is pseudo invex and $(\lambda_K)^T h_K(\cdot)$ is quasi invex at $\bar{x}$ with respect to a function η for any $(y, \lambda_J, \lambda_K)$ in $Y \times \mathbb{R}_+^m$;*
(ii) *$G_{11}^{v^*}, \ldots, G_{1p}^{v^*}, G_{21}^{v^*}, \ldots, G_{2m}^{v^*}$ are $(\mathbb{R}^n \times \mathbb{R}, F)$-differentiable at $(\bar{x}, 0)$ and $(\mathbb{R}^n \times \mathbb{R}, F)$-continuous at $(\bar{x}, 0)$;*
(iii) *$D((G_{11}^{v^*}, \ldots, G_{1p}^{v^*}, G_{21}^{v^*}, \ldots, G_{2m}^{v^*}))(\bar{x}, 0)(F) = \mathbb{R}^{p+m}$.*

Then there exists $(\bar{y}, \bar{\lambda}_J) \in Y \times \mathbb{R}_+^{|J|}$ such that $(\bar{x}, \bar{y}, \bar{\lambda}_J)$ is a saddle point of the incomplete Lagrange function L_G.

Proof. Because $\bar{x}$ is an optimal value of (GFP) with corresponding optimal value of the (GFP) objective equal to v^*, by Lemma 3.1, (x^*, q^*) is $(EGFP)_{v^*}$ optimal with corresponding optimal value of the $(EGFP)_{v^*}$ objective equal to 0. Then the problem

$$(L) \quad \begin{array}{l} \min H^{v^*}(x, q) \\ \text{subject to} \quad G_{1i}^{v^*}(x, q) \leqslant 0 \quad \text{and} \quad G_{2j}^{v^*}(x, q) \leqslant 0 \quad \forall (i, j) \in \{1, \ldots, p\} \times \{1, \ldots, m\} \end{array}$$

has an optimal solution (x^*, q^*) with $q^* = 0$.

Applying Theorem 2.2 in case $E = X = \mathbb{R}^n \times \mathbb{R}$, $U = S \times \mathbb{R}$ for H^{v^*} and $G_{11}^{v^*}, \dots, G_{1p}^{v^*}, G_{21}^{v^*}, \dots, G_{2m}^{v^*}$, there exists $(\bar{y}, \bar{\lambda}_J, \bar{\lambda}_K) \in \mathbb{R}_+^p \times \mathbb{R}_+^{|J|} \times \mathbb{R}_+^{|K|}$ such that

$$DH^{v^*}(\bar{x}, 0)(k, l) + \sum_{i=1}^{p} \bar{y}_i DG_{1i}^{v^*}(\bar{x}, 0)(k, l) + \sum_{j=1}^{m} \bar{\lambda}_j DG_{2j}^{v^*}(\bar{x}, 0)(k, l) = 0, \tag{$*$}$$

for every $(k, l) \in \mathbb{R}^n \times \mathbb{R}$ and

$$\bar{y}_i\big[f_i(\bar{x}) - v^* g_i(\bar{x})\big] = 0 \quad \forall i \in \{1, \dots, p\},$$
$$(\lambda_J)^T h_J(\bar{x}) = 0, \quad (\lambda_K)^T h_K(\bar{x}) = 0, \quad \bar{\lambda} = (\bar{\lambda}_J, \bar{\lambda}_K) \geqslant 0, \quad \bar{y} \geqslant 0.$$

If l in $(*)$ is equal to 0, we have

$$\nabla\big[\bar{y}^T f(\bar{x}) - v^* \bar{y}^T g(\bar{x}) + (\bar{\lambda}_J)^T h_J(\bar{x}) + (\bar{\lambda}_K)^T h_K(\bar{x})\big] = 0.$$

If k in $(*)$ is equal to 0, we have $\sum_{i=1}^{p} \bar{y}_i = 1$ or $\bar{y} \in Y$.

Now arguing as in the proof of Theorem 2.5 in [2], we get the theorem. □

We consider the following propositions:

(A3) $\bar{x}$ be optimal for (*GFP*) with corresponding optimal value of the (*GFP*) objective equal to v^*.

(A4) $\bar{x}$ has the following properties:

$$\nabla\big[\bar{y}^T f(\bar{x}) - v^* \bar{y}^T g(\bar{x}) + (\bar{\lambda}_J)^T h_J(\bar{x}) + (\bar{\lambda}_K)^T h_K(\bar{x})\big] = 0,$$
$$\bar{y}_i\big[f_i(\bar{x}) - v^* g_i(\bar{x})\big] = 0 \quad \forall i \in \{1, \dots, p\},$$
$$(\lambda_J)^T h_J(\bar{x}) = 0, \quad (\lambda_K)^T h_K(\bar{x}) = 0, \quad \bar{\lambda} = (\bar{\lambda}_J, \bar{\lambda}_K) \geqslant 0, \quad \bar{y} \geqslant 0,$$
$$\sum_{i=1}^{p} \bar{y}_i = 1.$$

Theorem 3.5. *Let S be an open subset of $X \equiv \mathbb{R}^n$, $f_1, \dots, f_p, g_1, \dots, g_p, h_1, \dots, h_m$ be real functions on S. Let $\bar{x}$ be in S.*

(i) *If the conditions* (ii), (iii) *of Theorem* 3.4 *are fulfilled then* (A3) *implies* (A4).

(ii) *If $\bar{y}^T f(\cdot) - v^* \bar{y}^T g(\cdot) + (\bar{\lambda}_J)^T h_J(\cdot) + (\bar{\lambda}_K)^T h_K(\cdot)$ is Gâteaux differentiable at $\bar{x}$ and invex with respect to the function η at $\bar{x}$ then* (A4) *implies* (A3).

Proof. Applying Theorem 2.2 in case $E = X = \mathbb{R}^n \times \mathbb{R}$, $U = S \times \mathbb{R}$ for H^{v^*} and $G_{11}^{v^*}, \dots, G_{1p}^{v^*}, G_{21}^{v^*}, \dots, G_{2m}^{v^*}$ (we define in p. 453) and with the same proof of Theorem 3.4, we get (i).

Consider (ii). Since the map $\bar{y}^T f(\cdot) - v^* \bar{y}^T g(\cdot) + (\bar{\lambda}_J)^T h_J(\cdot) + (\bar{\lambda}_K)^T h_K(\cdot)$ is invex with respect to the function η at $\bar{x}$, for every $x \in S_h$, we have

$$\begin{aligned}
&\bar{y}^T f(x) - v^* \bar{y}^T g(x) + (\bar{\lambda}_J)^T h_J(x) + (\bar{\lambda}_K)^T h_K(x) \\
&\quad - \big[\bar{y}^T f(\bar{x}) - v^* \bar{y}^T g(\bar{x}) + (\bar{\lambda}_J)^T h_J(\bar{x}) + (\bar{\lambda}_K)^T h_K(\bar{x})\big] \\
&\geqslant \eta(x, \bar{x})^T \nabla\big[\bar{y}^T f(\bar{x}) - v^* \bar{y}^T g(\bar{x}) + (\bar{\lambda}_J)^T h_J(\bar{x}) + (\bar{\lambda}_K)^T h_K(\bar{x})\big] = 0.
\end{aligned}$$

Thus

$$\bar{y}^T f(x) - v^* \bar{y}^T g(x) \geqslant \bar{y}^T f(x) - v^* \bar{y}^T g(x) + (\bar{\lambda}_J)^T h_J(x) + (\bar{\lambda}_K)^T h_K(x) \geqslant 0.$$

It implies that $\mathrm{Max}_{i=\overline{1,p}} \frac{f_i(x)}{g_i(x)} \geqslant v^*$, for every $x \in S_h$ and we get (ii). $\square$

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