Forbidding Kuratowski Graphs as Immersions

Archontia C. Giannopoulou[*,†] Marcin Kamiński[‡]
Dimitrios M. Thilikos[*,§]

Abstract

The immersion relation is a partial ordering relation on graphs that is weaker than the topological minor relation in the sense that if a graph G contains a graph H as a topological minor, then it also contains it as an immersion but not vice versa. Kuratowski graphs, namely K_5 and $K_{3,3}$, give a precise characterization of planar graphs when excluded as topological minors. In this note we give a structural characterization of the graphs that exclude Kuratowski graphs as immersions. We prove that they can be constructed by applying consecutive i-edge-sums, for $i \leq 3$, starting from graphs that are planar sub-cubic or of branch-width at most 10.

Keywords: graph immersions, Kuratowski graphs, tree-width, branch-width.

1 Introduction

A famous graph theoretic result is the theorem of Kuratowski which states that a graph G is planar if and only if it does not contain K_5 and $K_{3,3}$ (also known as the Kuratowski graphs) as a topological minor, that is, if K_5 and $K_{3,3}$ cannot be obtained from the graph by applying vertex and edge removals and edge dissolutions. It is well-known that the topological minor relation defines a (partial) ordering of the class of graphs.

In a similar way, the immersion and the minor orderings can be defined in graphs if instead of vertex dissolutions we ask for edge lifts and edge

[*]Department of Mathematics, National and Kapodistrian University of Athens, Panepistimioupolis, GR-15784, Athens, Greece. Emails: {arcgian,sedthilk}@math.uoa.gr

[†]Supported by a grant of the Special Account for Research Grants of the National and Kapodistrian University of Athens (project code: 70/4/10311).

[‡]Département d'Informatique, Université Libre de Bruxelles, Brussels, Belgium. Email: Marcin.Kaminski@ulb.ac.be

[§]Co-financed by the European Union (European Social Fund - ESF) and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF) - Research Funding Program: "Thales. Investing in knowledge society through the European Social Fund."

contractions respectively. (For detailed definitions see Section 2.) Notice here that the topological minor ordering is stronger than the minor and the immersion orderings in the sense that if a graph G contains a graph H as a topological minor then it also contains it as an immersion and as a minor but the inverse direction does not always hold.

In the celebrated Graph Minors theory, developed by Robertson and Seymour, it was proven that both the immersion and minor orderings are well-quasi-ordered, that is, there are no infinite sets of mutually non-comparable graphs [16,19] according to these orderings. This result has as a consequence the complete characterization of the graph classes that are closed under taking immersions or minors in terms of forbidden graphs, where a graph class is closed under taking immersions (respectively minors) if for any graph that belongs to the graph class all of its immersions (respectively minors) also belong to the graph class. For example, by an extension of the Kuratowski theorem (also known as Wagner's theorem), it is also known that a graph is planar if and only if it does not contain K_5 and $K_{3,3}$ as a minor.

Thus, a question that naturally arises is about the characterization of the structure of a graph G that excludes some fixed graph H as an immersion or as a minor. While this subject has been extensively studied for the minor ordering (see [6, 20, 18, 15, 9, 2, 3, 13, 17, 22]), the immersion ordering only recently attracted the attention of the research community [10, 4, 8, 1, 12]. In [4], DeVos et al. proved that for every positive integer t, every simple graph of minimum degree at least $200t$ contains the complete graph on t vertices as a (strong) immersion and in [7] Ferrara et al., given a graph H, provide a lower bound on the minimum degree of any graph G in order to ensure that H is contained in G as an immersion. More recently, in [21], Seymour and Wollan proved a structure theorem for graphs excluding complete graphs as immersions.

In terms of graph colorings, Abu-Khzam and Langston in [1] provide evidence supporting the analog of Hadwiger's Conjecture according to the immersion ordering, that is, the conjecture stating that if the chromatic number of a graph G is at least t then G contains the complete graph on t vertices as an immersion and prove it for $t \leq 4$. This conjecture is proven for $t = 5, 6$ and $t \leq 7$ by Lescure and Meyniel in [14] and by DeVos et al. in [5] independently. The most recent result on colorings is an approximation of the list coloring number on graphs excluding the complete graph as immersion [12].

Finally, in terms of algorithms, in [10], Grohe et al. gave a cubic time algorithm that decides whether a fixed graph H is contained in an input graph G as immersion and in [8] it was proved that the minimal graphs not belonging to a graph class closed under immersions can be computed when an upper bound on their tree-width and a description of the graph class in Monadic Second Order Logic are given.

In this note we characterize the structure of the graphs that do not contain K_5 and $K_{3,3}$ as immersions. As these graphs already exclude Kuratowski graphs as topological minors they are already planar. Additionally, we show that they have a more special structure: they can be constructed by repetitively, joining together simpler graphs, starting from either graphs of small decomposability or by planar graphs with maximum degree 3. In particular, we prove that a graph G that does not contain neither K_5 nor $K_{3,3}$ as immersions can be constructed by applying consecutive i-edge-sums, for $i \leq 3$, to graphs that are planar sub-cubic or of branch-width at most 10.

2 Definitions

For every integer n, we let $[n] = \{1, 2, \dots, n\}$. All graphs we consider are finite, undirected, and loopless but may have multiple edges. Given a graph G we denote by $V(G)$ and $E(G)$ its *vertex* and *edge set* respectively. Given a set $F \subseteq E(G)$ (resp. $S \subseteq V(G)$), we denote by $G \setminus F$ (resp. $G \setminus S$) the graph obtained from G if we remove the edges in F (resp. the vertices in S along with their incident edges). We denote by $\mathcal{C}(G)$ the set of the *connected components* of G. Given a vertex $v \in V(G)$, we also use the notation $G \setminus v = G \setminus \{v\}$. The *neighborhood* of a vertex $v \in V(G)$, denoted by $N_G(v)$, is the set of edges in G that are adjacent to v. We denote by $E_G(v)$ the set of the edges of G that are incident to v. The *degree* of a vertex $v \in V(G)$, denoted by $\deg_G(v)$, is the number of edges that are incident to it, i.e., $\deg_G(v) = |E_G(v)|$. Notice that, as we are dealing with multigraphs, $|N_G(v)| \leq \deg_G(v)$. The minimum degree of a graph G, denoted by $\delta(G)$, is the minimum of the degrees of the vertices of G, that is, $\delta(G) = \min_{v \in V(G)} \deg_G(v)$. A graph is called *sub-cubic* if all its vertices have degree at most 3. We also denote by K_r the *complete graph* on r vertices and by $K_{r,q}$ the *complete bipartite graph* with r vertices in its one part and q in the other. Let P be a path and $v, u \in V(P)$. We denote by $P[v, u]$ the sub-path of P with end-vertices v and u. Given two paths P_1 and P_2 who share a common endpoint v, we say that they are *well-arranged* if their common vertices appear in the same order in both paths.

We say that a graph H is a *subgraph* of a graph G if H can be obtained from G, after removing edges or vertices. An *edge cut* in a graph G is a non-empty set F of edges that belong to the same connected component of G and such that $G \setminus F$ has more connected components than G. If $G \setminus F$ has one more connected component than G then we say that F is a *minimal* edge cut. Let F be an edge cut of a graph G and let G' be the connected component of G containing the edges of F. We say that F is an *internal edge cut* if it is minimal and both connected components of $G' \setminus F$ contain at least 2 vertices. An edge cut is also called *i-edge-cut* if it has cardinality $\leq i$.

In this paper we mostly deal with lanai graphs, that is graphs that are embedded in the sphere $\mathbb{S}_0$. We call such a graph, along with its embedding, Σ_0-*embeddable graph*. Let C_1, C_2 be two disjoint cycles in a Σ_0-embeddable graph G. Let also Δ_i be the open disk of $\mathbb{S}_0 \setminus C_i$ that does not contain points of C_{3-i}, $i \in [2]$. The *annulus between* C_1 and C_2 is the set $\mathbb{S}_0 \setminus (\Delta_1 \cup \Delta_2)$ and we denote it by $A[C_1, C_2]$. Notice that $A[C_1, C_2]$ is a closed set. Let $\mathcal{A} = \{C_1, \ldots, C_r\}$ be a collection of cycles of a $\mathbb{S}_0$-embeddible graph G. We say that $\mathcal{A}$ is *nested* if for every $i \in [r-2]$, $A[C_i, C_{i+1}] \cup A[C_{i+1}, C_{i+2}] = A[C_i, C_{i+2}]$.

Contractions and minors. The *contraction of an edge* $e = \{x, y\}$ from G is the removal from G of all edges incident to x or y and the insertion of a new vertex v_e that is made adjacent to all the vertices of $(N_G(x) \setminus \{y\}) \cup (N_G(y) \setminus \{x\})$ such that edges corresponding to the vertices in $(N_G(x) \setminus \{y\}) \cap (N_G(y) \setminus \{x\})$ increase their multiplicity, that is, if there was a vertex $v \in (N_G(x) \setminus \{y\}) \cap (N_G(y) \setminus \{x\})$, k edges joining v and x and, l edges joining v and y then in the resulting graph there will be $k+l$ edges joining v with v_e. Finally, remove any loops resulting from this operation. Given two graphs H and G, we say that H is a *contraction* of G, denoted by $H \leq_c G$, if H can be obtained from G after a (possibly empty) series of edge contractions. Moreover, H is a *minor* of G if H is a contraction of some subgraph of G.

Topological minors. A *subdivision* of a graph H is any graph obtained after replacing some of its edges by paths between the same endpoints. A graph H is a *topological minor* of G (denoted by $H \leq_t G$) if G contains as a subgraph some subdivision of H.

Immersions. The *lift* of two edges $e_1 = \{x, y\}$ and $e_2 = \{x, z\}$ to an edge e is the operation of removing e_1 and e_2 from G and then adding the edge $e = \{y, z\}$ in the resulting graph. We say that a graph H can be *(weakly) immersed* in a graph G (or is an *immersion* of G), denoted by $H \leq_{im} G$, if H can be obtained from a subgraph of G after a (possibly empty) sequence of edge lifts. Equivalently, we say that H is an immersion of G if there is an injective mapping $f : V(H) \to V(G)$ such that, for every edge $\{u, v\}$ of H, there is a path from $f(u)$ to $f(v)$ in G and for any two distinct edges of H the corresponding paths in G are *edge-disjoint*, that is, they do not share common edges. Additionally, if these paths are internally disjoint from $f(V(H))$, then we say that H is *strongly immersed* in G (or is a *strong immersion of* G). The injective mapping f together with the edge-disjoint paths is called *a model of* H *in* G *defined by* f.

Edge sums. Let G_1 and G_2 be graphs, let v_1, v_2 be vertices of $V(G_1)$ and $V(G_2)$ respectively, and consider a bijection $\sigma : E_{G_1}(v_1) \to E_{G_2}(v_2)$ where

$E_{G_1}(v_1) = \{e_1^i \mid i \in [k]\}$. We define the *k-edge sum* of G_1 and G_2 on v_1 and v_2 as the graph G obtained if we take the disjoint union of G_1 and G_2, identify v_1 with v_2, and then, for each $i \in \{1, \ldots, k\}$, lift e_1^i and $\sigma(e_1^i)$ to a new edge e^i and remove the vertex v_1. (See Figures 1 and 2)

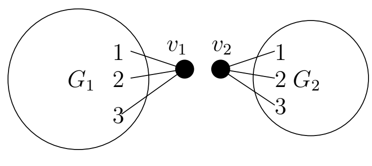

Figure 1: The graphs G_1 and G_2 before the edge-sum

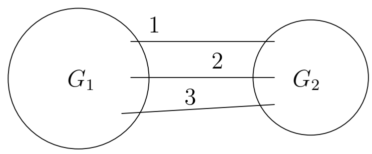

Figure 2: The graph obtained after the edge-sum

Let G be a graph, let F be a minimal i-edge cut in G, and let G' be the connected component of G that contains F. Let also C_1 and C_2 be the two connected components of $G' \setminus F$. We denote by C_i' the graph obtained from G' after contracting all edges of C_{3-i}' to a single vertex $v_i, i \in \{1, 2\}$. We say that the graph consisting of the disjoint union of the graphs in $\mathcal{C}(G) \setminus \{C_1, C_2\} \cup \{C_1', C_2'\}$ is the *F-split* of G and we denote it by $G|_F$. Notice that if G is connected and F is a minimal i-edge cut in G, then G is the result of the i-edge sum of the two connected components G_1 and G_2 of $\mathcal{C}(G|_F)$ on the vertices v_1 and v_2. From Menger's Theorem we obtain the following.

Observation 2.1. *Let k be a positive integer. If G is a connected graph that does not contain an internal i-edge cut, for some $i \in [k-1]$ and $v, v_1, \ldots, v_i \in V(G)$ are distinct vertices such that $\deg_G(v) \geq i$ then there exist i edge-disjoint paths from v to $v_1, v_2, \ldots, v_i$.*

Lemma 2.2. *If G is a $\{K_5, K_{3,3}\}$-immersion free connected graph and F is a minimal internal i-edge cut in G, for $i \in [3]$, then both connected components of $G|_F$ are $\{K_5, K_{3,3}\}$-immersion free.*

Proof. For contradiction assume that G is a $\{K_5, K_{3,3}\}$-immersion free connected graph and one of the connected components of $G|_F$, say C_1', contains K_5 or $K_{3,3}$ as an immersion, where F is a minimal internal i-edge cut in G, $i \in [3]$. Assume that $H \in \{K_5, K_{3,3}\}$ is immersed in C_1' and let

$f : V(H) \rightarrow V(C'_1)$ be a model of H in G. Let also v_1 be the newely introduced vertex of C'_1. Notice that if $v_1 \notin f(V(H))$ and v_1 is not an internal vertex of any of the edge-disjoint paths between the vertices in $f(V(H))$, then f is a model of H in C_1. As $C_1 \subseteq G$, f is a model of H in G, a contradiction to the hypothesis. Thus, we may assume that either $v_1 \in f(V(H))$ or v_1 is an internal vertex in at least one of the edge-disjoint paths between the vertives in $V(H)$. Note that, as neither K_5 nor $K_{3,3}$ contain vertices of degree 1, $\deg_{C'_1}(v_1) = 2$ or $\deg_{C'_1}(v_1) = 3$.

We first exclude the case where $v_1 \notin f(V(H))$, that is, v_1 only appears as an internal vertex on the edge-disjoint paths. Observe that, as $\deg_{C'_1}(v_1) \leq 3$, v_1 belongs to exactly one path P in the model defined by f. Let v_1^1 and v_1^2 be the neighbors of v_1 in P. Recall that, by the definition of an internal F-split, there are vertices v_2^1 and v_2^2 in C_2 such that $\{v_1^1, v_2^1\}, \{v_2^1, v_2^2\} \in E(G)$. Furthermore, as C_2 is connected, there exists a (v_2^1, v_2^2)-path P' in C_2. Therefore, be substituting the subpath $P[v_1^1, v_1^2]$ by the path defined by the union of the edges $\{v_1^1, v_2^1\}, \{v_2^1, v_2^2\} \in E(G)$ and the path P' in C_2 we obtain a model of H in G defined by f, a contradiction to the hypothesis.

Thus, the only possible case is that $v_1 \in f(V(H))$. As $\delta(K_5) = 4$ and $\deg_{C'_1}(v_1) \leq 3$, f defines a model of $K_{3,3}$ in C'_1. Let v_1^1, v_1^2 and v_1^3 be the neighbors of v_1 in C'_1. We claim that there is a vertex v in C_2 and edge-disjoint paths from v to v_1^1, v_1^2, v_1^3 in G, thus proving that there exists a model of $K_{3,3}$ in G as well, a contradiction to the hypothesis. By the definition of an internal F-split, there are vertices v_2^1, v_2^2 and v_2^3 in C_2 such that $\{v_1^i, v_2^i\} \in E(G)$, $i \in [3]$. Recall that C_2 is connected. Therefore, if for every vertex $v \in C_2$, $\deg_{C_2}(v) \leq 2$, C_2 contains a path whose endpoints, say u and u' belong to $\{v_2^1, v_2^2, v_2^3\}$ and internally contains the vertex in $\{v_2^1, v_2^2, v_2^3\} \setminus \{u, u'\}$, say u''. Then it is easy to verify that u'' satisfies the conditions of the claim. Assume then that there is a vertex $v \in C_2$ of degree at least 3. Let G' be the graph obtained from G after removing all vertices in $V(C_1) \setminus \{v_1^1, v_1^2, v_1^3\}$ and adding a new vertex that we make it adjacent to the vertices in $\{v_1^1, v_1^2, v_1^3\}$. As G does not contain an internal i-edge cut, $i \in [2]$, G' does not contain an internal i-edge cut, $i \in [2]$. Therefore, from Observation 2.1 and the fact that $v \notin \{v_1^1, v_1^2, v_1^3\}$, we obtain that there exist 3 edge-disjoint paths from v to v_1^1, v_1^2, v_1^3 in G' and thus in G. This completes the proof of the claim and the lemma follows. □

Let $r \geq 3$ and $q \geq 1$. A (r, q)-*cylinder*, denoted by $C_{r,q}$, is the cartesian product of a cycle on r vertices and a path on q vertices. (See, for example, Figure 2) A (r, q)-*railed annulus* in a graph G is a pair $(\mathcal{A}, \mathcal{W})$ such that $\mathcal{A}$ is a collection of r nested cycles $C_1, C_2, \ldots, C_r$ that are all met by a collection $\mathcal{W}$ of q paths $P_1, P_2, \ldots, P_q$ (called *rails*) in a way that the intersection of a rail and a path is always a (possibly trivial, that is, consisting of only one vertex) path. (See, for example, Figure 2) Notice that given a graph G embedded in the sphere and a (k, h)-*cylinder* ((r, q)-*railed annulus* respectively) of G,

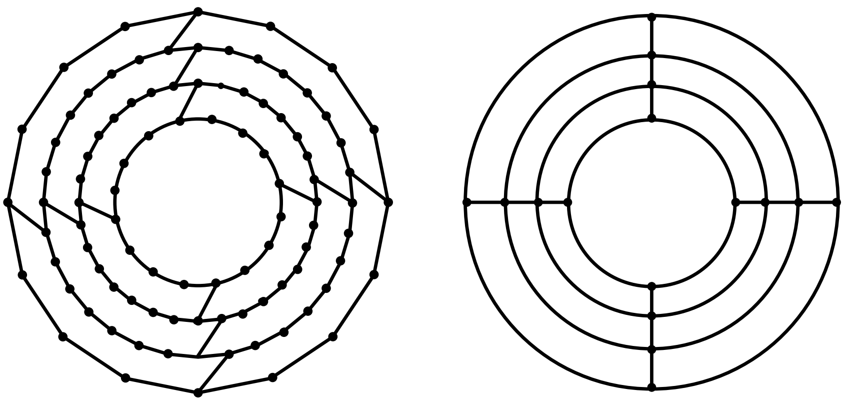

Figure 3: A (4,4)-railed annulus and a (4,4)-cylinder

then any two cycles of the (k, h)-*cylinder* ((r, q)-*railed annulus* respectively) define an annulus between them.

Branch decompositions. A *branch decomposition of a graph* G is a pair $B = (T, \tau)$, where T is a ternary tree and $\tau : E(G) \to \mathcal{L}(T)$ is a bijection of the edges of G to the leaves of T, denoted by $\mathcal{L}(T)$. Given a branch decomposition $\mathcal{B}$, we define $\sigma_B : E(T) \to \mathbb{N}$ as follows.

Given an edge $e \in E(T)$ let T_1 and T_2 be the trees in $T \setminus \{e\}$. Then $\sigma_B(e) = |\{v \mid \text{there exist } e_i \in \tau^{-1}(\mathcal{L}(T_i)),\ i \in [2], \text{ such that } e_1 \cap e_2 = \{v\}\}|$. The *width of a branch decomposition* B is $\max_{e \in E(T)} \sigma_B(e)$ and the *branch-width of a graph* G, denoted by $\mathbf{bw}(G)$, is the minimum width over all branch decompositions of G. In the case where $|V(T)| \leq 1$ the width of the branch decomposition is defined to be 0. The following has been proven in [11].

Theorem 2.3. *If G is a planar graph and k, h are integers with $k \geq 3$ and $h \geq 1$ then G either contains the (k, h)-cylinder as a minor or has branch-width at most $k + 2h - 2$.*

We now prove the following.

Lemma 2.4. *If G is a planar graph of branch-width at least 11, then G contains a (4,4)-railed annulus.*

Proof. Let G be a planar graph of branch-width at least 11. Then by Theorem 2.3, G contains $(4, 4)$-cylinder as a minor. By the definition of the minor relation, G contains a $(4, 4)$-railed annulus. □

Confluent paths Let G be a graph embedded in some surface Σ and let $x \in V(G)$. We define a *disk around* x as any open disk Δ_x with the property that each point in $\Delta_x \cap G$ is either x or belongs to the edges incident to x.

Let P_1 and P_2 be two edge-disjoint paths in G. We say that P_1 and P_2 are *confluent* if for every $x \in V(P_1) \cap V(P_2)$, that is not an endpoint of P_1 or P_2, and for every disk Δ_x around x, one of the connected components of the set $\Delta_x \setminus P_1$ does not contain any point of P_2. We also say that a collection of paths is *confluent* if the paths in it are pairwise confluent.

Moreover, given two edge-disjoint paths P_1 and P_2 in G we say that a vertex $x \in V(P_1) \cap V(P_2)$ that is not an endpoint of P_1 or P_2 is an *overlapping vertex of* P_1 *and* P_2 if there exists a Δ_x around x such that both connected components of $\Delta_x \setminus P_1$ contain points of P_2. For a family of paths $\mathcal{P}$, a vertex v of a path $P \in \mathcal{P}$ is called an *overlapping vertex of* P if there exists a path $P' \in \mathcal{P}$ such that v is an overlapping vertex of P and P'.

3 Preliminary results on the confluency of paths

Lemma 3.1. *Let G be a graph and $v, v_1, v_2 \in V(G)$ such that there exist edge-disjoint paths P_1 and P_2 from v to v_1 and v_2 respectively. If the paths P_1 and P_2 are not well-arranged then there exist edge-disjoint paths P_1' and P_2' from v to v_1 and v_2 respectively such that $E(P_1') \cup E(P_2') \subsetneq E(P_1) \cup E(P_2)$.*

Proof. Let $Z = V(P_1) \cap V(P_2) = \{v, u_1, u_2, \ldots, u_k\}$, where $(v, u_1, u_2, \ldots, u_k)$ is the order that the vertices in Z appear in P_1 and, $(v, u_{i_1}, u_{i_2}, \ldots, u_{i_k})$ is the order that they appear in P_2. As the paths are not well-arranged there exists $\lambda \in [k]$ such that $u_\lambda \neq u_{i_\lambda}$. Without loss of generality assume that λ is the smallest such integer. Without loss of generality assume also that $u_\lambda < u_{i_\lambda}$. We define

$$\begin{aligned} P_1' &= P_1[v, u_{\lambda-1}] \cup P_2[u_{\lambda-1}, u_{i_\lambda}] \cup P_1[u_{i_\lambda}, v_1] \\ P_2' &= P_2[v, u_{\lambda-1}] \cup P_1[u_{\lambda-1}, u_\lambda] \cup P_2[u_\lambda, v_2]. \end{aligned}$$

and observe that P_1' and P_2' satisfy the desired properties. (For an example, see Figure 3). □

Before proceeding to the statement and proof of the next proposition we need the following definition. Given a collection of paths $\mathcal{P}$ in a graph G, we define the function $f_\mathcal{P} : \bigcup_{P \in \mathcal{P}} V(P) \to \mathbb{N}$ such that $f(x)$ is the number of pairs of paths $P, P' \in \mathcal{P}$ for which x is an overlapping vertex. Let

$$g(\mathcal{P}) = \sum_{x \in \bigcup_{P \in \mathcal{P}} V(P)} f_\mathcal{P}(x).$$

Notice that $f(x) \geq 0$ for every $x \in \bigcup_{P \in \mathcal{P}} V(P)$ and thus $g(\mathcal{P}) \geq 0$. Observe also that $g(\mathcal{P}) = 0$ if and only if $\mathcal{P}$ is a confluent collection of paths.

Lemma 3.1 allows us to prove the main result of this section. We state the result for general surfaces as the proof for this more general setting does not have any essential difference than the case where Σ is the sphere $\mathbb{S}_0$.

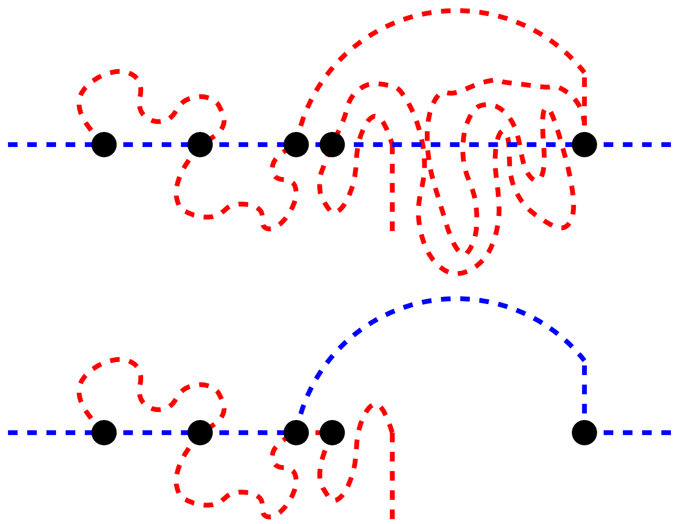

Figure 4: An example of the procedure in Lemma 3.1

Proposition 3.2. *Let r be a positive integer. If G is a graph embedded in a surface Σ, $v, v_1, v_2, \ldots, v_r \in V(G)$ and $\mathcal{P}$ is a collection of r edge-disjoint paths from v to $v_1, v_2, \ldots, v_r$ in G, then G contains a confluent collection $\mathcal{P}'$ of r well-arranged edge-disjoint paths from v to $v_1, v_2, \ldots, v_r$ where $|\mathcal{P}'| = |\mathcal{P}|$ and such that $E(\bigcup_{P \in \mathcal{P}'} P) \subseteq E(\bigcup_{P \in \mathcal{P}} P)$.*

Proof. Let $\hat{G}$ be the spanning subgraph of G induced by the edges of the paths in $\mathcal{P}$ and let G' be a minimal spanning subgraph of $\hat{G}$ that contains a collection of r edge-disjoint paths from v to $v_1, v_2, \ldots, v_r$. Let also $\mathcal{P}'$ be the collection of r edge-disjoint paths from v to $v_1, v_2, \ldots, v_r$ in G' for which $g(\mathcal{P}')$ is minimum. It is enough to prove that $g(\mathcal{P}') = 0$.

For a contradiction, we assume that $g(\mathcal{P}') > 0$ and we prove that there exists a collection $\tilde{\mathcal{P}}$ of r edge-disjoint paths from v to $v_1, v_2, \ldots, v_r$ in G' such that $g(\tilde{P}) < g(\mathcal{P}')$. As $g(\mathcal{P}') > 0$, then there exists a path, say $P_1 \in \mathcal{P}'$, that contains an overlapping vertex u. Let z_1 be the endpoint of P_1 which is different from v. Without loss of generality we may assume that u is the overlapping vertex of P that is closer to z_1 in P. Then there is a (v, z_2)-path $P_2 \in \mathcal{P}'$ such that u is an overlapping vertex of P_1 and P_2. Let $\tilde{P}_i = P_{3-i}[v, u] \cup P_i[u, z_i]$, $i \in [2]$ and $\tilde{P} = P$ for every $P \in \mathcal{P}' \setminus \{P_1, P_2\}$. As Lemma 3.1 and the edge-minimality of G' imply that the paths P_1 and P_2 are well-arranged, we obtain that $\tilde{P}_i$ is a path from v to v_i, $i \in [2]$. Let $\tilde{\mathcal{P}}$ be $\{\tilde{P} \mid P \in \mathcal{P}'\}$. It is easy to verify that $\tilde{\mathcal{P}}$ is a collection of r edge-disjoint paths from v to $v_1, v_2, \ldots, v_r$. We will now prove that $g(\tilde{\mathcal{P}}) < g(\mathcal{P}')$.

First notice that if $x \neq u$, then $f_{\tilde{\mathcal{P}}}(x) = f_{\mathcal{P}'}(x)$. Thus, it is enough to prove that $f_{\tilde{\mathcal{P}}}(u) < f_{\mathcal{P}'}(u)$. Observe that if $\{P, P'\} \subseteq \mathcal{P}' \setminus \{P_1, P_2\}$ and u is

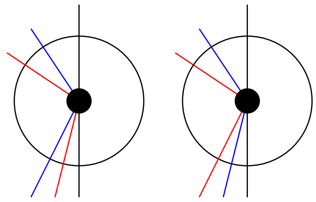

Figure 5: The paths P (black), P_1 (red) and P_2 (blue) and the paths $\tilde{P}_1$ (blue) and $\tilde{P}_2$ (red).

an overlapping vertex of P and P' then u is also an overlapping vertex of $\tilde{P}$ and P'. Furthermore, while u is an overlapping vertex in the case where $\{P, P'\} = \{P_1, P_2\}$, it is not an overlapping vertex of $\tilde{P}_1$ and $\tilde{P}_2$. It remains to examine the case where $|\{P, P'\} \cap \{P_1, P_2\}| = 1$. In other words, we examine the case where one of the paths P and P', say P', is P_1 or P_2, and $P \in \mathcal{P}' \setminus \{P_1, P_2\}$. Let Δ_u be a disk around u and Δ_1, Δ_2 be the two distinct disks contained in the interior of Δ_u after removing P. We distinguish the following cases.

Case 1. u is neither an overlapping vertex of P_1 and P, nor of P_2 and P (see Figure 3). Then it is easy to see that the same holds for the pairs of paths $\tilde{P}_1$ and P and, $\tilde{P}_2$ and P. Indeed, notice that for every $i \in [2]$, P_i intersects exactly one of Δ_1 and Δ_2. Furthermore, as u is an overlapping vertex of P_1 and P_2, both paths intersect the same disk. From the observation that $P_1 \cup P_2 = \tilde{P}_1 \cup \tilde{P}_2$, we obtain that u is neither an overlapping vertex of $\tilde{P}_1$ and P nor of $\tilde{P}_2$ and P.

Case 2. u is an overlapping vertex of P_i and P but not of P_{3-i} and P, $i \in [2]$ (see Figure 3). Notice that exactly one of the following holds.

- $P_i[v, u] \cup P_{3-i}[v, u]$ intersects exactly one of the disks Δ_1 or Δ_2, say Δ_1. Then $P_i[u, z_i]$ intersects Δ_2 and $P_{3-i}[u, z_{3-i}]$ intersects Δ_1. Therefore, it is easy to see that, u is not an overlapping vertex of P_i and P anymore but becomes an overlapping vertex of $\tilde{P}_{3-i}$ and P.

- $P_i[u, z_i] \cup P_{3-i}[u, z_{3-i}]$ intersects exactly one of the disks Δ_1 or Δ_2, say Δ_1. Then $P_i[v, u]$ intersects Δ_2 and $P_{3-i}[v, u]$ intersects Δ_1. Therefore, it is easy to see that u remains an overlapping vertex of $\tilde{P}_i$ and P and does not become an overlapping vertex of P_{3-i} and P.

Case 3. u is an overlapping vertex of both P_1 and P and, P_2 and P (see Figure 3). As above, exactly one of the following holds.

- $P_1[v, u] \cup P_2[v, u]$ intersects exactly one of the disks Δ_1 or Δ_2, say Δ_1. Then $P_1[u, z_1] \cup P_2[u, z_2]$ intersects Δ_2. It follows that u is an overlapping vertex of both $\tilde{P}_1$ and P and, $\tilde{P}_2$ and P

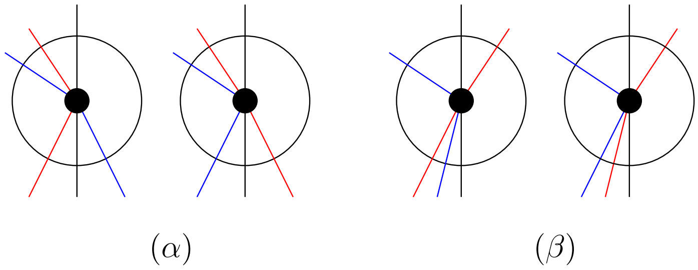

Figure 6: The paths P (black), P_1 (red) and P_2 (blue) and the paths $\tilde{P}_1$ (blue) and $\tilde{P}_2$ (red).

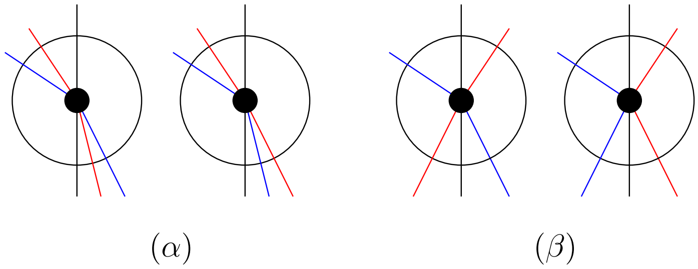

Figure 7: The paths P (black), P_1 (red) and P_2 (blue) and the paths $\tilde{P}_1$ (blue) and $\tilde{P}_2$ (red).

- $P_1[v,u] \cup P_2[u,z_2]$ intersects exactly one of the disks Δ_1 or Δ_2, say Δ_1. Then $P_1[u,z_1] \cup P_2[v,u]$ intersects Δ_2. It follows that u is neither an overlapping vertex of $\tilde{P}_1$ and P nor of $\tilde{P}_2$ and P.

From the above cases we obtain that $f_{\tilde{\mathcal{P}}}(u) < f_{\mathcal{P}'}(u)$ and therefore $g(\tilde{\mathcal{P}}) < g(\mathcal{P}')$, contradicting the choice of $\mathcal{P}'$. This completes the proof of the Proposition. $\square$

4 A decomposition theorem

We prove the following decomposition theorem for $(K_5, K_{3,3})$-immersion free graphs.

Theorem 4.1. *If G is a graph not containing K_5 or $K_{3,3}$ as an immersion, then G can be constructed by applying consecutive i-edge sums, for $i \leq 3$, to graphs that either are sub-cubic or have branch-width at most 10.*

Proof. Observe first that a $(K_5, K_{3,3})$-immersion-free graph is also $(K_5, K_{3,3})$-topological-minor-free, therefore, from Kuratowski's theorem, G is planar. Applying Lemma 2.2, we may assume that G is a $(K_5, K_{3,3})$-immersion-free graph G without any internal i-edge cut, $i \in [3]$. It is now enough to prove that G is either planar sub-cubic or has branch-width at most 10. For a contradiction, we assume that $\mathbf{bw}(G) \geq 11$ and that G contains some vertex v of degree ≥ 4. Our aim is to prove that G contains $K_{3,3}$ as an immersion. First, let G^s be the graph obtained from G after subdividing all of its edges once. Notice that G^s contains $K_{3,3}$ as an immersion if and only if G contains $K_{3,3}$ as an immersion. Hence, from now on, we want to find $K_{3,3}$ in G^s as an immersion.

From Lemma 2.4 G, and thus G^s, contains a $(4,4)$-railed annulus as a subgraph. Observe then that G^s also contains as a subgraph a $(2,4)$-railed annulus such that the vertex v of degree ≥ 4 does not belong in the annulus between its cycles (Figure 8 depicts the case where v is inside the annulus between the second and the third cycle). We denote by C_1 and C_2 the nested cycles and by R_1, R_2, R_3 and R_4 the rails of the above $(2,4)$-railed annulus. Let A be the annulus between C_1 and C_2. Without loss of generality we may assume that C_1 separates v from C_2 and that A is edge-minimal, that is, there is no other annulus A' such that $|E(A')| < |E(A)|$ and $A' \subseteq A$.

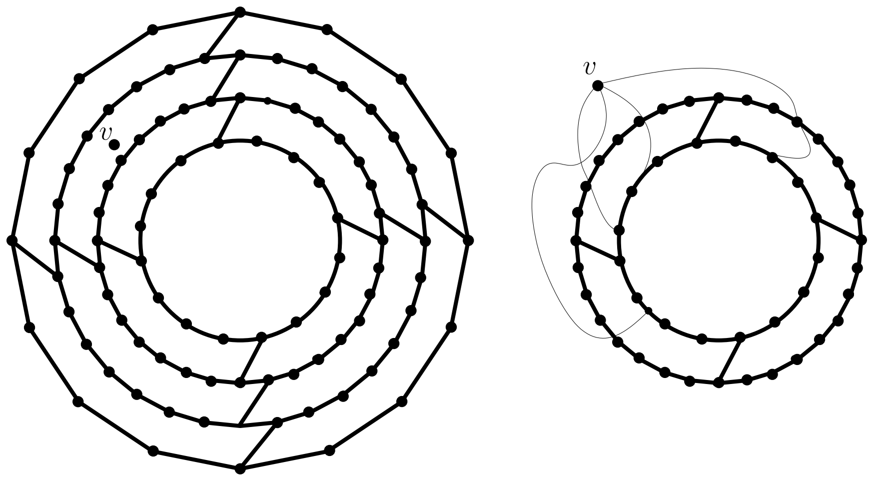

Figure 8: The $(4,4)$-railed annulus and the vertex v

Let now $G_1, G_2, \ldots, G_p$ be the connected components of $A \setminus (C_1 \cup C_2)$.

Claim 1. *For every $i \in [p]$ and every $j \in [2]$, $|N_{G^s}(V(G_i)) \cap V(C_j)| \leq 1$.*

Proof of Claim 1. Indeed, assume the contrary. Then there is a cycle C'_j such that C'_j and $C_{j \bmod 2+1}$ define an annulus A' with $A' \subseteq A$ and $|E(A')| < |E(A)|$, a contradiction to the edge-minimality of the annulus A. □

For every $l \in [p]$, we denote by u_1^l and u_2^l the unique neighbor of G_k in C_1 and C_2 respectively (whenever they exist). We call the connected components that have both a neighbor in C_1 and a neighbor in C_2 *substantial.* Let

$\mathcal{C} = \{\widehat{G}_i = G[V(G_i) \cup \{u_1^i, u_2^i\}] \mid G_i$ is a substantial connected component$\}$. That is, $\mathcal{C}$ is the set of graphs induced by the substantial connected components and their neighbors in the cycles C_1 and C_2. Note that every edge of G has been subdivided in G^s and thus every edge $e \in G$ for which $e \cap C_1 \neq \emptyset$ and $e \cap C_2 \neq \emptyset$ corresponds to a substantial connected component in $\mathcal{C}$.

We now claim that there exist four confluent edge-disjoint paths P_1, P_2, P_3 and P_4 from v to C_2 in G^s. This follows from the facts that G^s does not contain an internal i-edge cut, C_2 contains at least 4 vertices, $\deg_{G^s}(v) \geq 4$ combined with Observation 2.1. Moreover, from Proposition 3.2, we may assume that P_1, P_2, P_3 and P_4 are confluent.

Let P_i' be the subpath $P_i[v, v_i]$ of P_i, where v_i is the vertex in $V(P_i) \cap V(C_2)$ whose distance from v in P_i is minimum, $i \in [4]$. Recall that all edges of G have been subdivided in G^s. This implies that there exist four (possibly not disjoint) graphs in $\mathcal{C}$, say $\widehat{G}_1, \widehat{G}_2, \widehat{G}_3$ and $\widehat{G}_4$ such that $v_i = u_2^i$, $i \in [4]$. We distinguish two cases.

Case 1. The graphs $\widehat{G}_1, \widehat{G}_2, \widehat{G}_3$ and $\widehat{G}_4$ are vertex-disjoint.
This implies that the endpoints of P_1', P_2', P_3' and P_4' are disjoint. Let G' be the graph induced by the cycles C_1, C_2 and the paths P_1', P_2', P_3', P_4' and let $\widehat{P}_1, \widehat{P}_2, \widehat{P}_3$ and $\widehat{P}_4$ be confluent edge-disjoint paths from v to u_2^1, u_2^2, u_2^3 and u_2^4 in G' such that

(i) $\sum\{e \mid e \in \bigcup_{i \in [4]} E(\widehat{P}_i) \setminus E(A)\}$ is minimum, that is, the number of the edges of the paths that is outside of A is minimum, and

(ii) subject to i, $\sum\{e \mid e \in \bigcup_{i \in [4]} E(\widehat{P}_i)\}$ is minimum.

Let also $\widehat{G}$ be the graph induced by C_1, C_2, $\widehat{P}_1$, $\widehat{P}_2$, $\widehat{P}_3$ and $\widehat{P}_4$. From now on we work towards showing that $\widehat{G}$ contains $K_{3,3}$ as an immersion. For every $i \in [4]$ we call a connected component of $\widehat{P}_i \cap C_1$ non-trivial if it contains at least an edge.

Claim 2. *For every $i \in [4]$, $\widehat{P}_i \cap C_1$ contains at most one non-trivial connected component Q_i and u_1^i is an endpoint of Q_i.*

Proof of Claim 2. First, notice that any path from v to v_i in $\widehat{G}$ contains u_1^i and thus, $u_1^i \in V(\widehat{P}_i)$. Observe now that $\widehat{P}_i[u_1^i, u_2^i]$ is a subpath of $\widehat{P}_i$ whose internal vertices do not belong to C_1, thus if u_1^i belongs to a non-trivial connected component Q_i of $\widehat{P}_i \cap C_1$, then u_1^i is an endpoint of Q_i. We will now prove that any non-trivial connected component of $\widehat{P}_i \cap C_1$ contains u_1^i. Assume in contrary that there exists a non-trivial connected component P of $\widehat{P}_i \cap C_1$ that does not contain u_1^i. Let u be the endpoint of P for which $\mathbf{dist}_{\widehat{P}_i}(u, u_1^i)$ is minimum. Let also u' be the vertex in $\widehat{P}_i[u, u_1^i] \cap C_1$ such that $\mathbf{dist}_{\widehat{P}_i}(u, u')$ is minimum. Let P' be the subpath of C_1 with endpoints u, u' such that $\widehat{P}_i[u, u'] \cup P'$ is a cycle C with $C \cap P = \{u\}$. We further assume that the interior of $\widehat{P}_i[u, u'] \cup P'$ is the open disk that does not contain any vertices of $\widehat{P}_i$. We will prove that for every path $\widehat{P}_j$, $j \in [4]$ $\widehat{P}_j \cap P' \subseteq \{u, u'\}$. As this trivially holds for $j = i$ we will assume that $j \neq i$. Observe that, for every $j \in [4]$, $\widehat{P}_j[v, u_1^j] \cap A \subseteq C_1$ as for every connected component H of $A \setminus (C_1 \cup C_2)$ it holds that $|N_{G^s}(V(H)) \cap V(C_j)| \leq 1$. Furthermore, observe that $\widehat{P}_i[u, u'] \cup P'$ is a separator in $\widehat{G}$. This implies that v does not belong to the interior of $\widehat{P}_i[u, u'] \cup P'$. Thus, if there is a vertex z such that $z \in \widehat{P}_j \cap (P' \setminus \{u, u'\})$, $j \neq i$ there is a vertex $z' \in \widehat{P}_j \cap \widehat{P}_i[u, u']$, a contradiction to the confluency of the paths. We may then replace $\widehat{P}_i[u, u']$ by P', a contradiction to i. □

We denote by v_i the endpoint of Q_i that is different from u_1^i if Q_i is a non-trivial connected component of $\widehat{P}_i \cap C_1$, $i \in [4]$. Observe that $\widehat{P}_i = \widehat{P}_i[v, v_i] \cup Q_i \cup \widehat{P}_i[u_1^i, u_2^i]$, where we let $Q_i = \emptyset$ in the case where $\widehat{P}_i \cap C_1$ is edgeless, $i \in [4]$. We denote by T_i the subpath of C_1 with endpoints u_1^i and $u_1^{i \bmod 4+1}$ such that $T_i \cap \{\{u_1^1, u_1^2, u_1^3, u_1^4\} \setminus \{u_1^i, u_1^{i \bmod 4+1}\}\} = \emptyset$, $i \in [4]$. From the confluency of the paths $\widehat{P}_i$ and the fact that u_1^i is an endpoint of Q_i it follows that $Q_i \subseteq T_i$ or $Q_i \subseteq T_{i-1}$, $i \in [4]$ where $T_{i-1} = T_{3+i \bmod 4}$ if $i - 1 \notin [4]$.

Claim 3. *There exists an $i_0 \in [4]$ such that $T_{i_0} \cap (Q_{i_0}, Q_{i_0 \bmod 4+1}) \neq T_{i_0}$.*

Proof of Claim 3. Towards a contradiction assume that for every $i \in [4]$, it holds that $T_i \cap (Q_i, Q_{i \bmod 4+1}) = T_i$. It follows that either $Q_i = T_i = \widehat{P}_i[v_i, v_1^i]$, $i \in [4]$ or $Q_{i \bmod 4+1} = T_i$, $i \in [4]$. Notice then that either $v_i = u_1^{i \bmod 4+1}$, $i \in [4]$ or $v_{i \bmod 4+1} = u_1^i$, $i \in [4]$ respectively. Then, we let $\tilde{P}_{i \bmod 4+1} = \widehat{P}_i[v, v_i] \cup \widehat{P}_{i \bmod 4+1}[u_1^{i \bmod 4+1}, u_2^{i \bmod 4+1}]$ or $\tilde{P}_i = \widehat{P}_{i \bmod 4+1}[v, v_{i \bmod 4+1}] \cup \widehat{P}_i[u_1^i, u_2^i]$, $i \in [4]$ respectively. Notice that the paths $\tilde{P}_1, \tilde{P}_2, \tilde{P}_3$ and $\tilde{P}_4$ are confluent edge-disjoint paths from v to u_2^1, u_2^2, u_2^3 and u_2^4 such that $\cup_{i\in[4]}\tilde{P}_i$ is a proper subgraph of $\cup_{i\in[4]}\widehat{P}_i$. Therefore, we have that $\sum\{e \mid e \in \bigcup_{i\in[4]} E(\tilde{P}_i)\} < \sum\{e \mid e \in \bigcup_{i\in[4]} E(\widehat{P}_i)\}$, a contradiction to ii. □

It is now easy to see that $\widehat{G}$, and thus G, contains $K_{3,3}$ as an immersion. Indeed, first remove all edges of $C_1 \setminus T_{i_0}$ that do not belong to any path $\widehat{P}_i$, $i \in [4]$. Then lift the paths $\widehat{P}_i$ to a single edge where $i \neq i_0, i_0 \bmod 4 + 1$. Now let u_{i_0} ($u_{i_0 \bmod 4+1}$ respectively) be the vertex of T_{i_0} that belongs to $\widehat{P}_{i_0}$ ($\widehat{P}_{i_0 \bmod 4+1}$ respectively) whose distance from v in $\widehat{P}_{i_0}$ ($\widehat{P}_{i_0 \bmod 4+1}$ respectively) is minimum and lift the paths $\widehat{P}_{i_0}[v, u_{i_0}]$ and $\widehat{P}_{i_0 \bmod 4+1}[v, u_{i_0 \bmod 4+1}]$ to single edges. Notice now that $\widehat{G}$ contains the graph H_2 depicted in Figure 10 as an immersion. Thus, we get that $\widehat{G}$ contains $K_{3,3}$ as an immersion.

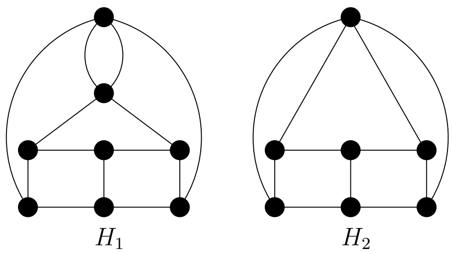

Figure 9: The graphs H_1 and H_2

Case 2. There exist $i_1, i_2 \in [4]$ such that $\widehat{G}_{i_1}$ and $\widehat{G}_{i_2}$ are not vertex-disjoint. Let G^μ be the graph induced by the cycles C_1 and C_2 and the graphs in $\mathcal{C}'$. We will show that G^μ contains $K_{3,3}$ as an immersion. First recall that the common vertices of $\widehat{G}_{i_1}$ and $\widehat{G}_{i_2}$ lie in at least one of the cycles C_1 and C_2. Without loss of generality assume that they have a common vertex in C_1. Recall that, as every edge of G has been subdivided in G^s, there does not exist an edge $e \in G^s$ such that $e \cap C_j \neq \emptyset$, $j \in [2]$. This observation and the fact that there exist four rails between C_1 and C_2 imply that there exist at least four graphs in $\mathcal{C}'$ that are vertex-disjoint. It follows that there exist three vertex-disjoint graphs, say $\widehat{G}_{i_3}, \widehat{G}_{i_4}, \widehat{G}_{i_5}$, in $\mathcal{C}'$ with the additional properties that $\widehat{G}_{i_{2+r}} \cap \widehat{G}_{i_1} \cap C_1 = \emptyset$, $r \in [3]$ and that at most one of the $\widehat{G}_{i_3}, \widehat{G}_{i_4}, \widehat{G}_{i_5}$ has a common vertex with one of the $\widehat{G}_{i_1}, \widehat{G}_{i_2}$. Note here that none of the $\widehat{G}_{i_3}, \widehat{G}_{i_4}, \widehat{G}_{i_5}$ can have a common vertex with one of the $\widehat{G}_{i_1}, \widehat{G}_{i_2}$ in C_2, in the case where $\widehat{G}_{i_1} \cap \widehat{G}_{i_2} \cap C_2 \neq \emptyset$. It is now easy to see that G^μ contains H_1 or (H_2 respectively) depicted in Figure 10 as a topological minor when $\widehat{G}_{i_1} \cap \widehat{G}_{i_2} \cap C_2 \neq \emptyset$ ($\widehat{G}_{i_1} \cap \widehat{G}_{i_2} \cap C_2 = \emptyset$ respectively). Observe now that H_1 contains H_2 as an immersion. Moreover, notice that H_2 contains $K_{3,3}$ as an immersion. Thus G^μ, and therefore G^s and G, contain $K_{3,3}$ as an immersion, a contradiction. □

Remark 4.2. It is easy to verify that our results hold for both the weak and strong immersion relations.

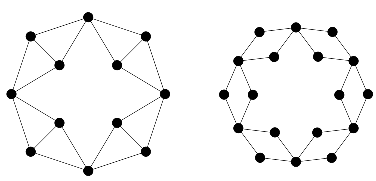

Figure 10: Simple non-sub-cubic graphs of branch-width 3 without K_5 or $K_{3,3}$ as immersions.

We believe that the upper bound on the branch-width of the building blocks of Theorem 4.1 can be further reduced, especially if we restrict ourselves to simple graphs. There are infinite such graphs that are not sub-cubic and have branch-width 3; some of them are depicted in Figure 10. However, we have not been able to find any simple non-sub-cubic graph of branch-width greater than 3 that does not contain K_5 or $K_{3,3}$ as an immersion.

References

[1] Faisal N. Abu-Khzam and Michael A. Langston. Graph coloring and the immersion order. In Tandy Warnow and Binhai Zhu, editors, *COCOON*, volume 2697 of *Lecture Notes in Computer Science*, pages 394–403. Springer, 2003.

[2] Stefan Arnborg, Andrzej Proskurowski, and Derek G. Corneil. Forbidden minors characterization of partial 3-trees. *Discrete Mathematics*, 80(1):1–19, 1990.

[3] Hans L. Bodlaender and Dimitrios M. Thilikos. Graphs with branch-width at most three. *Journal of Algorithms*, 32:167–197, 1999.

[4] M. DeVos, Z. Dvořák, J. Fox, J. McDonald, B. Mohar, and D. Scheide. Minimum degree condition forcing complete graph immersion. *ArXiv e-prints*, January 2011.

[5] Matt DeVos, Ken-Ichi Kawarabayashi, Bojan Mohar, and Haruko Okamura. Immersing small complete graphs. *Ars Math. Contemp.*, 3(2):139–146, 2010.

[6] Zdenek Dvorak, Archontia C. Giannopoulou, and Dimitrios M. Thilikos. Forbidden graphs for tree-depth. *Eur. J. Comb.*, 33(5):969–979, 2012.

[7] Michael Ferrara, Ronald J. Gould, Gerard Tansey, and Thor Whalen. On h-immersions. *Journal of Graph Theory*, 57(3):245–254, 2008.

[8] Archontia C. Giannopoulou, Iosif Salem, and Dimitris Zoros. Effective computation of immersion obstructions for unions of graph classes. In Fedor V. Fomin and Petteri Kaski, editors, *SWAT*, volume 7357 of *Lecture Notes in Computer Science*, pages 165–176. Springer, 2012.

[9] Archontia C. Giannopoulou and Dimitrios M. Thilikos. Optimizing the graph minors weak structure theorem. *CoRR*, abs/1102.5762, 2011.

[10] Martin Grohe, Ken ichi Kawarabayashi, Dániel Marx, and Paul Wollan. Finding topological subgraphs is fixed-parameter tractable. In *STOC*, pages 479–488, 2011.

[11] Qian-Ping Gu and Hisao Tamaki. Improved bounds on the planar branchwidth with respect to the largest grid minor size. In Otfried Cheong, Kyung-Yong Chwa, and Kunsoo Park, editors, *ISAAC (2)*, volume 6507 of *Lecture Notes in Computer Science*, pages 85–96. Springer, 2010.

[12] Ken ichi Kawarabayashi and Yusuke Kobayashi. List-coloring graphs without subdivisions and without immersions. In Yuval Rabani, editor, *SODA*, pages 1425–1435. SIAM, 2012.

[13] Athanassios Koutsonas, Dimitrios M. Thilikos, and Koichi Yamazaki. Outerplanar obstructions for matroid pathwidth. *Electronic Notes in Discrete Mathematics*, 38:541–546, 2011.

[14] F. Lescure and H. Meyniel. On a problem upon configurations contained in graphs with given chromatic number. In Carsten Thomassen Bjarne Toft Lars Dovling Andersen, Ivan Tafteberg Jakobsen and Preben Dahl Vestergaard, editors, *Graph Theory in Memory of G.A. Dirac*, volume 41 of *Annals of Discrete Mathematics*, pages 325 – 331. Elsevier, 1988.

[15] Neil Robertson and P. D. Seymour. Graph minors. XIII. The disjoint paths problem. *Journal of Combinatorial Theory. Series B*, 63(1):65–110, 1995.

[16] Neil Robertson and P. D. Seymour. Graph minors. XVI. Excluding a non-planar graph. *J. Combin. Theory Ser. B*, 89(1):43–76, 2003.

[17] Neil Robertson, P. D. Seymour, and Robin Thomas. Linkless embeddings of graphs in 3-space. *Bulletin of the Amer. Math. Soc.*, 28(1):84–89, 1993.

[18] Neil Robertson, Paul Seymour, and Robin Thomas. Quickly excluding a planar graph. *J. Combin. Theory Ser. B*, 62(2):323–348, 1994.

[19] Neil Robertson and Paul D. Seymour. Graph minors XXIII. Nash-Williams' immersion conjecture. *J. Comb. Theory, Ser. B*, 100(2):181–205, 2010.

[20] Juanjo Rué, Konstantinos S. Stavropoulos, and Dimitrios M. Thilikos. Outerplanar obstructions for a feedback vertex set. *Eur. J. Comb.*, 33(5):948–968, 2012.

[21] Paul Seymour and Paul Wollan. The structure of graphs not admitting a fixed immersion. *unpublished manuscript.*

[22] Atsushi Takahashi, Shuichi Ueno, and Yoji Kajitani. Minimal acyclic forbidden minors for the family of graphs with bounded path-width. *Discrete Mathematics*, 127(1/3):293–304, 1994.