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ScienceDirect
Neural Networks 20 (2007) 810–818
Neural Networks
www.elsevier.com/locate/neunet

An LMI approach to global asymptotic stability of the delayed Cohen–Grossberg neural network via nonsmooth analysis

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Received 6 October 2005; received in revised form 11 July 2007; accepted 11 July 2007

Abstract

In this paper, a linear matrix inequality (LMI) to global asymptotic stability of the delayed Cohen–Grossberg neural network is investigated by means of nonsmooth analysis. Several new sufficient conditions are presented to ascertain the uniqueness of the equilibrium point and the global asymptotic stability of the neural network. It is noted that the results herein require neither the smoothness of the behaved function, or the activation function nor the boundedness of the activation function. In addition, from theoretical analysis, it is found that the condition for ensuring the global asymptotic stability of the neural network also implies the uniqueness of equilibrium. The obtained results improve many earlier ones and are easy to apply. Some simulation results are shown to substantiate the theoretical results.

Keywords: Global asymptotic stability; Time delay; LMI technique; Lyapunov functional method; Cohen–Grossberg neural networks; Nonsmooth analysis

1. Introduction

In recent decades, much attention has been devoted to the studies of artificial neural networks partially due to the fact that neural networks can be applied to signal processing, image processing, pattern recognition, control and optimization problems. In particular, the Cohen–Grossberg neural network (Cohen & Grossberg, 1983) proposed in 1983, has been a focal research subject. There are many interesting phenomena in the dynamical behaviors of Cohen–Grossberg neural network. The asymptotic stability, exponential stability, robust stability, periodic bifurcation and chaos of the neural network have been hot topics since many applications of neural network require the knowledge of the dynamical behaviors of neural networks, such as the uniqueness and asymptotic stability (Arik & Orman, 2005; Cao & Liang, 2004; Cao & Song, 2006; Cao & Wang, 2003, 2004, 2005; Chen & Rong, 2003; Chen, 2001; Joy, 1999; Liao & Wang, 2000; Liao, Li, & Wong, 2004; Liao, Wong, Wu & Chen, 2001; Lu, 2000; H. Qi & L. Qi, 2004; Rong, 2005; Roska & Chua, 1990; Singh, 2004, 2005; Tu & Liao, 2005; Ye & Michel, 1996; Ye, Michel, & Wang, 1995; Yu, 2007, in press; Yu & Cao, 2006a, 2006b, 2007a, 2007b, in press; Yu, Cao, & Chen, 2007; Yu, Chen, Cao, Lü, & Parlitz, 2007; Yu & Yao, 2007; Yuan & Cao, 2005; Zeng & Wang, 2006; Zhang, Suda, & Komine, 2005). For example, when a neural network is applied as an optimization solver, the equilibrium points of the network characterize possible optimal solutions of the optimization problem, and the global asymptotic stability ensures the convergence to an optimal solution starting from any initial condition. Therefore, the stability analysis of neural networks has been an important topic for researchers.

It is well-known that time delays are ubiquitous in most physical, chemical, biological, neural, and other natural systems due to finite propagation speeds of signals, finite processing and reaction time. It has been observed both experimentally and numerically that time delay could derail the stability of the neural network and cause sustained oscillations such as bifurcation or chaos. Recently, there has been extensive studies on the effect of time delays on the collective dynamics of coupled models.

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doi:10.1016/j.neunet.2007.07.004

In this paper, the asymptotic stability of the Cohen–Grossberg neural network with time delay is studied by means of nonsmooth analysis and linear matrix inequality (LMI) technique. It should be pointed out that some functions (e.g., the piecewise linear approximation of a sigmoid) are nonsmooth. But they are of special interest since they are widely employed as activation functions in neural network models. The results herein require neither the smoothness of the behaved function or activation function nor the boundedness of the activation function. Several new sufficient conditions are presented to ascertain the uniqueness of the equilibrium point and the global asymptotic stability of the neural network. The obtained results are easy to apply and improve many earlier works.

The rest of the paper is organized as follows: In Section 2, some preliminaries are given. In Section 3, many sufficient conditions are presented for the uniqueness of the equilibrium point of the delayed Cohen–Grossberg neural network, where Theorem 1 is the basic theorem for ensuring the uniqueness of the equilibrium point of Cohen–Grossberg neural networks, Corollary 1 is a direct derivation without any unknown matrix, Corollaries 2–4 yield the results in H. Qi and L. Qi (2004). In Section 4, LMI criteria for ensuring the global asymptotic stability of the delayed Cohen–Grossberg neural network are given in Theorem 2. Corollaries 5–8 are given for choosing specific matrices in Theorem 2. Next, Corollary 9 is presented to show the condition for global asymptotic stability of equilibrium, which implies its uniqueness. Corollaries 10–12 are simplifications of Corollary 9. Many corollaries in this paper are existing theorems in other papers. In Section 5, simulation results aiming at substantiating the theoretical analysis are reported. In Section 6, the final conclusions are drawn.

2. Preliminaries

In this paper, a general delayed Cohen–Grossberg neural network model is considered as follows:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -a(x(t))[b(x(t)) - Af(x(t)) - Bf(x(t-\tau)) + u], \tag{1}$$

or

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i(x_i(t))[b_i(x_i(t)) - \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) - \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_{ij})) + u_i], \quad i = 1, 2, \ldots, n, \tag{2}$$

where n denotes the number of neurons, $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^{\mathrm{T}} \in \mathbb{R}^n$ is the state vector associated with the neurons, $u = (u_1, u_2, \ldots, u_n)^{\mathrm{T}} \in \mathbb{R}^n$ is the external input vector, $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t)))^{\mathrm{T}} \in \mathbb{R}^n$ is the activation functions of neurons, $\tau = \tau_{ij}$ $(i, j = 1, 2, \ldots, n)$ are the time delays. $a(x(t)) = \mathrm{diag}(a_1(x_1(t)), a_2(x_2(t)), \ldots, a_n(x_n(t))) \in \mathbb{R}^{n\times n}$ and $a_i(x_i(t))$ represents a vector amplification function. $b(x(t)) = (b_1(x_1(t)), b_2(x_2(t)), \ldots, b_n(x_n(t)))^{\mathrm{T}} \in \mathbb{R}^n$ and $b_i(x_i(t))$ is a value behaved function. $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ are the connection weight matrix and the delayed connection weight matrix, respectively. The initial conditions of (1) are given by $x_i(t) = \phi_i(t) \in \mathcal{C}([-r, 0], \mathbb{R})$ with $r = \max_{1\le i,j\le n}\{\tau_{ij}\}$, where $\mathcal{C}([-r, 0], \mathbb{R})$ denotes the set of all continuous functions from $[-r, 0]$ to $\mathbb{R}$.

To establish our main results, it is necessary to make the following assumptions:

A_1: Each amplification function $a_i(\cdot)$ is positive, continuous, and bounded.

A_2: Each behaved function $b_i : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and there exists $l_i > 0$ such that $b_i'(x) \ge l_i$ for all $x \in \mathbb{R}$ at which $b_i(\cdot)$ is differentiable.

A_3: Each activation function $f_i : \mathbb{R} \to \mathbb{R}$ is nondecreasing and globally Lipschitz with a constant $k_i > 0$, i.e.

$$|f_i(u) - f_i(v)| \le k_i|u - v| \quad \forall u, v \in \mathbb{R},\ i = 1, 2, \ldots, n. \tag{3}$$

Next, some notations to be used later are introduced for convenience.

For any vector $v = (v_1, v_2, \ldots, v_n)^{\mathrm{T}} \in R^n$, $|v| = (|v_1|, |v_2|, \ldots, |v_n|)$, $\|v\|^2 = v^{\mathrm{T}}v$. Similarly, for any matrix $W = (w_{ij})_{n\times n}$, $|W| = (|w_{ij}|)_{n\times n}$. For a symmetric matrix W, denote $\lambda_{\max}(W)$ and $\lambda_{\min}(W)$ as its largest and smallest eigenvalue, respectively. Then its norm is defined by: $\|W\| \doteq \sup\{\|Wx\| : \|x\| = 1\} = \sqrt{\lambda_{\max}(W^{\mathrm{T}}W)}$. Let $\rho(W)$ denote the spectral radius of W. It is known that $\rho(W) \le \rho(|W|)$. Moreover, $\rho(W) \ge \rho(\widetilde{W})$ if $W \ge \widetilde{W} \ge 0$ where $W \ge \widetilde{W}$ means $w_{ij} \ge \widetilde{w_{ij}}$ for all $i, j = 1, 2, \ldots, n$. W is called a $\mathcal{P}$ matrix ($\mathcal{P}_0$ matrix) if and only if all principal minors of W are positive (nonnegative) and is denoted by $W \in \mathcal{P}$ ($W \in \mathcal{P}_0$). $\mu_2(W) = \lambda_{\max}(W + W^{\mathrm{T}})/2$, i.e., $\mu_2(W)$ is the largest eigenvalue of the symmetric part of W. The notation $W > 0$ ($W < 0$) means that W is positive definite (negative definite). $L = \mathrm{diag}(l_1, l_2, \ldots, l_n)$, $K = \mathrm{diag}(k_1, k_2, \ldots, k_n)$, $l = \min_{1\le i\le n}\{l_i\}$ and $k = \max_{1\le i\le n}\{k_i\}$.

Next, we give the definition of the generalized Jacobian which is essential for conducting nonsmooth analysis on Lipschitz continuous functions. Let the function $F : \mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitz continuous. According to Rademacher's theorem (Rockafellar & Wets, 1998, Theorem 9, p. 60), F is differentiable almost everywhere. Let D_F denote the set of those points where F is differentiable and $F'(x)$ denote the Jacobian of F at $x \in D_F$. Then, the set D_F is dense in $\mathbb{R}^n$. For any given $x \in \mathbb{R}^n$ define

$$\mathrm{Lip}_x F := \lim_{y\to x} \sup_{x\ne y\in\mathbb{R}^n} \frac{\|F(y) - F(x)\|}{\|y - x\|}. \tag{4}$$

Since F is locally Lipschitz continuous, the constant Lip_x is finite and we have $\|F'(x)\| \le \mathrm{Lip}_x F$ for any $x \in D_F$. Now we are ready to define the generalized Jacobian in the sense of Clarke (Clarke, 1983):

Definition 1. For any $x \in \mathbb{R}^n$, let $\partial F(x)$ be the set of the following collection of matrices

$$\partial F(x) = \mathrm{co}\{W \mid \text{there exists a sequence of } \{x^k\} \subset D_F \text{ converging to } x \text{ with } \lim_{k\to\infty} F'(x^k) = W\},$$

where co Ω denotes the convex hull of the set Ω. We term $\partial F(x)$ as the generalized Jacobian.

It is easy to see that the above definition is well defined and $\|W\| \le \mathrm{Lip}_x F$ for any $W \in \partial F(x)$. We say that $\partial F(x)$ is invertible if every element W in $\partial F(x)$ is nonsingular. Though the generalized Jacobian $\partial F(x)$ has many nice properties, only a few of them need to be singled out for our purpose. For one thing, the collection $\partial F(x)$ reduces to a singleton $\{F'(x)\}$ whenever F is continuously differentiable at x. We stress that $\partial F(x)$ may contain other elements if F is not differentiable at x.

Lemma 1 (*Lebourg Theorem (Clarke, 1983, p. 41)*). *For any given $x, y \in \mathbb{R}^n$, there exists an element W in the union $\bigcup_{z\in[x,y]} \partial F(z)$ such that*

$$F(y) - F(x) = W(y - x), \tag{5}$$

where $[x, y]$ denotes the segment connecting x and y.

For more discussions on the generalized Jacobian and its various applications, please refer to books Clarke (1983), Rockafellar and Wets (1998). Now, we analyze (1) from the viewpoint of nonsmooth analysis. It is first recalled that a state $x^* \in \mathbb{R}^n$ is called an equilibrium point of (1) if it satisfies

$$-a(x^*)[b(x^*) - (A + B) f(x^*) + u] = 0. \tag{6}$$

Notice that $a_i(x_i(t))$ is positive, (6) is equivalent to

$$b(x^*) - (A + B) f(x^*) + u = 0. \tag{7}$$

We assume that model (1) has an equilibrium $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ for any given u. To simplify the proofs, we will shift the equilibrium point x^* of (1) to the origin by using the following common transformation

$$y(t) = x(t) - x^*, \qquad y(t-\tau) = x(t-\tau) - x^*. \tag{8}$$

Therefore, model (1) can be transformed into the following form:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = -\overline{a}(y(t))[\overline{b}(y(t)) - Ag(y(t)) - Bg(y(t-\tau))], \tag{9}$$

or

$$\frac{\mathrm{d}y_i(t)}{\mathrm{d}t} = -\overline{a}_i(y_i(t))[\overline{b}_i(y_i(t)) - \sum_{j=1}^{n} a_{ij} g_j(y_j(t)) - \sum_{j=1}^{n} b_{ij} g_j(y_j(t-\tau_{ij}))], \quad i = 1, 2, \ldots, n, \tag{10}$$

where $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^{\mathrm{T}} \in \mathbb{R}^n$, $\overline{a}(y(t)) = \mathrm{diag}(\overline{a}_1(y_1(t)), \overline{a}_2(y_2(t)), \ldots, \overline{a}_n(y_n(t)))^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, $\overline{b}(y(t)) = (\overline{b}_1(y_1(t)), \overline{b}_2(y_2(t)), \ldots, \overline{b}_n(y_n(t)))^{\mathrm{T}} \in \mathbb{R}^n$, $g(y(t)) = (g_1(y_1(t)), g_2(y_2(t)), \ldots, g_n(y_n(t)))^{\mathrm{T}} \in \mathbb{R}^n$, $\overline{a}_i(y_i(t)) = a_i(y_i(t) + x_i^*)$, $\overline{b}_i(y_i(t)) = b_i(y_i(t) + x_i^*) - b_i(x_i^*)$, $g_i(y_i(t)) = f_i(y_i(t) + x_i^*) - f_i(x_i^*)$.

It is easy to see that $\overline{b}_i(0) = 0$, $g_i(0) = 0$, $\forall i = 1, 2, \ldots, n$. Moreover, from (3), we know that

$$|g_i(y_i)| \le k_i |y_i| \quad \forall y_i \in \mathbb{R},\ i = 1, 2, \ldots, n, \tag{11}$$

and

$$\|g(y(t))\|^2 \le y^{\mathrm{T}}(t) K g(y(t)). \tag{12}$$

To obtain the main results, the following three lemmas are needed:

Lemma 2 (*Schur complement, Boyd, Ghaoui, Feron, and Balakrishnan (1994)*). *The following linear matrix inequality (LMI)*

$$\begin{pmatrix} \mathcal{Q}(x) & \mathcal{S}(x) \\ \mathcal{S}(x)^{\mathrm{T}} & \mathcal{R}(x) \end{pmatrix} > 0,$$

where $\mathcal{Q}(x) = \mathcal{Q}(x)^{\mathrm{T}}$, $\mathcal{R}(x) = \mathcal{R}(x)^{\mathrm{T}}$, is equivalent to one of the following conditions:

(i) $\mathcal{Q}(x) > 0$, $\mathcal{R}(x) - \mathcal{S}(x)^{\mathrm{T}}\mathcal{Q}(x)^{-1}\mathcal{S}(x) > 0$,
(ii) $\mathcal{R}(x) > 0$, $\mathcal{Q}(x) - \mathcal{S}(x)\mathcal{R}(x)^{-1}\mathcal{S}(x)^{\mathrm{T}} > 0$.

Lemma 3. *For any vectors $x, y \in \mathbb{R}^n$ and positive definite matrix $G \in \mathbb{R}^{n\times n}$, the following matrix inequality holds:*

$$2x^{\mathrm{T}}y \le x^{\mathrm{T}}Gx + y^{\mathrm{T}}G^{-1}y.$$

Lemma 4. *For any matrix $A \in \mathbb{R}^{n\times n}$ and positive definite matrix $G \in \mathbb{R}^{n\times n}$, the following matrix inequality holds:*

$$G + AG^{-1}A^{\mathrm{T}} - A - A^{\mathrm{T}} \ge 0.$$

Proof. For any vector $x \in \mathbb{R}^n$, from Lemma 3 we know that

$$x^{\mathrm{T}}Ax + x^{\mathrm{T}}A^{\mathrm{T}}x = 2x^{\mathrm{T}}A^{\mathrm{T}}x \le x^{\mathrm{T}}Gx + x^{\mathrm{T}}AG^{-1}A^{\mathrm{T}}x,$$

and we obtain

$$x^{\mathrm{T}}(G + AG^{-1}A^{\mathrm{T}} - A - A^{\mathrm{T}})x \ge 0,$$

for any vector $x \in \mathbb{R}^n$. This completes the proof. □

3. Uniqueness of the equilibrium point

In this section, we present several new sufficient conditions to ascertain the uniqueness of the equilibrium point of (9) by means of a new method based on the nonsmooth analysis and matrix inequality.

Theorem 1. *Under assumptions* A_1–A_3*, the origin is the unique equilibrium of model (9) if there is a positive definite diagonal matrix $D = \mathrm{diag}(d_1, d_2, \ldots, d_n) > 0$ such that*

$$2DLK^{-1} - (DA + A^{\mathrm{T}}D) - (DB + B^{\mathrm{T}}D) > 0. \tag{13}$$

Proof. We will prove the uniqueness of the equilibrium point using the method of contradiction. Consider the equilibrium equation of (9)

$$-\overline{a}(y^*)[\overline{b}(y^*) - (A + B)g(y^*)] = 0. \tag{14}$$

Notice that $\overline{a}_i(y_i(t))$ is positive, (14) is equivalent to

$$\overline{b}(y^*) - (A + B)g(y^*) = 0. \tag{15}$$

It is evident that if $\overline{b}(y^*) = 0$ and $g(y^*) = 0$, then $y^* = 0$. Now let $g(y^*) \neq 0$. Multiplying both sides of (15) by $2g^{\mathrm{T}}(y^*)D$ yields

$$2g^{\mathrm{T}}(y^*)D\overline{b}(y^*) - 2g^{\mathrm{T}}(y^*)DAg(y^*) - 2g^{\mathrm{T}}(y^*)DBg(y^*) = 0, \tag{16}$$

and then (16) can be written as

$$2g^{\mathrm{T}}(y^*)D\overline{b}(y^*) - g^{\mathrm{T}}(y^*)DAg(y^*) - g^{\mathrm{T}}(y^*)A^{\mathrm{T}}Dg(y^*) - g^{\mathrm{T}}(y^*)DBg(y^*) - g^{\mathrm{T}}(y^*)B^{\mathrm{T}}Dg(y^*) = 0. \tag{17}$$

From Lemma 1, we have

$$\overline{b}(y^*) = b(y^* + x^*) - b(x^*) = My^*, \quad M \in \bigcup_{y \in [x^*, y^*+x^*]} \partial b(y). \tag{18}$$

From the definition of b, matrix M is diagonal, and we denote $M = \mathrm{diag}(m_1, m_2, \ldots, m_n)$. It is obvious that $m_i \geq l_i$ for $i = 1, 2, \ldots, n$.

From assumption A_3, we get

$$y_i g_i(y_i) \geq 0 \quad \forall y_i \in \mathbb{R},\ i = 1, 2, \ldots, n, \tag{19}$$

and

$$g_i^2(y_i) \leq k_i y_i g_i(y_i) \quad \forall y_i \in \mathbb{R},\ i = 1, 2, \ldots, n. \tag{20}$$

According to (18) and (20), we can obtain the following inequality

$$\begin{aligned} g^{\mathrm{T}}(y^*)D\overline{b}(y^*) &= \sum_{i=1}^{n} g_i(y_i^*) d_i m_i y_i^* \\ &\geq \sum_{i=1}^{n} \frac{d_i l_i}{k_i} g_i^2(y_i^*) \\ &= g^{\mathrm{T}}(y^*) DLK^{-1} g(y^*). \end{aligned} \tag{21}$$

Substituting (21) into (17), we obtain

$$g^{\mathrm{T}}(y^*)[2DLK^{-1} - DA - A^{\mathrm{T}}D - DB - B^{\mathrm{T}}D]g(y^*) \leq 0. \tag{22}$$

Obviously, (22) contradicts with the condition (13) which in turn implies that at the equilibrium point $g(y^*) = 0$, as well as $y^* = 0$. Thus, we proved that the origin of model (9) is a unique equilibrium point. □

Corollary 1. *Under assumptions* A_1–A_3, *the origin is the unique equilibrium of model* (9) *if*

$$2LK^{-1} - (A + A^{\mathrm{T}}) - (B + B^{\mathrm{T}}) > 0. \tag{23}$$

Proof. Let $D = I$. We can have Corollary 1 easily from Theorem 1, where I is the identity matrix. □

Corollary 2. *Under assumptions* A_1–A_3, *the origin is the unique equilibrium of model* (9) *if*

$$\rho((|A + B|)KL^{-1}) < 1. \tag{24}$$

Proof. We will prove that if the condition (24) of Corollary 2 holds, the condition (23) of Corollary 1 can also hold.

From (24), it is easy to see that

$$2I \geq 2\rho((|A + B|)KL^{-1})I \geq [A + A^{\mathrm{T}} + B + B^{\mathrm{T}}]KL^{-1}. \tag{25}$$

Since K and L are both positive definite diagonal matrices, multiplying LK^{-1} on both sides of (25), we have condition (23). This completes the proof. □

Corollary 3. *Under assumptions* A_1–A_3, *the origin is the unique equilibrium of model* (9) *if*

$$-(A + B) \in \mathcal{P}_0. \tag{26}$$

Proof. From (26), we know that

$$-(A + B) \geq 0,$$

we can easily obtain that the condition (23) is satisfied. This completes the proof. □

Corollary 4. *Under assumptions* A_1–A_3, *the origin is the unique equilibrium of model* (9) *if*

$$\mu_2(A + B) < l/k. \tag{27}$$

Proof. It is known to us that

$$2\mu_2(A + B)I \geq (A + A^{\mathrm{T}}) + (B + B^{\mathrm{T}}),$$

and

$$2LK^{-1} \geq 2l/kI.$$

Then we obtain

$$\begin{aligned} &2LK^{-1} - (A + A^{\mathrm{T}}) - (B + B^{\mathrm{T}}) \\ &\quad \geq 2[l/k - \mu_2(A + B)]I > 0. \end{aligned}$$

The condition of Corollary 1 is satisfied. The proof is completed. □

Remark 1. Corollaries 2–4 are the same as Theorem 2 (H. Qi & L. Qi, 2004), which is a special case in Theorem 1 herein.

Remark 2. In Lu (2000, Theorem 1), the condition

$$\rho((|A| + |B|)KL^{-1}) < 1$$

was shown to be sufficient to ensure the global asymptotic stability under existence assumption of the equilibrium point. This condition was later shown in Chen (2001) to be also sufficient to ensure the existence of an equilibrium point under the assumption that all b_i's are continuously differentiable. We observe that

$$\rho((|A + B|)KL^{-1}) < \rho((|A| + |B|)KL^{-1}).$$

Hence, Corollary 2 in this paper can also guarantee the uniqueness of the equilibrium point under the assumption of the existence of equilibrium point, where the differentiability of b_i is not required.

Remark 3. In Arik and Orman (2005), the following criterion for the uniqueness of the equilibrium point of the Cohen–Grossberg neural network is given by

$$\|A\| + \|B\| < l/k.$$

Since

$$2(\|A\| + \|B\|) \geq (A + A^{\mathrm{T}}) + (B + B^{\mathrm{T}}),$$

and

$$2LK^{-1} \geq 2l/kI,$$

we obtain

$$2LK^{-1} - (A + A^{\mathrm{T}}) - (B + B^{\mathrm{T}}) \geq 2[l/k - (\|A\| + \|B\|)]I > 0.$$

Hence the result in Arik and Orman (2005) can be easily deduced.

4. Global asymptotic stability

In this section, new LMI conditions are presented for the global asymptotic stability of equilibrium point of model (9) via nonsmooth analysis. The new results improve and extend many earlier works.

Theorem 2. *Under assumptions* A_1–A_3*, the delayed Cohen–Grossberg neural network model* (9) *is globally asymptotically stable at the origin if there are positive definite diagonal matrices* $P = \mathrm{diag}(p_1, p_2, \dots, p_n) > 0$ *and* $Q = \mathrm{diag}(q_1, q_2, \dots, q_n) > 0$*, positive definite matrix* $H = (h_{ij})_{n\times n} > 0$*, positive real value* $\alpha > 0$*,* $\beta > 0$*, such that*

$$M = \begin{pmatrix} 2PL & -PA & -PB \\ -A^{\mathrm{T}}P & 2\alpha QLK^{-1} - \alpha QA - \alpha A^{\mathrm{T}}Q - \beta H & -\alpha QB \\ -B^{\mathrm{T}}P & -\alpha B^{\mathrm{T}}Q & \beta H \end{pmatrix} > 0. \tag{28}$$

Proof. We now prove that the condition given in (28) implies the global stability of the origin of (9). Consider the following Lyapunov functional candidate

$$V(y(t)) = \sum_{i=1}^{n} p_i \int_0^{y_i(t)} \frac{2s}{\overline{a}_i(s)}\mathrm{d}s + 2\alpha \sum_{i=1}^{n} q_i \int_0^{y_i(t)} \frac{g_i(s)}{\overline{a}_i(s)}\mathrm{d}s + \beta \int_{t-\tau}^{t} g^{\mathrm{T}}(y(s))Hg(y(s))\mathrm{d}s. \tag{29}$$

Taking the derivative of $V(y)$ along the trajectories of (9), we obtain

$$\begin{aligned} \dot{V}(y(t))|_{(9)} &= \sum_{i=1}^{n} p_i \frac{2y_i(t)}{\overline{a}_i(y_i(t))}\dot{y}_i(t) + 2\alpha \sum_{i=1}^{n} q_i \frac{g_i(y_i(t))}{\overline{a}_i(y_i(t))}\dot{y}_i(t) \\ &\quad + \beta g^{\mathrm{T}}(y(t))Hg(y(t)) - \beta g^{\mathrm{T}}(y(t-\tau))Hg(y(t-\tau)) \\ &= -2y^{\mathrm{T}}(t)P\overline{b}(y(t)) + 2y^{\mathrm{T}}(t)PAg(y(t)) \\ &\quad + 2y^{\mathrm{T}}(t)PBg(y(t-\tau)) - 2\alpha g^{\mathrm{T}}(y(t))Q\overline{b}(y(t)) \\ &\quad + 2\alpha g^{\mathrm{T}}(y(t))QAg(y(t)) + 2\alpha g^{\mathrm{T}}(y(t))QBg(y(t-\tau)) \\ &\quad + \beta g^{\mathrm{T}}(y(t))Hg(y(t)) - \beta g^{\mathrm{T}}(y(t-\tau))Hg(y(t-\tau)). \end{aligned} \tag{30}$$

Similarly to (18) and from Lemma 1, we have

$$\overline{b}(y(t)) = b(y(t) + x^*) - b(x^*) = \widehat{M}y(t),$$
$$\widehat{M} \in \bigcup_{z\in[x^*, y+x^*]} \partial b(z), \tag{31}$$

where $\widehat{M} = \mathrm{diag}(\widehat{m}_1, \widehat{m}_2, \dots, \widehat{m}_n)$. It is obvious that $\widehat{m}_i \geq l_i$ for $i = 1, 2, \dots, n$. We obtain

$$\begin{aligned} y^{\mathrm{T}}(t)P\overline{b}(y(t)) &= \sum_{i=1}^{n} y_i(t)p_i\widehat{m}_i y_i(t) \\ &\geq \sum_{i=1}^{n} p_i l_i y_i^2(t) \\ &= y^{\mathrm{T}}(t)PLy(t). \end{aligned} \tag{32}$$

Similarly to (21) and from assumption A_3, we have

$$\begin{aligned} g^{\mathrm{T}}(y(t))Q\overline{b}(y(t)) &= \sum_{i=1}^{n} g_i(y_i(t))q_i\widehat{m}_i y_i(t) \\ &\geq \sum_{i=1}^{n} \frac{q_i l_i}{k_i} g_i^2(y_i(t)) \\ &= g^{\mathrm{T}}(y(t))QLK^{-1}g(y(t)). \end{aligned} \tag{33}$$

Substituting (32) and (33) into (30), we obtain

$$\begin{aligned} \dot{V}(y(t))|_{(9)} &\leq -2y^{\mathrm{T}}(t)PLy(t) + 2y^{\mathrm{T}}(t)PAg(y(t)) \\ &\quad + 2y^{\mathrm{T}}(t)PBg(y(t-\tau)) - 2\alpha g^{\mathrm{T}}(y(t))QLK^{-1}g(y(t)) \\ &\quad + 2\alpha g^{\mathrm{T}}(y(t))QAg(y(t)) + 2\alpha g^{\mathrm{T}}(y(t))QBg(y(t-\tau)) \\ &\quad + \beta g^{\mathrm{T}}(y(t))Hg(y(t)) - \beta g^{\mathrm{T}}(y(t-\tau))Hg(y(t-\tau)) \\ &= -2y^{\mathrm{T}}(t)PLy(t) + y^{\mathrm{T}}(t)PAg(y(t)) \\ &\quad + g^{\mathrm{T}}(y(t))A^{\mathrm{T}}Py(t) + y^{\mathrm{T}}(t)PBg(y(t-\tau)) \\ &\quad + g^{\mathrm{T}}(y(t-\tau))B^{\mathrm{T}}Py(t) - g^{\mathrm{T}}(y(t)) \\ &\quad \times [2\alpha QLK^{-1} - \alpha QA - \alpha A^{\mathrm{T}}Q - \beta H]g(y(t)) \\ &\quad + \alpha g^{\mathrm{T}}(y(t))QBg(y(t-\tau)) \\ &\quad + \alpha g^{\mathrm{T}}(y(t-\tau))B^{\mathrm{T}}Qg^{\mathrm{T}}(y(t)) \\ &\quad - \beta g^{\mathrm{T}}(y(t-\tau))Hg(y(t-\tau)) \\ &= -\begin{pmatrix} y^{\mathrm{T}}(t) & g^{\mathrm{T}}(y(t)) & g^{\mathrm{T}}(y(t-\tau)) \end{pmatrix} \\ &\quad \times M \begin{pmatrix} y(t) \\ g(y(t)) \\ g(y(t-\tau)) \end{pmatrix}. \end{aligned} \tag{34}$$

Therefore, From (34), we know that under the given condition (28), $\dot{V}(y(t)) = 0$ if and only if $y(t) = g(y(t)) = g(y(t-\tau)) = 0$, otherwise $\dot{V}(y(t)) \leq 0$. Moreover, on the other hand, $V(y)$ is radially unbounded since $V(y(t)) \to \infty$ as $\|y(t)\| \to \infty$. We have proved that the equilibrium of (9) is globally asymptotically stable. This completes the proof. □

Corollary 5. *Under assumptions* A_1–A_3*, the delayed Cohen–Grossberg neural network model* (9) *is globally asymptotically* *stable at the origin if there are positive definite diagonal matrices* $P = \mathrm{diag}(p_1, p_2, \ldots, p_n) > 0$ *and* $Q = \mathrm{diag}(q_1, q_2, \ldots, q_n) > 0$*, positive definite matrix* $H = (h_{ij})_{n\times n} > 0$*, positive real value* $\alpha > 0$*, such that*

$$M_1 = \begin{pmatrix} 2PL & -PA & -PB \\ -A^{\mathrm{T}}P & \alpha(2QLK^{-1} - QA - A^{\mathrm{T}}Q - H) & -\alpha QB \\ -B^{\mathrm{T}}P & -\alpha B^{\mathrm{T}}Q & \alpha H \end{pmatrix} > 0. \tag{35}$$

Proof. Let $\beta = \alpha$ in Theorem 2, we can obtain Corollary 5 directly. □

Corollary 6. *Under assumptions* A_1–A_3*, the delayed Cohen–Grossberg neural network model* (9) *is globally asymptotically stable at the origin if there are positive definite diagonal matrix* $Q = \mathrm{diag}(q_1, q_2, \ldots, q_n) > 0$ *and positive definite matrix* $H = (h_{ij})_{n\times n} > 0$*, such that*

$$N = \begin{pmatrix} 2QLK^{-1} - QA - A^{\mathrm{T}}Q - H & QB \\ B^{\mathrm{T}}Q & H \end{pmatrix} > 0. \tag{36}$$

Proof. We prove that under the condition (36) of Corollary 6, the condition (35) in Corollary 5 is satisfied.

Let $\underline{W} = [-PA \;\; -PB]$, $N = \begin{pmatrix} 2QLK^{-1} - QA - A^{\mathrm{T}}Q - H & QB \\ B^{\mathrm{T}}Q & H \end{pmatrix} > 0$, and (35) is equivalent to

$$M_1 = \begin{pmatrix} 2PL & \underline{W} \\ \underline{W}^{\mathrm{T}} & \alpha N \end{pmatrix} > 0. \tag{37}$$

According to Lemma 2, we have

$$PL > 0, \quad \alpha N - \frac{1}{2}\underline{W}^{\mathrm{T}}(PL)^{-1}\underline{W} > 0.$$

From the condition (36) given in Corollary 6, we know that $N > 0$, we choose a sufficiently large value of α, we can see that (35) is satisfied. For example, we choose $\alpha > \gamma_1/\gamma_2$, where γ_1 denotes the maximum eigenvalue of $\frac{1}{2}\underline{W}^{\mathrm{T}}(PL)^{-1}\underline{W}$ and γ_2 denotes the minimum eigenvalue of N. This completes the proof. □

Corollary 7. *Under assumptions* A_1–A_3*, the delayed Cohen–Grossberg neural network model* (9) *is globally asymptotically stable at the origin if there are positive definite diagonal matrix* $Q = \mathrm{diag}(q_1, q_2, \ldots, q_n) > 0$ *and positive definite matrix* $H = (h_{ij})_{n\times n} > 0$*, such that*

$$2QLK^{-1} - QA - A^{\mathrm{T}}Q - H - QBH^{-1}B^{\mathrm{T}}Q > 0. \tag{38}$$

Proof. According Lemma 2, (36) is equivalent to (38), we can easily have our result based on Corollary 6. □

Corollary 8. *Under assumptions* A_1–A_3*, the delayed Cohen–Grossberg neural network model* (9) *is globally asymptotically stable at the origin if there is a positive definite matrix* $H = (h_{ij})_{n\times n} > 0$*, such that*

$$2LK^{-1} - A - A^{\mathrm{T}} - H - BH^{-1}B^{\mathrm{T}} > 0. \tag{39}$$

Proof. Let $Q = I$, where I is the identity matrix, we can obtain Corollary 8 directly from Corollary 7.

Next, a simple corollary is presented to show the condition in Corollary 7 for global asymptotic stability of equilibrium point also, which implies its uniqueness in Theorem 1. □

Corollary 9. *Under assumptions* A_1–A_3*, the origin is the unique equilibrium of model* (9)*, and it is globally asymptotically stable, provided that there are positive definite diagonal matrix* $Q = \mathrm{diag}(q_1, q_2, \ldots, q_n) > 0$ *and positive definite matrix* $H = (h_{ij})_{n\times n} > 0$*, such that*

$$2QLK^{-1} - QA - A^{\mathrm{T}}Q - H - QBH^{-1}B^{\mathrm{T}}Q > 0. \tag{40}$$

Proof. From Lemma 4, we obtain

$$H + QBH^{-1}B^{\mathrm{T}}Q \geq QB + B^{\mathrm{T}}Q,$$

for a positive definite matrix H. Then

$$\begin{aligned} 2QLK^{-1} - (QA + A^{\mathrm{T}}Q) - (QB + B^{\mathrm{T}}Q) &\geq 2QLK^{-1} \\ - QA - A^{\mathrm{T}}Q - H - QBH^{-1}B^{\mathrm{T}}Q &> 0. \end{aligned}$$

Condition in Theorem 1 is also satisfied. This completes the proof. □

Corollary 10. *Under assumptions* A_1–A_3*, the origin is the unique equilibrium of model* (9)*, and it is globally asymptotically stable, provided that there is a positive definite diagonal matrix* $Q = \mathrm{diag}(q_1, q_2, \ldots, q_n) > 0$*, such that*

$$2QLK^{-1} - QA - A^{\mathrm{T}}Q - I - QBB^{\mathrm{T}}Q > 0. \tag{41}$$

Proof. Let $H = I$ in Corollary 9, the proof is completed. □

Corollary 11. *Under assumptions* A_1–A_3*, the origin is the unique equilibrium of model* (9)*, and it is globally asymptotically stable, provided that the following condition holds:*

$$2LK^{-1} - A - A^{\mathrm{T}} - I - BB^{\mathrm{T}} > 0. \tag{42}$$

Proof. Let $H = Q = I$ in Corollary 9, the proof is complete. □

Corollary 12. *Under assumptions* A_1–A_3*, the origin is the unique equilibrium of model* (9)*, and it is globally asymptotically stable, provided that the following conditions hold:*

$$2\frac{l}{k} - 2\mu_2(A + B) - \|B - I\|^2 > 0. \tag{43}$$

Proof. We will show that under the condition (43) of Corollary 12, condition (42) in Corollary 11 is also satisfied.

$$\begin{aligned} &2LK^{-1} - A - A^{\mathrm{T}} - I - BB^{\mathrm{T}} \\ &\quad = 2LK^{-1} - [(A + B) + (A + B)^{\mathrm{T}}] - (B - I)(B - I)^{\mathrm{T}} \\ &\quad \geq 2\frac{l}{k} - 2\mu_2(A + B) - \|B - I\|^2 > 0. \end{aligned}$$

This completes the proof. □

Remark 4. Corollaries 10 and 12 are the main results in (H. Qi & L. Qi, 2004; Yuan & Cao, 2005). As the main results in (H. Qi & L. Qi, 2004; Yuan & Cao, 2005) are the corollaries of this paper, the new results are more general about uniqueness and global asymptotic stability of equilibrium point.

Remark 5. In Joy (1999), it is shown that if

$$2\gamma QLK^{-1} - \gamma(QA - A^{\mathrm{T}}Q) - I - \gamma^2 QB(B)^{\mathrm{T}}Q > 0, \quad (44)$$

where $\gamma = \max_i\{k_i/l_i\}$, then the equilibrium point is globally asymptotically stable. But in Joy (1999), all activation functions are assumed to be monotone increasing or bounded and $b(x)$ is a linear function. We note that if we choose $H = I/\gamma$ in Corollary 9, we can obtain (44) directly. So it is a special case in our paper. Also, in this paper we have weaker conditions that the activation function is nondecreasing and function b is allowed to be nondifferentiable.

Remark 6. In Chen and Rong (2003), Chen and Rong have considered delay-independent stability analysis of Cohen–Grossberg neural networks using Lyapunov functional method and LMI approach. The main results are the same as Corollary 6, but H is a positive definite diagonal matrix in Chen and Rong (2003), it is easy to see that the theorem is more general.

Remark 7. In Cao and Liang (2004), Liao et al. (2001), Tu and Liao (2005), Ye et al. (1995), Zhang et al. (2005), the criteria are explicit and easily verified in practice. But they neglect the signs of the entries in the connection matrices, and thus, the difference between the neuronal excitatory and inhibitory effects might be ignored. In recent years, some improvements have been obtained to overcome this disadvantage (Chen & Rong, 2003; Rong, 2005). In this paper, by combining Lyapunov functional and linear matrix inequality (LMI) approaches, new criteria on global asymptotic stability for delayed Cohen–Grossberg neural networks are presented.

Remark 8. In Rong (2005), robust and asymptotic stability of Cohen–Grossberg neural networks is considered. But the following Lyapunov functional is employed

$$V(y(t)) = \sum_{i=1}^{n} p_i \int_0^{y_i(t)} \frac{2s}{\overline{a}_i(s)}\mathrm{d}s + \int_{t-\tau}^{t} g^{\mathrm{T}}(y(s))Hg(y(s))\mathrm{d}s.$$

When $\alpha = \beta = 1$ and $q_i = \varepsilon_i$, it is a special case in this paper.

5. Numerical examples

In this section, two examples are constructed to show the effectiveness of the obtained results.

Example 1. Consider the following delayed Cohen–Grossberg neural network for $i = 1, 2$:

$$\begin{cases} \dfrac{\mathrm{d}x_1(t)}{\mathrm{d}t} = -(2+\cos x_1(t)) \\ \quad \times [b_1 x_1(t) - 0.1x_1(t) + 0.1x_2(t-1) + 2], \\ \dfrac{\mathrm{d}x_2(t)}{\mathrm{d}t} = -(2+\sin x_2(t)) \\ \quad \times [b_2 x_2(t) - 0.4x_1(t) - 0.1x_2(t) - 0.5x_1(t-1) + 1], \end{cases} \quad (45)$$

where

$$b_i(u) = \begin{cases} u, & \text{if } u \ge 0, \\ 2u, & \text{if } u < 0. \end{cases}$$

It is easy to see that assumptions A_1–A_3 hold. $a_1(x_1(t)) = 2 + \cos x_1(t)$ and $a_2(x_2(t)) = 2 + \sin x_2(t)$ are bounded, positive continuous functions, $b_i(x_i(t))$ is locally Lipschitz but not differentiable, $f_i(x_i(t)) = x_i(t)$ is globally Lipschitz and nondecreasing but not bounded. Obviously, we have $K = L = I$.

$$A = \begin{pmatrix} 0.1 & 0 \\ 0.4 & 0.1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -0.1 \\ 0.5 & 0 \end{pmatrix}, \qquad u = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

From (42) in Corollary 11, we obtain

$$2LK^{-1} - A - A^{\mathrm{T}} - I - BB^{\mathrm{T}} = \begin{pmatrix} 0.79 & -0.4 \\ -0.4 & 0.55 \end{pmatrix} > 0.$$

But

$$2\frac{l}{k} - 2\mu_2(A+B) - \|B - I\|^2 = -0.244.$$

Corollary 12 in Yuan and Cao (2005) cannot be applied. Fig. 1 shows the simulation results of Example 1 where some constant initial functions are chosen. We can see that model (45) is globally asymptotically stable at a unique equilibrium, which substantiates Corollary 11 herein.

Example 2. Consider the following delayed Cohen–Grossberg neural network for $i = 1, 2$:

$$\begin{cases} \dfrac{\mathrm{d}x_1(t)}{\mathrm{d}t} = -(2+\cos x_1(t))[b_1 x_1(t) - 0.2\tanh(x_1(t)) \\ \quad + 0.2\tanh(x_2(t-1)) + 2], \\ \dfrac{\mathrm{d}x_2(t)}{\mathrm{d}t} = -(2+\sin x_2(t))[b_2 x_2(t) - 0.4\tanh(x_1(t)) \\ \quad - 0.1\tanh(x_2(t)) - 0.4\tanh(x_1(t-1)) \\ \quad + 0.5\tanh(x_2(t-1)) + 1], \end{cases} \quad (46)$$

where

$$b_i(u) = \begin{cases} u, & \text{if } u \ge 0, \\ 3u, & \text{if } u < 0. \end{cases}$$

It is easy to see that the assumptions A_1–A_3 hold. $a_1(x_1(t)) = 2 + \cos x_1(t)$ and $a_2(x_2(t)) = 2 + \sin x_2(t)$ are bounded, positive continuous functions, $b_i(x_i(t))$ is locally Lipschitz but not differentiable, $f_i(x_i(t)) = \tanh(x_i(t))$ is globally Lipschitz and nondecreasing. Obviously, we have $K = L = I$, where I is the identity matrix.

$$A = \begin{pmatrix} 0.2 & 0 \\ 0.4 & 0.1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -0.2 \\ 0.4 & -0.5 \end{pmatrix}, \qquad u = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

From (42) in Corollary 11, we obtain

$$2LK^{-1} - A - A^{\mathrm{T}} - I - BB^{\mathrm{T}} = \begin{pmatrix} 0.56 & -0.5 \\ -0.5 & 0.39 \end{pmatrix},$$

for simply calculating all the principal minors of the above matrix, which is not a positive definite matrix. However, if we choose an appropriate positive definite matrix $H = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.6 \end{pmatrix}$ in Corollary 9 herein, we obtain

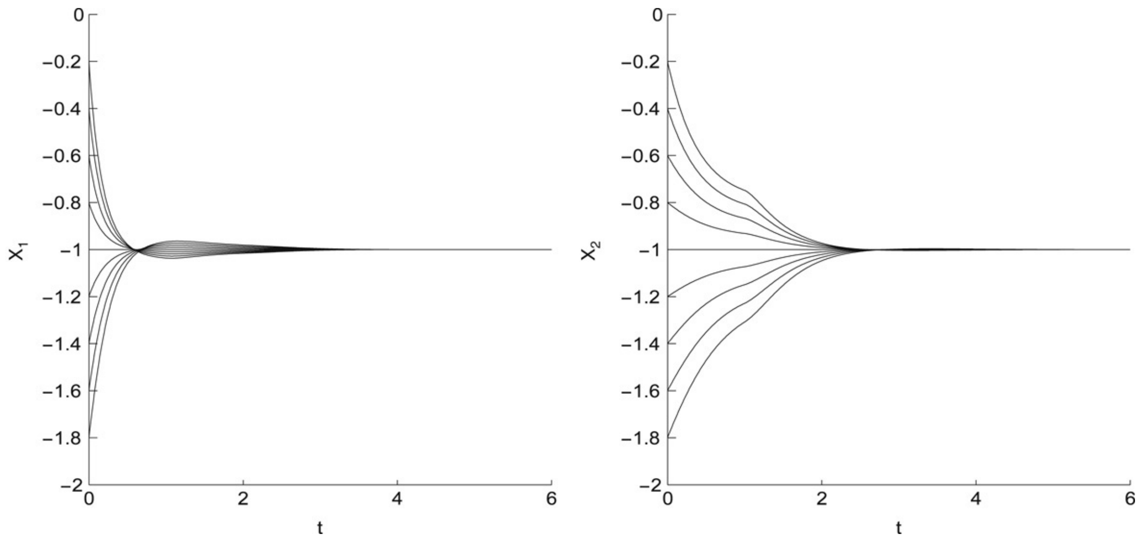

Fig. 1. Trajectories of state variables $x_1(t)$ and $x_2(t)$.

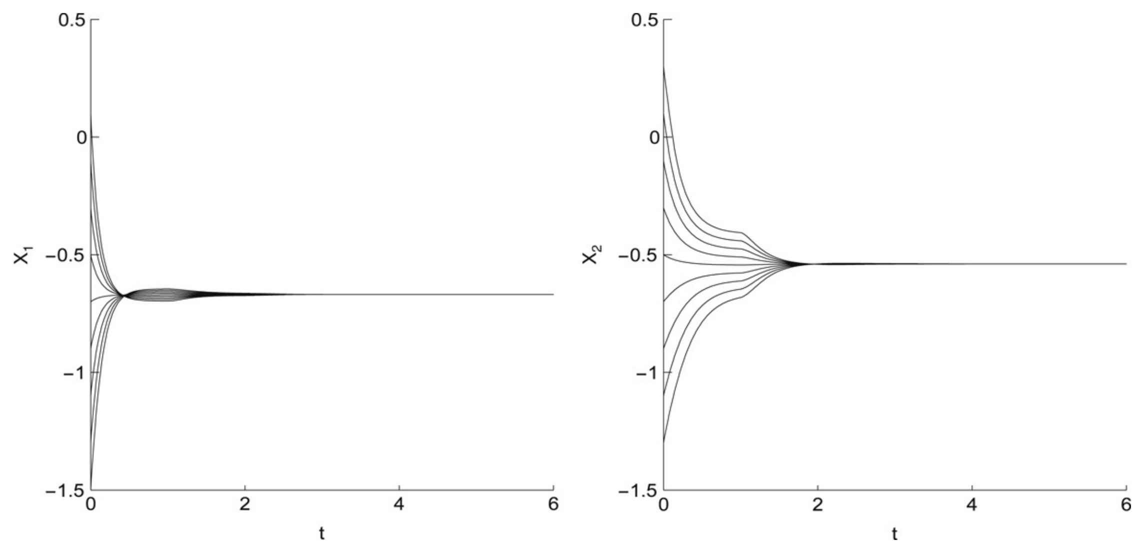

Fig. 2. Trajectories of state variables $x_1(t)$ and $x_2(t)$.

$$2LK^{-1} - A - A^{\mathrm{T}} - H - BH^{-1}B^{\mathrm{T}} = \begin{pmatrix} 0.9333 & -0.5667 \\ -0.5667 & 0.5167 \end{pmatrix} > 0.$$

So the result in H. Qi and L. Qi (2004) is a special case in this paper.

In the numerical simulation, we choose some constant initial functions as shown in Fig. 2. We can see that model (46) is globally asymptotically stable at a unique equilibrium point, which substantiates Corollary 9 herein. But the criteria in many other papers cannot be used since the behaved function b is nondifferentiable.

6. Conclusions

Several sufficient conditions in LMI are derived to ascertain the uniqueness of the equilibrium point and the global asymptotic stability of the delayed Cohen–Grossberg neural network via nonsmooth analysis. The obtained results improve and extend many earlier works.

Acknowledgements

This work was jointly supported by the National Natural Science Foundation of China under Grant No. 60574043, and the Natural Science Foundation of Jiangsu Province of China under Grant No. BK2006093.

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