A Creative Review on Coprime (Prime) Graphs

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Abstract.
Coprime labelings and Coprime graphs have been of interest since 1980s and got popularized by the *Entringer-Tout Tree Conjecture*. Around the same time *Newman's coprime mapping conjecture* was settled by Pomerance and Selfridge. This result was further extended to integers in arithmetic progression.

Since then coprime graphs were studied for various combinatorial properties. Here, coprimality of graphs for classes of graphs under the themes: Bipartite with special attention to Acyclicity, Eulerian and Regularity. Extremal graphs under non-coprimality and Eulerian properties are studied. Embeddings of coprime graphs in the general graphs, the maximum coprime graph and the Eulerian coprime graphs are studied as subgraphs and induced subgraphs. The purpose of this review is to assimilate the available works on coprime graphs. The results in the context of these themes are reviewed including embeddings and extremal problems.

Terminology

The terms *coprime labelings or coprime graphs* are used throughout the paper as was also used by P. Erdos, G.N. Sarkozy (1997) which otherwise are known as *prime lebelings* and *prime graphs* in the literature. The term *coprime* justifies the nomenclature in the definition based on the coprimality. This also avoids the confusion with other types of *prime graphs* defined in the literature in other contexts based on the primality.

A graph of order n is *coprime* if one can bijectively label its nodes with integers 1,...,n so that any two adjacent nodes get coprime labels. A graph which is not coprime is called *non-coprime graph*. A maximal coprime graph called *Maximum coprime graph* $\mathcal{M}_n$ of order n has $V(\mathcal{M}_n)=\{1,2,...,n\}$ and two nodes $uv \varepsilon V(\mathcal{M}_n)$ are adjacent if and only if $(u,v)=1$. Every coprime graph of order $\leq n$ is an induced subgraph of $\mathcal{M}_n$ and so is maximal and unique hence the name.

x-Multiple (x-m) is a number divisible by x. A number n is *x-factor free* (x-ff) if n has no factor x. For an even number e that is not a power of 2 define '*odd part of e'* denoted by $o(e)=e/2^{\alpha}$, as the odd number resulting after dividing e by the maximal 2 power in e. Denote the *prime decomposition* of x by pd(x) and the set of *distinct primes* in the prime factorization of x by $\mathcal{DP}(x)$. Let $\mathcal{P}_{px}$ denote the set of all *powers of* $x \leq n$. That is, $\mathcal{P}_{px}=\{x,x^2,x^3,\ldots\leq n\}$ with threshold n. Let $\mathcal{P}_n=\{x \varepsilon N_n : x\leq n$ and x is prime$\}$ be the *set of all primes* $\leq n$. Define, $\mathcal{P}_{>2}=\{x\in\mathcal{P}_n:2x>n\}$. Lastly, define *prime product* with a threshold of m by $\prod_m=\{ p_1p_2\ldots p_\beta \leq m: p_i \in\mathcal{P}_n, p_i$'s are distinct, $\beta>1\}$. Define $\pi_{i,j}$ to be the product of primes p, $i\leq p\leq j$ for $i,j \varepsilon \mathcal{P}_n$. When i=j=p then $\pi_{i,j}=\pi_p=p$. Note that, $\pi_{2,p}$ is even for any $p\varepsilon\mathcal{P}_n$. A *strict composite* is a composite number that is not a prime power. Two numbers n_1 and n_2 are said to be *equal upto primes* if they have the same distinct prime number set. That is, $\mathcal{DP}(n_1)= \mathcal{DP}(n_2)$. N_1 and N_2 subsets of N and are said to be *equal upto prime* if $\mathcal{DP}(N_1)= \mathcal{DP}(N_2)$.

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Beginnings

In the last 3 decades investigation of various *graphs on the integers* has received significant attention. The most popular graphs being the *maximum coprime graph and coprime graphs*, although there were many problems and results concerning the divisor graphs.

Newman, Pomerance and Selfridge (1980)

Newman's coprime mapping conjecture in this connection was settled and now known by the name *Coprime Mapping Theorem:*

- If n is a positive integer and I is a set of n consecutive integers, then there is a bijection $f:\{1,2,...,n\}\to I$ such that $\gcd(i,f(i))=1$ for $1\le i\le n$.

Leanne Robertson, Ben Small (2009)

A natural extension was made by replacing the set I with a set S of n integers in arithmetic progression and determining when there exist coprime mappings $f:\{1,2,...,n\}\to S$ and $g:\{1,3,...,2n-1\}\to S$.

Theorems

- Let n be a positive integer and $S=\{a+tb|0\le t\le n-1\}$ be a set of n integers in arithmetic progression with leading term a and common difference b. Then there is a coprime mapping $g:\{1,2,...,n\}\to S$ if and only if every common prime divisor of a and b is greater than n.
- Let n be a positive integer, $O=\{1+2t|0\le t\le n-1\}$ be the set of the first n positive odd integers, and $S=\{a+tb|0\le t\le n-1\}$. Then there is a coprime mapping $h:O\to S$ if and only if every common odd prime divisor of a and b is greater than $2n-1$.

The above three coprime mapping theorems were used to prove Coprimality of various families of trees presented in the following sections.

More works on coprime mappings may be found in: C Pomerance (1983), C Szabo, G Toth,(1985), R Ahlswede, L H Khachatrian (1994), R Ahlswede, L H Khachatrian (1995),

Elementary Properties of $\mathcal{M}_n$

Here, we list some interesting graph theoretical properties of $\mathcal{M}_n$. Note that the study of structural properties of $\mathcal{M}_n$ is significant in the study of coprime graphs. The results in this section were reported in Rao Hebbare (1981) & (1984) and SNRao (2002,2003,2009). See also BDAcharya (1981), P. Erdos, G.N. Sarkozy (1997) and Babujee (2010).

$\mathcal{M}_n$ satisfies the following:

- *Hereditary property*. Every spanning subgraph of a coprime graph of $\mathcal{M}_n$ is coprime. $\mathcal{M}_n$ is an induced subgraph of $\mathcal{M}_{n+1}$. Every coprime graph of order $\le$n is a subgraph of $\mathcal{M}_n$. Finding all distinct coprime graphs of order n is equivalent to finding all the distinct spanning subgraphs of $\mathcal{M}_n$.
- Every supergraph of non-coprime graph of order n is non-coprime. So, a non-coprime graph of order n cannot be embedded in a coprime graph of order n.
- Number of edges m in $\mathcal{M}_n$ is given by $m=\sum\varphi(x)$, x=1,..,n-1 where $\varphi(1)=0$, $\varphi(x)=\{t\le x: (x,t)=1\}$. φ is the Euler's φ-function a necessary condition for an (n,m)-graph to be coprime.

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Table-1 $\mathcal{M}_n$ (n,m)-graph.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
m	0	1	3	5	9	11	17	21	27	31	41	45	57	63	71
n	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
m	79	95	101	119	127	139	149	171	179	199	211	229	241	269	277

- A clique of size at least two in a coprime graph G yields a set of *mutually coprime labels*.
- For any $i\leq n-1$, (i-1,i) and (i,i+1) are coprime including (1,n). This implies that $d(x)\geq 2$, for any x in $\mathcal{M}_n$.
- $\mathcal{M}_n$ is pancyclic: The pairs (i,i+1) for i=1,…,n-1 and (1,n) are edges and form an n-cycle making $\mathcal{M}_n$ Hamiltonian. Since 1 is a full degree node (a node of degree n-1), cycles of length n-2,…,3 can be realized.
- $\mathcal{M}_n$ has exactly $|\mathcal{P}_{>2}|+1$ full degree nodes and this set is precisely $\mathcal{P}_{>2}\cup 1$.
- The prime labels p satisfying 3p>n correspond to the nodes having exactly one non-adjacency and that is the label 2p (<n) itself.
- For each prime $p\in\mathcal{P}_n$, the set of nodes labeled p, $p^2,\ldots,p^\alpha$ have same degree where $p^\alpha\leq n$ and α is maximum with this property.
- Minimum degree in $\mathcal{M}_n$ is given by $\delta(\mathcal{M}_n)=d(\pi_{2,m})$.
  - Node label corresponding to the maximal product of distinct primes $\leq$n.
  - A graph G of order n with $\delta(G)>\delta(\mathcal{M}_n)$ is not coprime.
  - An r-regular subgraph in $\mathcal{M}_n$ satisfies that $r\leq\delta(\mathcal{M}_n)$.
- In $\mathcal{M}_n$, for any $p\varepsilon\mathcal{P}_n$, $N(p)=N(p^2)=\ldots=N(p^\alpha)$ and α is maximum such that $p^\alpha\leq n$. Also, for $p_1,p_2\varepsilon\mathcal{P}_n$, $N(p_1p_2) = N(p_1)\cup N(p_2)$. So $d(p_1p_2)= |N(p_1)|+|N(p_2)|-|N(p_1)\cap N(p_2)|$. Combining the above two results, for $n_1=p_1{}^{\alpha}{}_1p_2{}^{\alpha}{}_2\ldots p_c{}^{\alpha}{}_c$, that $N(n_1)=N(p_1{}^{\alpha}{}_1)\cup\ldots\cup N(p_c{}^{\alpha}{}_c)$, where $2\leq n_1\leq n$. $d(n_1)$ then can be obtained from the general inclusion exclusion principle.
- Let $2\leq n_1,n_2\leq n$ be such that one of n_1 and n_2 is not prime. Then $\mathcal{DP}(n_1)= \mathcal{DP}(n_2)$ iff $N(n_1)=N(n_2)$ and so $d(n_1)=d(n_2)$.

SNRao (2009)

Construction of coprime graphs requires knowledge of coprime pairs in an orderly way.

Theorem: The following hold:

- (i,i+1)=1 for i=1,2,… ; (i,i+2)=1 and (i,i+4)=1, for i=1,3,5,… with the labels reduced (mod n) for a fixed n.
- For p prime (p,i)=1 holds for i=1,2,…,p-1.
- Further, (p,j(p+i))=1, for i=1,..,p-1 and j=1,2,… with the labels reduced (mod n) for a fixed n.
- For any prime p, (p,x)=1 iff $(p^\alpha,x)=1$, $\alpha>1$.
- For any x,y such that $x\neq y$ and for $\alpha\neq\beta>1$, (x,y)=1 iff $(x^\alpha,y^\beta)=1$ holds.
- (x,y)=1 iff (x',y')=1 holds whenever pd(x)=pd(x') and pd(y)=pd(y').

For cliques in $\mathcal{M}_n$ the following hold:

- $<\{\mathcal{P}_n\cup\{1\}\}>$ is a maximum clique in $\mathcal{M}_n$.
- A maximum clique Q in $\mathcal{M}_n$ is $\{\mathcal{P}_n\cup\{1\}\}$ or $\mathcal{DP}(Q)=\{\mathcal{P}_n\cup\{1\}\}$.
  - A maximum clique size in $\mathcal{M}_n$ is $|\mathcal{P}_n|+1$.
- A maximal clique Q in $\mathcal{M}_n$ with a strict composite number satisfies: $\mathcal{DP}(Q)\subset\{\mathcal{P}_n\cup\{1\}\}$.
  - A maximal clique Q size in a coprime graph G of order n is at most $|\mathcal{P}_n|+1$.

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- A coprime graph G of order n satisfies: $\beta_o(G)\geq[n/2]$. In particular, $\beta_o(\mathcal{M}_n)=[n/2]$, where $\beta_o(G)$ stands for the *independence number* of G.

Degree sequence of $\mathcal{M}_n$

No general rule is so far known for the degree sequence of $\mathcal{M}_n$. However, the degree sequences for $n\leq41$ are tabulated. It gives additionally the degree of each node. So, the possible minimum degree $\delta(\mathcal{M}_n)$ and the nodes attaining this minimum may be read. Denote the degree of i in $\mathcal{M}_n$ by $d_n(i)$. In particular, the notation $d_i(i)$ then stands for degree of i in $\mathcal{M}_i$.

Theorems

- $\mathcal{M}_n$ has at least one node of even and one node of odd degree for all $n\geq4$.
  - $\mathcal{M}_n$ is not Eulerian for $n\geq4$.
- For p prime $d_p(p)=p-1$ in $\mathcal{M}_p$.
- For a prime power, $d_p{}^n(p^n)=p^{(n-1)}d_p(p)=p^{n-1}(p-1)$.
- If $n= p_1{}^{\alpha}{}_1\ p_2{}^{\alpha}{}_2 \ldots p_s{}^{\alpha}{}_s$. Then $d_n(n)=p_1{}^{(\alpha_1-1)}\ldots p_s{}^{(\alpha_s-1)}(p_1-1)\ldots(p_s-1)$.

Carlson (2001) presented several new classes of coprime graphs with degree sequences of coprime graphs.

Primality Under Acyclicity Theme

A new concept under study in Graph Theory especially difficult for general classes of graphs focuses on the simplest class of connected graphs- trees. The notion of coprime graphs was introduced by Entringer who conjectured jointly with Tout around 1980 that *every tree is coprime*. We shall refer to it as the *Entringer–Tout Conjecture* as suggested in **Penny Haxell, Oleg Pikhurko, Anusch Taraz (2010).**

Conjecture (Entringer-Tout (1980)). All trees are coprime.

This conjecture was verified for some very special classes of trees: small trees, caterpillars, spiders, complete binary trees, olive trees, palm trees, banana trees, twigs, binomial trees, bistars, etc. Although the Entringer–Tout Conjecture for trees of small order n seems quite amenable with the current record $n\leq206$ claimed in a recent manuscript of Kuo and Fu.

Entringer's conjecture was verified for all trees of order:

- ≤15 by Rao Hebbare (1981), using tree complete graphs.
- ≤15 by Fu and Ching (1994), using Hall's SDR.
- ≤50 and '*every tree is almost coprime*' by Pikhurko (2002,2007).

A successful attempt was made by taking a large choice for the node labels.

Classes of Coprime Trees

Paths, Stars, Caterpillars, Complete Binary Trees, Spiders (i.e., trees with one vertex of degree at least 3 and with all other nodes with degree at most 2), Olive Trees (i.e., a rooted tree consisting of k branches such that the i-th branch is a path of length i),

Tout, R., A.N.Dabboucy and K. Howalla (1980, 1982)

Theorems:

- The following classes of trees are coprime:
  1. Complete binary trees.
  2. A caterpillar in which all the internal nodes are of the same degree.

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3. A tree having exactly one node whose removal results in a forest each component of which is a caterpillar with maximum degree<5.

Leanne Robertson, Ben Small (2009)
Theorems
Using coprime mappings it is proved that various families of trees are coprime, including palm trees, banana trees, binomial trees, and certain families of spider colonies.

Palm Trees. The palm tree $PT_{n,k}$ is the tree obtained from the concatenation of n stars with k nodes each by linking one leaf from each star. *Spider Colonies*. A *spider* is a tree with at most one vertex of degree greater than two. The vertex of degree greater than two if exists is the root of the spider. The paths from a spider's leaves to its root are called legs. A *regular spider* is a spider whose legs all have the same length. In ecology, a group of spiders that live together and build their web in a single tree is called a spider colony. Thus we are inclined to make the following definitions. *Binomial Trees*. Binomial trees are a family of *regular spider colonies* and defined recursively: The binomial tree B_0 is of order 0 and consists of a single vertex; the binomial tree B_n of order n has a root vertex whose children are the roots of the binomial trees of order 0,1,2,...,n−1. The name comes from the fact that B_n has height n and the number of nodes at level i is equal to the binomial coefficient $\binom{n}{i}$, i.e., the coefficient of x^i in $(1+x)^n$. Similarly, the number of nodes at level i of the regular spider colony $SC_n(k,l)$ is equal to the coefficient of x^i in $(1 + kx + kx^2 + \cdots + kx^\ell)^n$.
A *banana tree* is a tree obtained by joining one leaf of each of any number of stars to a new root vertex. If all the stars have the same number of nodes, then the banana tree is said to be regular. In this section we prove the following theorem.

Theorems:

- Palm trees are coprime. (Proof based on a result from Number Theory by Pillai (1940) and Brauer (1941)).
- Let T be a coprime tree with vertex set V , N = |V |, and $L:V\to\{1,2,...,N\}$ be a coprime labeling of T. Let $n\le N$ be a positive integer and $W = \{L^{-1}(1),L^{-1}(2),...,L^{-1}(n)\}$ be the set of n nodes of T that are labeled from 1 to n. Let S be a spider with $k \ge 1$ legs of lengths $l_1,l_2,...,l_k$. Suppose $\gcd(l_1,N) = 1$ or $\gcd(l_1,N+1) = 1$ and for $2\le i\le k$, that either $\gcd(l_i,N+n(l_1+l_2+\ldots+l_{i-1})) = 1$ or $\gcd(li,N+1+n(l_1+l_2+\ldots+l_{i-1})) = 1$. Then the spider colony Col(T,W, S) obtained by colonizing W by the spider S is coprime. Moreover, Col(T,W, S) has a coprime labeling that agrees with L on the nodes of T.
    - Let T be a coprime tree with N nodes and S be a regular spider whose legs have length l. If gcd(l,N)=1 or gcd(l,N+1)=1 then the tree Col(T,S) obtained by colonizing T by the spider S is coprime. In particular, Col(T,S) is coprime if l=1or l is a power of a coprime.
    - Regular spider colonies are coprime, i.e., $SC_n(k,l)$ is coprime for all positive integers n,k, and l.
    - For $n\ge1$, the binomial tree B_n can be obtained from B_{n-1} by attaching a child to every vertex of Bn−1, that is, B_n is equal to the regular spider colony $SC_n(1,1)$.
- All binomial trees are coprime.
- Banana trees are coprime.

    - Let T be a coprime tree with vertex set V, N=|V|, and $L:V\to\{1,2,...,N\}$ be a coprime labeling of T. Let $n\le N$ be a positive integer such that $2n-1\le N$, and W=

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$\{L^{-1}(1), L^{-1}(3), \ldots, L^{-1}(2n-1)\}$ be the set of n nodes of T that are labeled with the n odd integers from 1 to 2n−1. Let S be a spider with k≥1 legs of lengths $l_1, l_2, ..., l_k$. Suppose $\gcd(l_1,N)=1$ or $\gcd(l_1,N+1)=1$, and, for $2\le i \le k$, that either $\gcd(li,N+n(l_1+l_2+\ldots+l_{i-1}))=1$ or $\gcd(l_i,N+1+n(l_1+l_2+\ldots+l_{i-1}))=1$. Then the spider colony Col(T,W,S) obtained by colonizing W by the spider S is coprime. Moreover, Col(T,W,S) has a coprime labeling that agrees with L on the nodes of T.

Hung-Lin Fu and Kuo-Ching Huang (1994)
Theorems:

- If G is a coprime graph of order v, then $\beta_o(G) \ge \lfloor v/2 \rfloor$
- For each graph G, if $|V(K(G))| < \lfloor v/2 \rfloor$, then G is not a coprime graph.
- A complete bipartite graph of order v, G=(A,B), $|A|\le|B|$, is coprime if and only if $|A|\le|P(v/2,v)|+1$.
  - Let G=(A,B) be a bipartite graph with $|A|<|B|$ and $|A|<|P(v/2,v)|+1$. Then G is coprime.
- A tree T(A,B) of order v with $|A|<|P(v/2,u)|+1$ is coprime.
  - If T is a tree of order less than 9, then T is coprime.
- Let T be a tree of order u with T=(A,B) and $|A|\le|P(u/2,v)|+|S|+1$, where S is a subset of {1,2,...,v}. If there exists a subset W of A such that $|W|=|S|$ and $|B|-d(W)\ge|n(S)|$, then T is coprime.
- All trees of order ≤15 are coprime. (Proof using Hall's system of distinct representatives**)**
- A complete binary tree of level n, T(n), is coprime for all n>1.
- A complete t-partite graph $K(k_1,k_2,...,k_t)$, $k_1\ge k_2 \ge ... \ge k_t$, and Σi=1 to t k_i, is coprime if and only if $k_1 \ge L_k/2J$ and $|P(k/2,k))| \ge k-k_i-1$.
  - Let G be a graph of order u. If a(G)>v- |P(v/2,v)|-1, then G is coprime.

Ho, Huang, Lee, Wang, Wang and Wui (To Appear)
Theorems

- A palm tree P(n,k) is coprime whenever k≤5.
- For each k≤14, a palm tree P(n, k) is coprime.

Yao, Cheng, Zhongfu, and Yao, preprint.
Theorems

- A tree of order p with maximum degree at least p/2 is coprime;
- A tree of order p with maximum degree at least p/2 has a vertex subdivision that is coprime;
- If a tree T has an edge u_1u_2 such that the two components T_1 and T_2 of $T-u_1u_2$ have the properties that $dT_1(u_1) > |T_1|/2$ and $dT_2(u_2) > |T_2|/2$, then T is coprime when $|T_1|+|T_2|$ is coprime;
- If a tree T has two edges u_1u_2 and u_2u_3 such that the three components T_1, T_2, and T_3 of $T-\{u_1u_2,u_2u_3\}$ have the properties that dT_1 $(u_1)>|T_1|/2$, dT_2 (u2)$>|T_2|/2$, and dT_3 (u3)$>|T_3|/2$, then T is coprime when $|T_1|+|T_2|+|T_3|$ is coprime.

Salmasian (2000)
Theorem

- Every tree of order n ≥ 50 can be labelled with n integers from {1,...,4n} such that two adjacent nodes have co-prime labels.

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Tree Conjecture for Large N

Pikhurko (2002)

For any c>0 there is an N such that any tree of order n>N can be labeled with n integers between 1 and (1+c)n such that labels of adjacent nodes are relatively prime.

Penny Haxell, Oleg Pikhurko, Anusch Taraz (2010)

Prime Number Theorem and very basic results about divisibility and primality of integers were used. A graph G is *d-degenerate* if every non-empty subgraph of G has a vertex of degree at most d. For example, a graph is 1-degenerate if and only if it is acyclic. A graph G is *s-separable* if for every subgraph G'⊆G there is a set S ⊆ V (G') such that |S|≤s and each component of G'−S has at most |V(G')|/2 nodes. Given an integer d≥1, define a function s = s(n) by $s(n)=n^{1-10^6.d/\ln\ln n}$

Theorems

- There exists n' such that every tree with n≥n' nodes is coprime.
- For every d≥1 there exists n'' such that every s(n)-separable bipartite d-degenerate graph F of order n≥n'' is coprime.
- All bipartite d-degenerate graphs with separators of size at most $n^{1-Od(1/\ln\ln n)}$ are coprime.
  - All large trees are coprime.
- For every h, all sufficiently large bipartite graphs without a K_h-minor are coprime.

An application of the lemmas is discussed in the ensuing sections.

Primality and Bipartite graphs

Every coprime labeling of a graph induces a natural node bipartition (called a label (L-) bipartition) with reference to even ($\mathcal{E}$) and odd ($\mathcal{O}$) labels, V=$\mathcal{E}\cup\mathcal{O}$. This is pure L-bipartition (PB) of G. A partition of V which is not pure is called mixed L-bipartition (MB). When n is even$|\mathcal{E}|=|\mathcal{O}|$holds and when n is odd $|\mathcal{O}|=|\mathcal{E}|+1$ holds. The induced graph $<\mathcal{E}>$ is an independent set in G. $<\mathcal{O}>$ may not be an independent set. Conversely, an L-bipartition of a coprime bipartite graph induces an L-bipartition of the node labels $N=N_1\cup N_2$. In general, N_1 or N_2may not be pure that is even $\mathcal{E}$ or $\mathcal{O}$.

Similar to $\mathcal{M}_n$ a *maximum bipartite coprime graph* $\mathcal{B}_{m,n}$ is defined as follows: This may be done in two ways. Given a bipartite graph, consider the node bipartition and label the nodes so that the labeling is a coprime labeling. This induces a label bipartition $N=N_1\cup N_2$. The maximal coprime bipartite graph with this label bipartition is the $\mathcal{B}_{m,n}$ with $m=|N_1|$, $n=|N_2|$. Given a label bipartition of {1,2,…,m+n} construct a maximal coprime bipartite graph $\mathcal{B}_{m,n}$ with the label bipartition same as the node bipartition.

Properties of $\mathcal{B}_{m,n}$

We shall investigate the question: Does a coprime labeling of a bipartite graph always lead to a L-bipartition with one of the partitions being $\mathcal{E}$? Unfortunately this may not true. In fact, this is the case even if the bipartite coprime graph is maximal. For example, consider the bipartite graph with label bipartition for order 6: ({1,3,5},{2,4,6}) and ({1,5} and {2,4,3,6}) are a (6,8)-maximal bipartite coprime graphs. For order 8 the L-bipartitions ({1,3,5,7}, {2,4,6,8}) and ({1,7,5}, {2,4,6,8,3}) give rise to maximal bipartite coprime (8,15)-graphs. Another example is: the maximal bipartite coprime graphs with label bipartitions ({1,3,5,7,9}, {2,4,6,8}) and ({1,5,7}, {2,4,6,8,3,9}) are bipartite coprime (9,18)-graphs.

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SNRao (2009)

Observation. $\mathcal{B}_{m,n}$ satisfies the following:

- $\mathcal{B}_{m,n}$ asserts maximality of edges in the bipartite coprime graphs.
- A pure L-bipartition can be written as: $V=\{\mathcal{E}_1,\mathcal{E}_2\}\cup\{\mathcal{O}_1,\mathcal{O}_2\}$ with $E(G)=E(\mathcal{E},\mathcal{O})$ where $E=\mathcal{E}_1\cup\mathcal{E}_2$, $\mathcal{O}=\mathcal{O}_1\cup\mathcal{O}_2$. We shall now compare the number of edges amongst these L-bipartitions. First consider the structure of bipartite coprime graphs with mixed L-bipartition. Such an L-bipartition may be written as: $V=\{\mathcal{E}_1,\mathcal{O}_2\}\cup\{\mathcal{O}_1,\mathcal{E}_2\}$. $E(G)=E(\mathcal{E}_1,\mathcal{O}_1)\cup E(\mathcal{O}_2,\mathcal{E}_2)\cup E(\mathcal{O}_1,\mathcal{O}_2)\}$. Note that there are no edges between $\mathcal{E}_1$ and $\mathcal{E}_2$. The edges between $(\mathcal{E}_1,\mathcal{O}_1)$ and $(\mathcal{O}_2,\mathcal{E}_2)$ remain same as in pure bipartition.
- For an edge of the type (e_1,o_1), the following is true: $(e_1,o_1)=1$ iff $(o(e_1),o_1)=1$, where $o(e_1)$ is the odd part of e_1. Further, let $e_1\varepsilon\mathcal{E}_1$ and $o_1\varepsilon\mathcal{O}_1$. But then, $o(e_1)$ being odd is either in $\mathcal{O}_1$ or $\mathcal{O}_2$. If $o(e_1)\varepsilon\,\mathcal{O}_2$ by maximality, $(o(e_1),o_1)$ is an edge between $\mathcal{O}_1,\mathcal{O}_2$. If $o(e_1)\varepsilon\mathcal{O}_1$ then it cannot be an edge in G. Similar argument applies to an edge of type (o,e) with $o\varepsilon\mathcal{O}_2$ and $e\varepsilon\mathcal{E}_2$. Conversely, for any $(o_1,o_2)=1$ and an even number e satisfying that $e*o_1$ and $e*o_2$ εN, it follows that $(e*o_1,o_2)=1$ or $(o_1,e*o_2)=1$.
- Consider an edge of the type (o_1,o_2) between $(\mathcal{O}_1,\mathcal{O}_2)$. Suppose o_1 or $o_2\le n/2$. Then, $(2o_1,o_2)$ $(o_1,2o_2)$ are edges between $E,\mathcal{O}$. Suppose o_1 and $o_2>n/2$. If o_1 or o_2 is a prime then it is coprime with each label of E. If one of them say o_1 is composite odd number. Then $o_1=p_1,,,p_\alpha$, $p_i>2$, for each $i=1,\ldots,\alpha$. But then, $o_1/p_i<n/2$, for each i.
- We shall prove in the ensuing section that a regular coprime graph of even order is bipartite. As a consequence, a regular graph of odd degree is non-bipartite irrespective of n is even or not. Whether a graph is bipartite or not, the degree sequence restricted to the bipartition is of interest. As noted earlier induced bipartition of a coprime graph with respect to even labels is unique and so properties of DS of such a graph is of interest. Tables are constructed giving coprime pairs and bipartite DS for $n\le 64$ of $\mathcal{EO}$, $\mathcal{OE}$, $\mathcal{OO}$ cases and for $\mathcal{EO}$, $\mathcal{OE}$, $\mathcal{OO}$ cases in $\mathcal{M}_n$.

SNRao (2009)

Theorems

- A coprime graph with pure L-bipartition $(\mathcal{E},\mathcal{O})$ has at least as many edges as in any mixed L-bipartition. That is, $E(\mathcal{E}_1,\mathcal{O}_2)+E(\mathcal{E}_2,\mathcal{O}_1)\ge E(\mathcal{O}_1,\mathcal{O}_2)$.
- $d_{bn}(e)=d_n(u)$, for any $e\varepsilon E$ where degree of u in $\mathcal{B}_{m,n}$ is denoted by $d_{bn}(u)$.

Bipartite Coprime Graphs with Pure L-Bipartition

Are there bipartite coprime graphs with the only L-bipartition $(\mathcal{E},\mathcal{O})$ type? We shall answer affirmatively for connected regular bipartite graphs.

Theorems

- A connected regular bipartite coprime graph allows only pure L-bipartition.
- Every coprime labeling of cube Q_n is of the type pure L-bipartition.

Problems

- What happens in the Eulerian case?
- Study Eulerian bipartite coprime graphs for properties with respect to $(\mathcal{E},\mathcal{O})$ L-bipartition?

Complete Bipartite Subgraphs

Complete bipartite subgraphs in $\mathcal{M}_n$ arise in several ways. For orders ≤ 10 see:

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Table-2 Complete bipartite subgraphs of $\mathcal{M}_n$.

n	Km,n	Sets	
2	$K_{1,1}$	1; 2	
3	$K_{1,2}$	1,3; 2	
4	$K_{2,2}$	1,3; 2,4	
5	$K_{2,3}$	1,3,5; 2,4	
6	$K_{2,3}$	1,3,5; 2,4	1,5; 2,4,6
7	$K_{3,3}$; $K_{2,4}$	1,5,7; 2,4,6	1,3,5,7; 2,4
8	$K_{3,4}$	1,5,7; 2,4,8,6	1,3,5,7; 2,4,8
9	$K_{3,4}$; $K_{3,5}$	1,3,5,7,9; 2,4,8	1,5,7; 2,4,8,6
10	$K_{3,5}$	1,3,5,7,9; 2,4,8	1,3,5,7,9; 2,4,8

Theorems

- A largest complete bipartite subgraph in $\mathcal{M}_n$ is:
  $K_{|\mathscr{P}p2|,t}$ or $K_{|\mathscr{P}>2|+1,t)}$, when n=2t,
  $K_{|\mathscr{P}p2|,t+1}$ or $K_{|\mathscr{P}>2|+1,t+1}$, when n=2t+1,

Tout, R., A.N. Dabboucy and K. Howalla (1980,1982)

Consider the bipartite graph B(r,k), r,k integers>1, with bipartition $X_o(r)$, $X_k(r)$ where $X_o(r)$=r and $X_k(r)=k_r+1,..., k_r+r$ and $i\varepsilon X_o(r)$ is defined to be adjacent to $j\varepsilon X_k(r)$ if and only (i,j)=1.

Theorems

- B(r,k) has a perfect matching $f_{r,k}$: $X_o(r)\rightarrow X_k(r)$.
  This was used to prove:
- The graph $C_n(t)$ obtained from C_n by connecting each of its nodes to exactly t>1 new nodes is coprime.

A wheel graph W_n, for n even is not coprime as its independence number is <n/2.

- The wheel $W_n=K_1+C_{n-1}$, n an integer >4, is coprime if and only if $n\equiv 1 \pmod 2$.

BDAcharya (1981)

As an extension the graph G(t) obtained from a graph G by connecting each node of G to exactly t>1 new nodes.

Theorems

- If G is a coprime graph so is G(t) for any t>1.
- r-page book B_r satisfies $\beta_o(B_r)$=r+1 and is coprime for all rεN.
- $L_{m,n}$ were verified to be coprime for m≤3.

Conjecture (BDAcharya (1981))

- $L_{m,n}$ are coprime for all m,n.

Cycles in Coprime Graphs (1997)

Paul Erdos, Gabor N. Sarkozy

Cycles in the coprime graph of integers are studied. *f(n, k)* denotes the number of positive integers m≤n with a prime factor among the first k primes.

Theorems

- There exists a constant c such that if A ⊂{1,2,...,n} with |A|>f(n,2) (if 6|n then f(n,2)=(2/3)n), then the coprime graph induced by A not only contains a triangle, but also a cycle of length 2l+1 for every positive integer l≤cn.

Complete tripartite subgraphs in the coprime graph of integers

Gábor N. Sárközy-(2001)

Theorems

- Answer to a question of Paul Erdos: If there exist constants c, n_0 such that if $n\geq n_0$,

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A⊂{1,2,…,n} with $|A| > f(n,2)$ (if $6|n$ then $f(n,2) = \frac{2}{3}n$), then the coprime graph induced by A contains a complete tripartite graph on $2\lfloor c\frac{\log n}{\log\log\log n}\rfloor + 1$ nodes where one of the classes is a simple vertex and the other two classes each have $\lfloor c\frac{\log n}{\log\log\log n}\rfloor$ nodes.

On the Kernel of the Coprime Graph of Integers
J.W. Sander and T. Sander (2009)
The authors determine the *kernels* of the coprime graph and its loopless counterpart as well as *simple bases* for them (in case such bases exist), which means that basis vectors have entries only from {−1,0,1}. For the loopless version knowledge about the value distribution of Mertens' function is required.
Coprime Graphs with Cycles

Let us consider a complete bipartite graph and let *P(t,v)* be the set of all primes x such that t<x≤v. Then we have the following proposition.

Borosh, Hensley and Hobbs (1997)
Let G=∪ C_{2n_i}, for i=1,..,t and N=∑n_i, for i=1,…,t. There is a positive constant n' such that the conjecture of Deretsky et al. is true for the following cases:
Theorems
- G is the disjoint union of at most seven cycles;
- G is a union of cycles all of the same even length 2n where n≤150000 or where $n \geq n_0$; $n_i \geq (\log N)^{4 \log\log\log n}$ for all i=1,..,t; and when each C_{2n_i} is repeated at most n_i times.

In the end they discuss graphs whose components are all even cycles and with some components that are not cycles and some components that are odd cycles.

Seoud and Youssef (1999)
Theorems
- The following graphs are non-coprime:
    - C_m+C_n; C_{2n} for n>4;
    - P_{2n} for n=6 and for n>8; and
    - Mobius ladders M_n for n even.
- Formulae are given for:
    - the maximum number of edges in a coprime graph of order n
    - an upper bound for the chromatic number of a coprime graph.

Conjecture (Seoud and Youssef (1999) and Rao Hebbare (1981)))
- Unicycle graphs are coprime.

Baskar Babujee and Vishnupriya (2006)
Theorems
- The following graphs have coprime labelings: nP_2, $P_n \cup P_n \cup \cdots P_n$, bistars
- The graphs obtained by joining the centers of two identical stars with an edge), and the graph obtained by subdividing the edge joining edge of a bistar.
    - $(P_m \cup nK_1)+K_2$, $(C_m \cup nK_1)+K_2$, $(P_m \cup C_n \cup K_r)+K_2$, $C_n \cup C_n+1$,
    - $(2n-2)C_{2n}$ (n >1), $C_n \cup mP_k$
- The graph obtained by subdividing each edge of a star once.

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Youssef and Elsakhawi (2007)

Corona of a G of order n with H, $G\odot H$ is the graph obtained by taking one copy of G and n copies of H and joining the ith vertex of G with an edge to every vertex in the ith copy of H.

Theorems

- The union of stars $S_m \cup S_n$, are coprime;
- The union of cycles and stars $C_m \cup S_n$ are coprime;
- $K_m \cup P_n$ is coprime if and only if m is at most 3 or if m=4 and n is odd;
- $K_n \odot K_1$ is coprime if and only if $n \le 7$;
- $K_n \odot K_2$ is coprime if and only if $n \le 16$;
- $6K_m \cup S_n$ is coprime if and only if the number of primes less than or equal to m+n+1 is at least m; and
- The complement of every coprime graph of order $p \ge 20$ is non-coprime.

Coprime Graphs, B.Nirmala Gnanam Pricilla, 2008 Thesis

It is proved that the graph C_n^t (isomorphic), for all $n \ge 3$, $t \ge 1$ C_n^t (non isomorphic), for all n, $n \ge 3$, $t \ge 1$, W_n where n is odd and $C_n.K_1$, for all n are coprime graphs.

Theorems

- The one vertex union of t isomorphic copies of any cycle is coprime graph
- The one vertex union of t non isomorphic cycles of different length is coprime.
- Every cycle with pendent nodes is coprime.
- Every Wheel graph W_n is coprime graph when n is odd and $n \ge 4$.
- The graph $C_{nt}(C_n)$ for all $n \ge 3$, $t \ge 3$ is coprime graph.

S K Vaidya, K K Kanani (2010)

They investigate coprime labelings for some cycle related graphs in the context of some graph operations namely fusion, duplication and vertex switching in cycle C_n. Illustrations are given.

Theorems

- The graph obtained by identifying any two nodes v_i and v_j (where $d(v_i,v_j) \ge 3$) of cycle C_n is a coprime graph.
- The graph obtained by duplicating arbitrary vertex of cycle C_n is a coprime graph.
- The switching of any vertex in cycle C_n produces a coprime graph.
- The graph obtained by the path union of finite number of copies of cycle C_n is a coprime graph except for odd n.
- The graph obtained by joining two copies of cycle C_n by a path P_k is a coprime graph except n and k both are odd.

J. Baskar Babujee and L. Shobana (2010)

If every edge of graph G is subdivided, then the resulting graph is called *barycentric subdivision* of graph G (A problem posed in Rao Hebbare (1981)). In other words barycentric subdivision is the graph obtained by inserting a vertex of degree two into every edge of original graph. Consider barycentric subdivision of cycle and join each newly inserted nodes of incident edges by an edge. We denote the new graph by $C_n(C_n)$ as it look like *C_n inscribed in C_n*. The graph obtained by joining two copies of $C_n(C_n)$ by a path of length one is denoted by $K_2\Theta C_n(C_n)$. $G_1,G_2,\ldots,G_n$, $n \ge 2$ be n copies of the graph G. Adding an edge between G_i to G_{i+1} for $i=1,2,\ldots,n-1$ is called the *path union of G*.

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Theorems

- The barycentric subdivision of cycle C_n is coprime.
- The graph G obtained by two copies of $C_n(C_n)$ is coprime only when $n \equiv 0 \bmod 3$ joined by a path P_k, $k \geq 3$ and k is odd of arbitrary length is coprime.
- The graph $K_{1,m}\Theta K_{1,n}$ for all m, $n \geq 1$ is coprime labeling.

S. K. Vaidya and U. M. Prajapati (2011)

Let $G=<W_n:W_m>$ be the graph obtained by joining apex nodes of wheels W_n and W_m to a new vertex w. A *t-ply $P_t(u,v)$* is a graph with t paths, each of length at least two and such that no two paths have a vertex in common except for the end nodes u and v. The authors illustrate the theorems with examples.

Theorems:

- Let n_1 and n_2 be two even positive integers such that $n_1+n_2+3=p$ where p and $p-2$ are twin primes. Then the graph $G=<W_{n1}:W_{n2}>$ is a coprime graph.
- If $n_1 \geq 4$ is an even integer and $n_2 \in N$ then the disjoint union of the wheel W_{n1} and the path graph P_{n2} is a coprime graph.
- If $n_1 \geq 4$ is an even integer and $n_2 \in N$ then the graph obtained by identifying any of the rim nodes of a wheel W_{n1} with an end vertex of a path graph P_{n2} is a coprime graph.
- If n_1 is even then the graph G obtained by identifying the apex vertex of a wheel graph W_{n1} with an end vertex of P_{n2} is a coprime graph.
- Let G_1 be a coprime graph of order n_1 with a coprime labeling f and having nodes u_1 and u_{n1} with the labels 1 and n_1 respectively. Then the graph G obtained by identifying an end vertex of a path P_{n2} with either u_1 or u_{n1} of G_1 is a coprime graph.
- A graph G obtained by identifying all the apex nodes of m fans $f_{n1},f_{n2},\cdots,f_{nm}$ (is called a *multiple shell*) is a coprime graph.
- A graph G obtained by identifying all the apex nodes of m wheels $W_{n1},W_{n2},\cdots,W_{nm}$ is a coprime graph if each $n_i \geq 4$ is an even integer for each $i \in \{1,2,\cdots,m\}$ and n_i-1 is relatively prime with
  $$2+\sum_{k=1}^{i-1} n_k, \text{ for each } i \in \{2,3,\cdots,m\}.$$
- A t-ply graph $P_t(u,v)$ is a coprime graph if the order of $P_t(u,v)$ is a prime number.

Varkey and Singh, preprint

A *twig* as a graph obtained from a path by attaching exactly two pendent edges to each internal vertex of the path. A *lotus* graph is obtained from a nontrivial path of even length by joining every other vertex to one isolated vertex.The r-page book is the graph $B_r=K_{1,r} \times K_2$.

Theorems

The following graphs have coprime labelings:

- Ladders, crowns, cycles with a chord, books,
- One node unions of C_n, and L_n+K_1.
- A graph obtained by connecting two nodes with internally disjoint paths of equal length are coprime.
- Twigs obtained from a path of odd length ≥ 3 and lotus inside circle graphs is coprime.

Gallion (2011) regularly updates *dynamic survey of graph labeling problems* which in particular includes works coprime labelings. We shall also include works not referred in Gallion's survey.

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Joseph A. Gallian (2010)-Survey

For $m,n \geq 3$, define $S(m)_n$, the (m,n)-gon star, as the graph obtained from the cycle C_n by joining the two end nodes of the path P_{m-2} to every pair of consecutive nodes of the cycle such that each of the end nodes of the path is connected to exactly one vertex of the cycle.

Theorems

- All cycles and the disjoint union of C_{2k} and C_n.
- W_n is coprime if and only if n is even.
- Fans; Helms; Flowers, Stars;
- K_n is not a coprime for $n>4$
- $K_{2,n}$; and $K_{3,n}$ unless n=3 or 7.
- P_n+K_m (m>3) is not coprime.
- $C_m \odot K_n$ is coprime for all m and n.
- Books;
- Spanning subgraph of a coprime graph is coprime.
- Every graph is a subgraph of a coprime graph.
- *$S(m)_n$;*
- $C_n \odot P_m$;
- P_n+K_2 if and only if n=2 or n is odd; and
- $C_n \odot K_1$ with a complete binary tree of order $2k-1$ (k>2) attached at each pendant vertex.

MINIMAL NON-COPRIME GRAPHS

Rao Hebbare (1984), SNRao(2002)

Paul Erdös asked the following question at the Third MATSCIENCE Conference on Label Theory (3-6 June, 1981) at Mysore.

Erdös Problem. What is the minimum number of edges in a non-coprime graph of order n?

We answer this in two cases according as the graph is connected or not. Let $\mu(n)$ and $\mu'(n)$ denote the minimum number of edges a non-coprime graph can have in the respective cases. Below we obtain upper bounds for $\mu(n)$ and $\mu'(n)$ and conjecture that the bounds are attained. The results were reported in **(**Rao Hebbare (1984)). Recent results showing that the conjecture holds good for large values of n are included.

Connected Case

K_4 and K_5 are the only non-coprime graphs of order at most five. $\mu(n)$, $n\leq 3$ is undefined, $\mu(4)=6$ and $\mu(5)=10$. $\mu(6)=7$ and there is a unique (6,7)-graph which is not coprime. For n=7, $\mu(7)\leq 10$ holds and conjecture that $\mu(7)=10$ holds.

Theorem

$$\text{If } n\geq 8 \text{ then } \mu(n) \leq \begin{cases} n+1, & \text{if } n \text{ is even,} \\ n+2, & \text{otherwise.} \end{cases}$$

Conjecture (Rao Hebbare (1984))

- For a connected graph of order n (≥ 8), equality above holds for $\mu(n)$.

Truth of the conjecture implies:

  - The truth of Entringer's coprime tree conjecture and
  - All unicyclic graphs are coprime.

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Rao Hebbare (1984)
Disconnected Case

All disconnected graphs with at most five nodes are coprime and hence $\mu'(n)$, $n\leq 3$ is undefined and $\mu'(n)=\mu(n)$ for n=4,5. $\mu'(6)=6$ and there is a unique disconnected non-coprime graph namely, 2K3, that is, two copies of K3. $\mu'(7)\leq 9$ follows from the non-coprime graph $K3\cup K4$, and conjecture that $\mu'(7)=9$ holds. We prove the following:

Theorem

- If $n\geq 8$ then $\mu'(n)\leq n$.

Conjectures (Rao Hebbare (1984))

- For any graph of order n, $n\geq 8$, $\mu'(n)=n$ holds.
- 0% of all regular graphs are coprime.

Rao Hebbare (1981)
Strong Coprime Tree Conjecture

- For every tree of order n, a coprime labeling assigning n to a pendant node exists.

Penny Haxell, Oleg Pikhurko, Anusch Taraz (2010)

The results obtained earlier are useful in proving further results on coprime labeling or more general embedding problems. One example is:

Theorem

- $\mu(n)= 2\lceil n/2\rceil + 1$ for large n. That is, the Conjecture for $\mu(n)$ is true for all large n.

Not much progress on another conjecture of Rao Hebbare (1981) that the value of $\mu'(n)$, the smallest size of a not necessarily connected non-coprime graph of order n. However, they **claim that their** methods can prove *strong coprime tree conjecture* for all large n.

Tree Completeness and Primality
Rao Hebbare (1981)

A graph G of order n is said to be `tree-complete' if every tree of order n is a subgraph of G. Tree-complete graphs, which are not complete, are of interest (**Nebesky (1975)**). Entringer-Tout's Conjecture is equivalent to the following:

Conjecture (Rao Hebbare (1981))

- Maximum coprime graph $\mathcal{M}_n$ is tree-complete for all $n\geq 1$.

Theorems

- Let $i\varepsilon\{1,2\}$ and let $\mathcal{M}_n$ be such that every tree T_o of order n with $\Delta(T_o)\leq[(n+i)/2]$ is isomorphic to a spanning subgraph of G. Then $\mathcal{M}_{n+1}$ and $\mathcal{M}_{n+2}$ are tree complete. Further, if n+1 and n+3 are twin primes then $\mathcal{M}_{n+1},\ldots,\mathcal{M}_{n+5}$ are tree-complete.
    - Let $\mathcal{M}_n$ be a tree-complete graph such that n+1 is prime. Then $\mathcal{M}_{n+1}$ and $\mathcal{M}_{n+2}$ are tree complete. Further, if n+1 and n+3 are twin-primes then $\mathcal{M}_{n+1},\ldots,\mathcal{M}_{n+5}$ are tree-complete.

Entringer's Conjecture was verified for all $n\leq 15$. Further, if $\mathcal{M}_{16}$ is proved to be tree-complete then since 17 and 19 are twin-primes, it follows from above that $\mathcal{M}_i$ is tree-complete for i=17, 18, 19, 20, 21.

Problem.

Is $\mathcal{M}_{16}$ tree-complete?

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Primality Under Planarity Theme

SNRao (2009)
Maximal Planar Coprime (MPC) graphs
We consider the question: What is the maximal planar coprime graph in $\mathcal{M}_n$?
Theorems
- A maximal planar (p,q)-graph satisfies q=3p-6. For p≤6, an MPC (p,q)-graph exists for the pairs (2,1), (3,2), (4,5), (5,9), (6,11). The graphs are: K_2, K_3, K_4-e, W_5+e, $\mathcal{M}_6$. The node labeled 6 in any coprime labeling of the graph $\mathcal{M}_6$ has degree 2. Hence no MPC graph of order 6 exists with 12 edges. For p=7 there exists an MPC graph satisfying the equality with the coprime labeling using the node labels {1,2,…,7}.
- An MPC (p,q)-graph of order p and size q=3p-6 exists for every p≥7.

SNRao (2009)
Maximal Planar Bipartite Coprime (MPBC) graphs
Theorems
- An MPBC graph of order p and size q=2p-4 with diameter 2 exists for every p≥ 3.
- An MPBC graph of order p and size q=2p-4, exists for every p≥ 3 with arbitrarily large diameter.

Primality under Eulerian Theme

We consider extremal problems amongst coprime graphs with reference to Eulerian property.

SNRao (2009)
Maximal Eulerian coprime subgraph in $\mathcal{M}_n$
Theorems
- $\mathcal{M}_n$ is Eulerian for n=3 and $\mathcal{M}_n$, n≥4 is not Eulerian.
- A maximal Eulerian (n,q-od)-graph can always be derived from $\mathcal{M}_n$, where od is the number of odd degree nodes in $\mathcal{M}_n$.

A class of 4-regular Eulerian non-coprime graphs of order p≥6 is constructed. For p=6, a 6-cycle with any 3 alternate nodes adjacent results in a minimal 4-regular non-coprime graph.
- There exists a 4-regular Eulerian non-coprime graph of order p for every p≥6.
- An Eulerian non-coprime graph exists for every even order p, p≥8 and q=p+2.
- An Eulerian non-coprime graph exists for every odd order p, p≥ 9 and q=p+3.
- η(p) the number of edges in a minimal Eulerian non-coprime graph of order p satisfies:

$$\eta(p) \leq \begin{cases} p+2, & \text{for } p \text{ even}, p\geq 8; \\ p+3, & \text{for } p \text{ odd}, p\geq 9. \end{cases}$$

Conjecture (SNRao (2009))
- For Eulerian graphs of order p (≥8), equality above holds for η(p).

For Eulerian coprime graphs for small orders we have:
- Graphs with degree sequences DS:4,2,…,2 and DS:6,2,…2 are coprime graphs.

Problem
- What is an upper bound for disconnected non-coprime graphs with Eulerian components?

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The Class of Graphs E3
The DS:4,4,2,…,2 represents a (p,p+2)-graph consisting of three cycles C(r), C(s), C(t), where p=r+s+t-2 and q=r+s+t. these cycles are in chain with exactly one node in common, Let u,v be the nodes in common. The cycle C(s) has two parts say $s_o \geq 0$ and $s_e \geq 0$ so that $s_o+s_e \geq 1$ and $s=s_o+s_e+2$. Since s is odd exactly one of s_o and s_e is odd say s_o. Denote this family of graphs by E_3.
Theorems
Cases: E,E,O or E,O,E
- A graph in E_3 with exactly two of r,s,t are even is a coprime graph.

Case: O,O,O. Now we shall consider the case where all three of r,s,t are odd. In this case, we have only partial results as given in the following:
- A graph G in E_3 is coprime if $s_o=1$, $s_e \geq 0$.
- A graph G in E_3 is coprime if $s_e=0$ and $s_o>1$ if (t-1) doesn't divide r+s-1.
- A graph G from E_3 is a coprime graph if:
  - r+s+t-2 is prime.
  - $r \geq t$ and r+s+1 is prime
  - r+s>t and r+s+1 is prime
  - s>r+t and s is prime
  - r>t is prime and r is coprime with $r+s_e+1$, $r+s_o+1$or r+s-1.

Conjecture (SNRao (2009))
- Every graph of the class E_3 with all r,s,t odd is coprime.

Primality under Regularity Theme
BDAcharya (1981)
Theorems
- For every odd integer $r \geq 3$, all nonbipartite r-regular graphs are non-coprime. (A conjecture of Rao Hebbare for cubic graphs).
- Let r and s be any two positive integers exactly one of which is odd. Let G be an r-regular graph and H be any s-regular graph at least one of which is nonbipartite. Then GxH is non-coprime.
  - If G is a nonbipartite r-regular graph with $2<r\equiv 0 \pmod 2$ then GxK2 is non-coprime.

SNRao (2009)
Cubic coprime graphs of order p exist for p>6. There exists a cubic coprime graph of order 8 and is the graph Q_3. It is a planar bipartite coprime (8,12)-graph. A construction of cubic coprime graphs of order n, for $n \geq 10$ is considered first.
Theorems
- A necessary condition for the existence of an r-regular coprime graph of order n is $r \leq \delta(\mathcal{M}_n)$).
- A regular coprime graph of even order is bipartite.
  - If there exists an r-regular coprime graph of even order p then s-regular coprime graph of even order p exists for each s, $0 \leq s \leq r$.
- Regular coprime graphs of odd order are non-bipartite and so exist only for even degree.
- A cubic coprime graph of order:
  - n=4p exists whenever p>3 is prime.
  - $n=2^s p$ exists whenever p>3 is prime and $p \equiv 1 \pmod 3$.
  - n=2t exists for every $t \geq 4$.
- For any even n>1, say n=2t and $0 \leq \alpha \leq t-1$, the $t-\alpha$ pairs $(i+\alpha, 2t-(i-1))$, for $i=1,\ldots,t-\alpha$; and the α pairs $(i, t+\alpha-(i-1))$, for $i=1,\ldots,\alpha$ are coprime pairs if $2t+\alpha+1$ and $t+\alpha+1$ are prime.
- A necessary condition that $2t+\alpha+1$ and $t+\alpha+1$ are prime is that $n \equiv 0 \pmod 4$.

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- An (r+2)-regular coprime graph G of order n=2t=4s exists where r is the number of prime pairs in the Theorem above for values of $0\leq\alpha<2t$.
- An r-regular coprime graph of order n exists where r is the number of odd primes p≤n so that n+p is prime.

Regular Non-Bipartite Coprime (RNBP) Graphs
SNRao (2009)
As a regular coprime graph of even order is bipartite we have:
Theorems
- A nonbipartite coprime graph is of odd order and so is of even degree.

C_{2n+1}, n>0 is a 2-RNBP graph. Also, $\delta(\mathcal{M}_n)\geq 4$ for n≥11. So order of an RNBP graph is ≥11.
- A 4-RNBP graph of order:
  - n=4t, t>0 exists whenever r≥2.
  - n exists for every n prime.

Almost Regular Coprime Graphs
SNRao (2009)
An *almost regular graph* has two different degrees, say d_1 and d_2. Degree sequence of an AR-graph of order n with f_1 and f_2 nodes of degree d_1 and d_2 respectively is denoted by DS:$(n:d_1,d_2;f_1,f_2)$. An AR-graph which is coprime is denoted by ARC-graph.

Theorems
- (n=2t:2,4;t,t)- and (n=2t+1:2,4;t,t+1)- ARC graphs exist for n≥ 4.

Near r-regular graph is a graph with all nodes of degree r except for one.
- Existence of an r-regular graph of order n=2t implies existence of (r,r+2)-AR Graph of order n.
- Near regular coprime graphs of odd order 2t-1 and 2t+1, for t prime t≥ 3 exist of degrees 3 and 2 with a unique node of degree 2.

Primality of Hypercubes
SNRao (2009)
Cubes are defined recursively as follows: $Q_0=K_1$, $Q_1=K_2$, $Q_2=K_2 \times Q_1$, …, $Q_n=K_2 \times Q_{n-1}$. Hypercubes have several characterizations, See Harary, Hayes and Wu (1988). We refer to Laborde and Rao (1982) for a characterization of hypercubes and its symmetric structure. Cubes Q_n for n≤3 are coprime. The conjecture '*All cubes are coprime*' (by BDAcharya (1981) and SBRao(2009)) is answered in negative for orders ≥4.
Theorems
- Q_3 is the unique coprime graph of order 8.
- $K_{2,3}$ is forbidden as an induced subgraph in a cube Q_n, n≥3.
- Q_4 is not coprime.

This proof technique cannot be extended to higher order cubes although Q_n, n>1 has no induced $K_{2,3}$. Counting 3-factors in $\mathcal{E}$ and $\mathcal{O}$ helps in settling the primality of cubes.
- By definition Q_n is a highly symmetric n-regular bipartite graph.
- A necessary condition for a labeling of a bipartite graph to be coprime is that: Span((i-m of $\mathcal{O}$') in $\mathcal{E}$)≤ |i-ff subset in $\mathcal{E}$|.
- A necessary condition for a labeling of a bipartite graph to be coprime is that: Span($\mathcal{O}$') in $\mathcal{E}$≤|union of i-ff subsets for each i∈ $\mathcal{O}$' in $\mathcal{E}$|.

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By symmetry, the property holds for any subset of nodes in $\mathcal{O}$. In particular, it holds even for 3-m in $\mathcal{O}$. For the graph to be coprime there are at least as many 3-ff labels in $\mathcal{E}$ as span(3-m of $\mathcal{O}$) in $\mathcal{E}$.

- Every coprime labeling of Q_n satisfies that span(3-m in $\mathcal{O}$) in $\mathcal{E}$ >3-ff subset in $\mathcal{E}$.
- Q_n is not a coprime graph for $n \geq 4$.

Embedding and Coprimality
BDAcharya(1981), SNRao(2009)
Theorem

- Every graph order m can be embedded:
  - in a coprime graph.
  - as an induced subgraph in $\mathcal{M}_n$, for some n>m.
  - as an induced subgraph in a coprime graph of a suitable large order m>n.

Mutually Coprime Number (MCN) Set
SNRao (2009)
There are many ways of constructing MCN sets.
Algorithm 12.1
$a_1=5$, $a_2=21$, $a_3=a_1*a_2+1=5*21+1=106$, …, $a_i= \pi(a_1*\ldots*a_i-1)+1$, …

- The set of numbers $\{a_1,a_2,a_3,\ldots\}$ forms a set of mutually co-prime numbers.

Primality Under Graph Operations

Lee, Wui, and Yeh (1988)
Define Amal$\{(G_i,v_i)\}$, the amalgamation of $\{(G_i,v_i):i=1,..n\}$, as the graph obtained by taking the union of the G_i and identifying $v_1,v_2,..,v_n$ for a collection of graphs $G_1,..,G_n$ and some fixed vertex vi from each G_i,
Theorems

- Amal$\{(G_i,v_i)\}$ has a coprime labeling when G_i are paths and when G_i are cycles.
- The amalgamation of any number of copies of W_n, n odd, with a common vertex is not coprime.

Conjecture (Lee, Wui, and Yeh (1988))

- For any tree T and any vertex v from T, the amalgamation of two or more copies of T with v in common is coprime.
- Further, the amalgamation of two or more copies of W_n that share a common node is coprime when n is even ($n \neq 4$).

Carlson (2006)
The edge amalgamation Edgeamal$\{(G_i,u_iv_i)\}$ is the graph obtained by taking the union of all the G_i and identifying their fixed edges for any finite collection $\{G_i,u_iv_i\}$ of graphs G_i, each with a fixed edge u_iv_i. The case where all the graphs are cycles are called generalized books.
Theorems

- All generalized books are coprime graphs.
- Graphs obtained by taking the union of cycles and identifying in each cycle the path P_n are coprime.
- C_m-snakes are coprime.

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Vilfred, Somasundaram and Nicholas, preprint
Theorems

- Conjecture (Lee, Wui, and Yeh (1988)) holds for the case that $n \equiv 2 \pmod 4$ where the central nodes are identified.
- Helms are coprime;
- The grid $P_m \times P_n$ is coprime when $m \leq 3$ and n is a prime greater than m;
- The double cone $C_n + \overline{K_2}$ is coprime only for n=3;
- The double fan $P_n \times \overline{K_2}$ ($n \neq 2$) is coprime if and only if n is odd or n=2; and
- Every cycle with a P_k-chord is coprime.
- **Conjecture** (Vilfred, Somasundaram and Nicholas ()) The grid $P_m \times P_n$ is coprime when n is prime and n>m.

Sundaram, Ponraj, and Somasundaram (2006)
Theorems

- The grid $P_m \times P_n$ is coprime when n is prime and n>m settling the conjecture of Vilfred, Somasundaram and Nicholas.
- $P_m \times P_n$ is coprime when n is prime.

Kanetkar (2009)
Theorems

- $P_6 \times P_6$ is coprime;
- $P_{n+1} \times P_{n+1}$ is coprime when n is a prime with $n \equiv 3 \text{or} 9 \pmod{10}$ and $(n+1)^2+1$ is also prime;
- $P_n \times P_{n+2}$ is coprime when n is an odd prime with $n \not\equiv 2 \pmod 7$.

Problems on Coprime Graphs (SNRao (2009))

- Investigate generalized Petersen graphs for primality.
- Does every regular coprime graph of even order have only the bipartition (E,O)? If not give an example.
- Construction of r-regular coprime graphs for all $n \geq 8$ for $r \leq \delta$, where $\delta(\mathcal{M}_n)=\delta$. It is known that such graphs don't exist for some δ. E.g. δ=6 and n=18; δ=7 and n=20.
- Determine r for which there exists an r-connected coprime graph.
- Investigate $mK_{n,n}$, $K_{n,n} \cup K_{m,m}, \ldots$ for coprimality.
- Diameter and radius of $\mathcal{M}_n$ is 1. Describe or construct coprime graphs with large radii and large diameters. In other words, maximize radius and diameter with edges.
- Characterize:
    - Maximum planar coprime graphs.
    - Maximum planar bipartite coprime graphs.
    - Study the class of $B_{m,n}$ graphs.

Closely related to Coprime Sets

On the number of co-prime-free sets, **Neil J. Calkin and Andrew Granville** (2007).
For a variety of arithmetic properties P (such as the one in the title) the authors investigate the number of subsets of the positive integers x, having that property. The authors answer some questions posed by Cameron and Erdos.

Closely related to Coprime Labelings and Coprime Graphs

k−Coprime Graphs (S. K. Vaidya, U. M. Prajapati 2011)

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A *k−coprime labeling* of a graph G is an injective function $f:V\to\{k,k+1,k+2,k+3,\cdots,k+|V|-1\}$ for some positive integer k that induces a function $f:E(G)\to N$ of the edges of G defined by $f(uv)=\gcd(f(u),f(v))$, $\forall e=uv\in E(G)$ such that $\gcd(f(u),f(v))=1$, $\forall e=uv\in E(G)$. A *k−coprime graph* admits a k−coprime labeling. A *tadpole* is a graph obtained by identifying a vertex of a cycle to an end vertex of a path. The authors illustrate the theorems with examples.

Theorems

- For each positive integer m the path graph P_m is a k−coprime graph for each positive integer k.
- The graph G obtained by disjoint union of a coprime graph G_1 of order n_1 and a (n_1+1)−coprime graph G_2 is a coprime graph.
- Let G_1 be a coprime graph of order n_1 with a coprime labeling f_1 and having nodes u_1 and u_{n1} with $f_1(u_1)=1$ and $f_1(u_{n1})=n_1$. Let G_2 be a n_1−coprime graph of order n_2 with a n_1-coprime labeling f_2 having a vertex v_1 with $f_2(v_1)=n_1$. Then the graph G obtained by identifying the vertex v_1 of G_2 with either to u_1 or to u_{n1} of G_1 is a coprime graph.
  - A tadpole is a coprime graph.

Vertex Coprime Labeling

Deretsky, Lee, and Mitchem (1991)

A dual of coprime labeling a graph with edge set E has a *vertex coprime labeling* if its edges can be labelled with distinct integers 1,..,|E| such that for each vertex of degree at least 2 the greatest common divisor of the labels on its incident edges is 1.

Theorems

The following graphs have vertex coprime labelings:

- Forests;
- All connected graphs;
- $C_{2k}\cup C_n$; $C_{2m}\cup C_{2n}\cup C_{2k+1}$; $C_{2m}\cup C_{2n}\cup C_{2t}\cup C_k$; and $5C_{2m}$.
- A graph with exactly two components, one of which is not an odd cycle
- A 2-regular graph with at least two odd cycles *does not* have a vertex coprime labeling.

Conjecture (Deretsky, Lee, and Mitchem (1991)). A 2-regular graph has a vertex coprime labeling if and only if it does not have two odd cycles.

J. Baskar Babujee (2010)

The graph $\mathbf{Pl_n}(V,E)$ where $V=\{1,2,...,n\}$ and $E=E(K_n)\setminus\{(k,l):3\le k\le n-2 \text{ and } k+2\le l\le n\}$ is a planar graph having maximum number of edges, with n nodes. The planar graph Pl_n having maximum number of edges with n nodes is obtained by removal of $[(n-4)(n-3)]/2$ edges from K_n. The number of edges in Pl_n, $n\ge5$ is $3(n-2)$. It is interesting to check whether coprime labeling is possible for Pl_n class. Investigations show that Pl_n does not admits coprime labeling for n even.

Theorem

- The class of Planar graphs Pl_n: n odd admits vertex coprime labeling.

Coprime Magic Labelings

Baca, M., Hollander, I. (1990), Baca, M (2003)

Theorems

- Coprime-magic labeling for the complete bipartite graph $K_{4,4}$ and for $K_{n,n}$, $n\ge5$,

They also formulate a conjecture.

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Coprime Cordial Labelings
J. Baskar Babujee and L. Shobana (2010)
A *coprime cordial labeling* of a graph G with vertex set V is a bijection f from V to {1,2,...,|V|} such that if each edge uv is assigned the label 1 if gcd(f(u),f(v))=1 and 0 if gcd(f(u),f(v))>1, and the number of edges labeled with 0 and the number of edges labeled with 1 differ by atmost 1.

Theorems

- The graph $K_2\Theta C_6(C_6)$ admits coprime cordial labeling if n≡0,2(mod 3). The graph $K_2\Theta C_6(C_6)$ is coprime cordial.
- The graph<$K_{1,n,n}$>for n ≥ 3 admits coprime cordial labeling.
- The graph S(1) n:S(2) n for all n>2 admits coprime cordial labeling.
- The full binary tree admits coprime cordial labeling.

Closely Related Concepts
In this section we shall include some works closed related to Coprime graphs like extensions, other definitions, …

Coprime Directed Graphs
Directed Acyclic Graph (DAG) could be used for modeling subsumption hierarchies. Several labeling schemes have been proposed or tailored for indexing DAG in order to efficiently explore relationships in such hierarchy. However few of them can satisfy all the requirements in response time, space, and effect of updates simultaneously. In this paper, the coprime number labeling scheme is extended for DAG. The scheme invests intrinsic mapping between integer divisibility and subsumption hierarchy, which simplifies the transitive closure computations and diminishes storage redundancy, as well as inherits the dynamic labeling ability from original scheme. Performance is further improved by introducing some optimization techniques. Our extensive experimental results show that coprime number labeling scheme for DAG outperforms interval-based and prefix-based labeling schemes in most cases.

Prime Distance Graphs

2-odd Graphs and Prime Distance Graphs
Starr, C. and Laison, J. (2011), 2-odd Graphs and Prime Distance Graphs, Paper presented at the annual meeting of The Mathematical Association of America MathFest, Portland Marriott Downtown Waterfront, Portland.
http://www.allacademic.com/meta/p377971_index.html
A graph G is a *prime-distance graph* if the nodes can be labeled with distinct integers in such a way that the differences between the labels on adjacent nodes are all prime. A graph is 2-odd if the differences are either exactly 2 or odd.

- They characterize 2-odd graphs and give a family of 2-odd circulant graphs. They propose a conjecture relating prime distance graphs and 2-odd graphs.

Prime Number Labeling for Dynamic Ordered XML Trees

"A Prime Number Labeling Scheme for Dynamic Ordered XML Trees," Xiaodong Wu, Mong Li Lee, Wynne Hsu, Data Engineering, International Conference on, pp. 66, 20th International Conference on Data Engineering (ICDE'04), 2004. Xiaodong Wu, Mong Li Lee, Wynne Hsu, National University of Singapore.
http://doi.ieeecomputersociety.org/10.1109/ICDE.2004.1319985

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Efficient evaluation of XML queries requires the determination of whether a relationship exists between two elements. A number of labeling schemes have been designed to label the element nodes such that the relationships between nodes can be easily determined by comparing their labels. With the increased popularity of XML on the web, finding a labeling scheme that is able to support order-sensitive queries in the presence of dynamic updates becomes urgent. In this paper, we propose a new labeling scheme that takes advantage of the unique property of prime numbers to meet this need. The global order of the nodes can be captured by generating simultaneous congruence values from the prime number node labels. Theoretical analysis of the label size requirements for the various labeling schemes is given. Experiment results indicate that the prime number labeling scheme is compact compared to existing dynamic labeling schemes, and provides efficient support to order-sensitive queries and updates.

https://files.pbworks.com/download/e3PLMrC3tq/mathfest/20530376/MathFest09-IPS-CPS-schedule.pdf
Graph Labeling Problems, Appropriate for Undergraduate Research
Distance labeling requires an assignment of labels (positive integers) to the nodes of the graph so as to satisfy one of several conditions involving the differences between the labels and the distances between the nodes. (The motivating context is that of assigning frequencies to transmitters so that interference is avoided.)

Many labeling problems are immediately accessible to undergraduates and are valuable to their learning what research is and how it may be conducted. We'll discuss solved problems to which undergraduates contributed and some open problems ripe for undergraduate exploration.

IPS5: Research with undergraduates;
Graph Labeling Problems Appropriate, **Cynthia Wyels,** CSU Channel Islands

International Workshop on Graph Labelings 2010

The seventh workshop in the series took place at University of Minnesota Duluth campus on October 20–22, 2010. IWOGL 2010 was immediately followed by MIGHTY L, held on October 22–23 at University of Wisconsin-Superior campus just across the bridge.

The workshop consisted of keynote and invited lectures by distinguished experts in graph labelings and contributed talks by other participants.

Selected papers will be published in

Journal of the Indonesian Mathematical Society—JIMS
(Formerly: **Majalah Ilmiah Himpunan Matematika Indonesia - MIHMI)**

University Success
The field of graph labeling is a rapidly expanding field with contributions from all over the world. Many of the published papers come from India and China, but UMD has two leading scholars as well. "We are very proud that UMD is the first American university to host this workshop," Froncek said. "It just shows the strength of our department and our university."
For information contact: iwogl@d.umn.edu
http://www.d.umn.edu/math/iwogl/
http://www.d.umn.edu/unirel/homepage/10/graphlabeling.html

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