Dirichlet polynomials: some old and recent results, and their interplay in number theory

Michel WEBER

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Abstract

In the first part of this expository paper, we present and discuss the interplay of Dirichlet polynomials in some classical problems of number theory, notably the Lindelöf Hypothesis. We review some typical properties of their means and continue with some investigations concerning their supremum properties. Their random counterpart is considered in the last part of the paper, where a analysis of their supremum properties, based on methods of stochastic processes, is developed.

1 Introduction

This is an expository article on the interplay of Dirichlet polynomials in some classical problems of number theory, notably the Lindelöf Hypothesis (LH), the Riemann Hypothesis (RH), as well as on their typical means and supremum properties. Some of the efficient methods used in this context are also sketched. In recent works [26],[27],[28],[47],[48],[49],[50] lying at the interface of probability theory, the theory of Dirichlet polynomials and of the one of the Riemann zeta-function, we had to apply and combine these results. The growing interaction between various specialities of analysis, further motivated us in this project to put in the same framework a certain number of basic and very important results and tools arising from the theory of Dirichlet polynomials and of the Riemann zeta-function; and to let them at disposal to analysts and probabilists who are not necessarily number theorists. In doing so, our wish is to spare their time in the sometimes tedious enterprise of finding the relevant results with the mostly appropriate methods to establish them.

The interplay with the LH and RH is presented in Section 2, where some equivalent reformulations of the LH in terms of approximating Dirichlet polynomials, arising notably from Túran's works, are discussed. The link between the RH and the absence of zeros of the approximating Dirichlet polynomials in some regions of the complex plane was thoroughly investigated by Túran in [41],[42],[43],[44] and later by Montgomery in [29],[30]. Some of the most striking results are presented.

In Section 3, we investigate the behavior of the mean value of Dirichlet polynomials. The used reference sources are [21], [30], [36] and naturally [40]. To begin, we follow an approach based on the Fourier inversion formula. Next, the mean value estimates are established by means of a version of Hilbert's inequality due to Montgomery and Vaughan [29]. A simple argument is provided for establishing the lower bound. The corresponding results for the zeta-function are briefly mentionned and discussed. Some basic results concerning suprema of Dirichlet polynomials are presented in Section 4. The analysis of the suprema of their random counterpart is made in the two last sections. These are notably built on works of Halász [13], Bayart, Konyagin, Queffélec [1],[23],[33],[34],[35] and Lisfhits, Weber [26],[27],[47].

2 Interplay in Number Theory

To any real valued function d defined on the integers, we may associate the Dirichlet polynomials

$$D_N(s) = \sum_{n=1}^{N} \frac{d(n)}{n^s}, \qquad\qquad (s = \sigma + it). \tag{1}$$

The particular case $d(n) \equiv 1$ is already of crucial importance, since it is intimately related to the behavior of the Riemann Zeta-function. We shall first investigate this link. Recall that the Riemann Zeta-function is defined on the half-plane $\{s : \Re s > 1\}$ by the series $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, which admits a meromorphic continuation to the entire complex plane.

And we have the following classical approximation result ([40], Theorem 4.11)

$$\zeta(s) = \sum_{n \le x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + \mathcal{O}(x^{-\sigma}), \tag{2}$$

uniformly in the region $\sigma \ge \sigma_0 > 0$, $|t| \le T_x := 2\pi x/C$, C being a constant > 1. The celebrated Lindelöf Hypothesis claiming that

$$\zeta(\frac{1}{2} + it) = \mathcal{O}_\varepsilon(|t|^\varepsilon) \tag{3}$$

can be reformulated in terms of Dirichlet polynomials. This was observed since quite a long time and Turán [42] had shown that the truth of the inequality

$$\Big|\sum_{n=1}^{N} \frac{(-1)^n}{n^{it}}\Big| \le C N^{1/2+\varepsilon}(2 + |t|)^\varepsilon \tag{4}$$

with an arbitrary small $\varepsilon > 0$, is equivalent to the LH. Alternatively, the equivalent reformulation ([40], Chap. XIII)

$$\frac{1}{T}\int_1^T \big|\zeta(\frac{1}{2} + it)\big|^{2k} \mathrm{d}t = \mathcal{O}_\varepsilon\big(T^\varepsilon\big), \qquad k = 1, 2, \ldots \tag{5}$$

which reduces to

$$\int_{T/2}^T \big|\zeta(\frac{1}{2} + it)\big|^{2k} \mathrm{d}t = \mathcal{O}_\varepsilon\big(T^{1+\varepsilon}\big), \qquad k = 1, 2, \ldots \tag{6}$$

yields when combined with (2), the equivalence with

$$\int_{n/2}^{n} \Big| \sum_{m=1}^{n} \frac{1}{m^{1/2+it}} \Big|^{2k} \mathrm{d}t = \mathcal{O}\big(n^{1+\varepsilon}\big), \qquad k = 1, 2, \ldots \tag{7}$$

or, by using Euler-MacLaurin formula, with

$$\int_{n/2}^{n} \Big| s \int_{0}^{n} y^{-1-s} B_1(y) \mathrm{d}y \Big|^{2k} \mathrm{d}t = \mathcal{O}\big(n^{1+\varepsilon}\big), \qquad k = 1, 2, \ldots \tag{8}$$

where $B_1(y) = \{y\} - 1/2$, mod 1 is the first Bernoulli function. The term $n^{1-s}/(1-s)$ can be indeed neglected since

$$\int_{\frac{n}{2}}^{n} \Big| \frac{n^{\frac{1}{2}-it}}{\frac{1}{2}-it} \Big|^{2k} \mathrm{d}t \le C n^k \int_{\frac{n}{2}}^{\infty} \frac{\mathrm{d}t}{(\frac{1}{4}+t^2)^k} \le C_k \, n^k n^{-2k+1} \le C_k \, n^{1-k}.$$

We see with (7) that all the mystery of the LH is hidden in the Dirichlet sum $\sum_{m=1}^{n} m^{-1/2-it}$, $n/2 \le t \le n$.

Remark 1 The best known result is due to Huxley [19],

$$\zeta(\frac{1}{2} + it) = \mathcal{O}_\varepsilon(t^{32/205+\varepsilon}). \tag{9}$$

In the recent work [28], in order to understand the behavior of $\zeta(\frac{1}{2} + it)$ as t tends to infinity, the time t is modelized by a Cauchy random walk, namely the sequence of partial sums $S_n = X_1 + \ldots + X_n$ of a sequence of independent Cauchy distributed random variables $X_1, X_2, \ldots$ (with characteristic function $\varphi(t) = e^{-|t|}$). The almost sure asymptotic behavior of the system

$$\zeta_n := \zeta(\frac{1}{2} + iS_n), \qquad n = 1, 2, \ldots$$

is investigated. Put for any positive integer n

$$\mathcal{Z}_n = \zeta(1/2 + iS_n) - \mathbf{E}\, \zeta(1/2 + iS_n) = \zeta_n - \mathbf{E}\, \zeta_n.$$

The crucial preliminary study of second order properties of the system $\{\mathcal{Z}_n, n \ge 1\}$ yields the striking fact that this one nearly behaves like a system of non-correlated variables, i.e. the variables $\mathcal{Z}_n$ are weakly orthogonal. More precisely, there exist constants C, C_0

$$\mathbf{E}\, |\mathcal{Z}_n|^2 = \log n + C + o(1), \qquad n \to \infty,$$

and for $m > n+1$,

$$\big|\mathbf{E}\, \mathcal{Z}_n \overline{\mathcal{Z}_m}\big| \le C_0 \max\Big(\frac{1}{n}, \frac{1}{2^{m-n}}\Big). \tag{10}$$

The proof is very technical. And the main result of [28], which follows from a convergence criterion, states

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \zeta(\frac{1}{2} + iS_k) - n}{n^{1/2} (\log n)^b} \overset{(a.s.)}{=} 0, \tag{11}$$

for any real $b > 2$.

Now we pass to the interplay with the Riemann Hypothesis. As is well-known, the Riemann Zeta-function has 1 as unique and simple pole in the whole complex plane. In the half-plane $\Re z \leq 0$, the Riemann Zeta-function has simple zeros at $-2, -4, -6, \ldots$, and only at these points which are called trivial zeros. There exist also non-trivial zeros in the band $\{s : 0 < \Re s < 1\}$. The Riemann Hypothesis asserts that all non-trivial zeros of the ζ-function have abscissa $\frac{1}{2}$. The validity of RH implies ([40], p.300) that

$$\zeta(\frac{1}{2} + it) = \mathcal{O}\Big(\exp\big\{A\frac{\log t}{\log\log t}\big\}\Big),$$

A being a constant, which is even a stronger form of LH; the latter being strictly weaker than RH.

In several papers, Turán investigated the interconnection between the RH and the absence of zeros of the approximating Dirichlet polynomials in some regions of the complex plane. For instance, he proved in [41] that if for $n > n_0$, none of the Dirichlet polynomials $\sum_{m=1}^{n} m^{-s}$ vanishes in a half-strip

$$\sigma \geq 1 + \frac{\log^3 n}{\sqrt{n}}, \qquad \gamma_n \leq t \leq \gamma_n + e^{n^3},$$

with a suitable real γ_n, then the RH is true.

However Montgomery has shown in [30] that for $n > n_0$, any interval $[\gamma, \gamma + e^{n^3}]$ contains the imaginary part of a zero. Another definitive result established by Montgomery in the same paper is that if $0 < c < \frac{4}{\pi} - 1$, then for all $n > n(c)$, $\sum_{m=1}^{n} m^{-s}$ has zeros in the half-plane

$$\sigma > 1 + c\frac{\log\log n}{\log n}, \tag{12}$$

whereas if $c > \frac{4}{\pi} - 1$, $n > n(c)$, then $\big|\sum_{m=1}^{n} m^{-s} - \zeta(s)\big| \leq |\zeta(s)|/2$, so that $\sum_{m=1}^{n} m^{-s}$ do not vanish in a half-strip (12).

In the other direction, Turán showed ([44], Satz II) that if the RH is true, none of the Dirichlet polynomials $\sum_{m=1}^{n} m^{-s}$ vanishes in a half-strip

$$\sigma \geq 1, \qquad c_1 \leq t \leq e^{c_2\sqrt{\log n \log\log n}}, \tag{13}$$

when $n \geq c_3$.

There are also results for Dirichlet polynomials expanded over the primes. Their local suprema are intimately connected with zerofree regions of the Riemann zeta-function. Among several results proved in [43], we may quote the following. Suppose there are constants $\alpha \geq 2$, $0 < \beta \leq 1$, $\tau(\alpha, \beta)$, such that for a $\tau > \tau(\alpha, \beta)$ the inequality

$$\Big|\sum_{N_1 \leq p \leq N_2} \frac{1}{p^{i\tau}}\Big| \leq \frac{N \log^{10} N}{\tau^\beta} \tag{14}$$

holds for all N_1, N_2 integers with $\tau^\alpha \leq N \leq N_1 \leq N_2 \leq 2N \leq e^{\tau^{\beta/10}}$. Then ([43], Theorem 1) $\zeta(\sigma + i\tau)$ does not vanish if $\sigma > 1 - \beta^3/(e^{10}\alpha^2)$. For the sake of orientation, Turán also remarked that for the sum

$$S = \sum_{N \leq n \leq 2N} \frac{1}{n^{it}}, \qquad \tau \geq 2$$

the elementary formula $|(\nu+1)^{1+i\tau} - \nu^{1+i\tau} - (1+i\tau)\nu^{i\tau}| \ll \tau^2/\nu$, gives at once

$$|S| \ll (N \log N)/\tau \tag{15}$$

if $N \geq \tau^2$. But the relevant sum is $\sum_{n=1}^{N} \frac{(-1)^n}{n^{it}}$ according to (4).

Farag recently showed in [10] that these sums possess zeros near every vertical line in the critical strip. The proof is notably based on a "localized" version of Kronecker's Theorem (section 3).

The likely best known result concerning zerofree regions is due to Ford [11]: $\zeta(\sigma+it) \neq 0$ whenever $|t| \geq 3$ and

$$\sigma \geq 1 - \frac{1}{57.54(\log|t|)^{2/3}(\log\log|t|)^{1/3}}. \tag{16}$$

Remark 2 Speaking of the RH, it is difficult not mentionning the striking equivalent reformulation proved by Robin in [37], which is at the same time likely the most simple. Let an integer n be termed "colossally abundant" if, for some $\varepsilon > 0$, $\sigma(n)/n^{1+\varepsilon} \geq \sigma(m)/m^{1+\varepsilon}$ for $m < n$ and $\sigma(n)/n^{1+\varepsilon} > \sigma(m)/m^{1+\varepsilon}$ for $m > n$, where $\sigma(n)$ is the sum of divisors of n. Using colossally abundant numbers, Robin showed that the RH is true if and only if

$$\frac{\sigma(n)}{n} < e^{\gamma} \log\log n,$$

for $n > 5040$, where γ is Euler's constant. Let $\{x_n, n \geq 1\}$ be the sequence of colossally abundant numbers. In the same paper, he also showed that the sequence $\{\sigma(x_n)/x_n \log\log x_n, n \geq 1\}$ contains an infinite number of local extrema. In relation with Robin's result, Lagarias showed in [24] that the RH is true if and only if

$$\sigma(n) \leq H_n + e^{H_n} \log H_n,$$

where $H_n = \sum_{j \leq n} 1/j$ is the n-th harmonic number.

Grytczuk [12] investigated the upper bound for $\sigma(n)$ with some different n. Let $(2,n) = 1$ and $n = \prod_{j=1}^{k} p_j^{\alpha_j}$, where the p_j are prime numbers and $\alpha_j \geq 1$. Then, for all odd positive integers $n > 3^9/2$,

$$\sigma(2n) < \frac{39}{40} e^{\gamma} 2n \log\log 2n, \qquad \text{and} \qquad \sigma(n) < e^{\gamma} n \log\log n.$$

Some other criteria equivalent to the RH can be found in [6].

3 Mean Values of Dirichlet Polynomials

Let k be some positive integer. Considerable efforts were made to finding good estimates for the mean integrals

$$\frac{1}{2T} \int_{-T}^{T} \Big| \sum_{n=1}^{N} \frac{1}{n^{\sigma+it}} \Big|^{2k} \mathrm{d}t,$$

because of Bohr's theory of almost periodic functions, and of the "mean value" equivalent reformulations of the LH. First notice that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \Big| \sum_{n=1}^{N} \frac{1}{n^{\sigma+it}} \Big|^{2k} \mathrm{d}t = \sum_{m=1}^{\infty} \frac{d_{k,N}^2(m)}{m^{2\sigma}}, \tag{17}$$

where $d_{k,N}(m)$ denotes the number of representations of m as a product of k factors less or equal to N. We propose to deduce this from the Fourier inversion formula. This seems to be a natural approach, although we could not refer to some book or paper. If ν is a distribution on $\mathbf{R}$ and $\widehat{\nu}(t) = \int_{\mathbf{R}} e^{itx}\nu(dx)$ denotes its characteristic function, then

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-itx_0}\widehat{\nu}(t)dt = \nu\{x_0\}. \tag{18}$$

From this also follows that $\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\widehat{\nu}(t)|^2 dt = \sum_{x\in\mathbf{R}} \nu(\{x\})^2$ and more generally

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\widehat{\nu}(t)|^{2k} dt = \sum_{x\in\mathbf{R}} \nu^{*k}(\{x\})^2, \tag{19}$$

for any positive integer k. Apply (19) to the measure $\nu = \sum_{n=1}^{N} \frac{1}{n^\sigma}\delta_{\{-\log n\}}$, where $\delta_{\{x\}}$ is the Dirac measure at point x, then $\widehat{\nu}(t) = \sum_{n=1}^{N} n^{-(\sigma+it)}$ and

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \Big|\sum_{n=1}^{N} \frac{1}{n^{\sigma+it}}\Big|^{2k} dt = \sum_{x\in\mathbf{R}} \Big(\sum_{n_1\ldots n_k = e^x} \frac{1}{n_1^\sigma \ldots n_k^\sigma}\Big)^2$$

$$= \sum_{\substack{Y=e^x\\ Y\in\mathbf{N}}} \frac{\#\{(n_1,\ldots,n_k), n_i \le N : \prod_{i=1}^{k} n_i = Y\}^2}{Y^{2\sigma}} = \sum_{Y\in\mathbf{N}} \frac{d_{k,N}^2(Y)}{Y^{2\sigma}}.$$

Clearly

$$\lim_{N\to\infty} \sum_{Y\in\mathbf{N}} \frac{d_{k,N}^2(Y)}{Y^{2\sigma}} = \sum_{m=1}^{\infty} \frac{d_k^2(m)}{m^{2\sigma}},$$

where $d_k(m)$ denotes the number of representations of m as a product of k factors. An equivalent formulation of the LH being that for $\sigma > \frac{1}{2}$, $k = 1, 2, \ldots$

$$\lim_{T\to\infty} \frac{1}{T}\int_{1}^{T} \big|\zeta(\sigma+it)\big|^{2k} \mathrm{d}t = \sum_{m=1}^{\infty} \frac{d_k^2(m)}{m^{2\sigma}}, \tag{20}$$

the LH can be interpreted as a kind of generalized Fourier inversion formula for the infinite measure $\sum_{n=1}^{\infty} \frac{1}{n^\sigma}\delta_{\{-\log n\}}$. This approach is investigated in [48], see also [51] section 13.7.

One can precise a little more (18). Let

$$\mathbf{M}_T(\nu, x_0) = \frac{1}{2T}\int_{-T}^{T} e^{-itx_0}\widehat{\nu}(t)dt.$$

Proposition 3 *For any non-decreasing sequence $\{T_p, p \ge 1\}$ of positive reals, any x_0*

$$\sum_{k=1}^{\infty} \big|\mathbf{M}_{T_{k+1}}(\nu, x_0) - \mathbf{M}_{T_k}(\nu, x_0)\big|^2 \le 24\nu^2(\mathbf{R}). \tag{21}$$

The total mass of the measure appears this time, unlike in (18). Notice also that (21) alone already implies that $\mathbf{M}_T(\nu, x_0)$ converges, as T tends to infinity.

Proof. Consider the kernels

$$\mathcal{V}_T(\vartheta) = \frac{e^{iT\vartheta}-1}{iT\vartheta}, \qquad V_T(y) = \Re\big(\mathcal{V}_T(y)\big) = \frac{\sin Ty}{Ty}.$$

By the Cauchy-Schwarz inequality, we first observe that

$$\begin{aligned}\big|\mathbf{M}_{T_2}(x_0) - \mathbf{M}_{T_1}(x_0)\big|^2 &= \Big|\int_{\mathbf{R}} \big[V_{T_2}(x-x_0) - V_{T_1}(x-x_0)\big]\nu(dx)\Big|^2 \\ \leq\ & \nu(\mathbf{R}).\int_{\mathbf{R}} \big[V_{T_2}(x-x_0) - V_{T_1}(x-x_0)\big]^2\nu(dx).\end{aligned}$$

Introduce a new measure $\hat{\nu}$, a regularization of ν defined as follows:

$$\frac{d\hat{\nu}}{dx}(x) = \int_{|\vartheta|<|x|} |x|^{-3}\vartheta^2\nu(d\vartheta) + \int_{|\vartheta|\geq|x|} |\vartheta|^{-1}\nu(d\vartheta).$$

From the basic elementary inequalities

$$\begin{aligned}|\mathcal{V}_{T_2}(\vartheta) - \mathcal{V}_{T_1}(\vartheta)| &\leq \min\Big\{\frac{T_2-T_1}{2}|\vartheta|, \frac{2(T_2-T_1)}{T_2}\Big\}, \qquad T_2 \geq T_1, \\ |\mathcal{V}_{T_1}(\vartheta)| &\leq \frac{2}{T_1|\vartheta|},\end{aligned}$$

we have ([25], Section 4)

$$\|\mathcal{V}_{T_2} - \mathcal{V}_{T_1}\|_{2,\nu}^2 \leq 8\hat{\nu}(\frac{1}{T_2}, \frac{1}{T_1}].$$

Thereby

$$\begin{aligned}\int_{\mathbf{R}} \big[V_{T_2}(x-x_0) - V_{T_1}(x-x_0)\big]^2\nu(dx) &\leq \int_{\mathbf{R}} \big|\mathcal{V}_{T_2}(y) - \mathcal{V}_{T_1}(y)\big|^2\nu_{x_0}(dy) \\ &\leq 8\widehat{\nu}_{x_0}(\frac{1}{T_2}, \frac{1}{T_1}],\end{aligned}$$

where we write $\nu_y(A) = \nu(A-y)$, for each $A \in \mathcal{B}(\mathbf{R})$. And

$$\widehat{\nu}_{x_0}(\mathbf{R}) = \int \left(\int_{|\vartheta|<|x|} |x|^{-3}dx\ \vartheta^2 + \int_{|x|<|\vartheta|} dx|\vartheta|^{-1}\right)\nu(d\vartheta) \leq 3\nu_{x_0}(\mathbf{R}) = 3\nu(\mathbf{R}).$$

Therefore

$$\big|\mathbf{M}_{T_2}(x_0) - \mathbf{M}_{T_1}(x_0)\big|^2 \leq 24\hat{\nu}(\frac{1}{T_2}, \frac{1}{T_1}].$$

The claimed inequality follows easily.

■

Let k, N be fixed but arbitrary positive integers. Put for $T > 0$

$$M_T = \frac{1}{2T}\int_{-T}^{T}\Big|\sum_{n=1}^{N}\frac{1}{n^{\sigma+it}}\Big|^{2k}\mathrm{d}t.$$

As an immediate consequence of the preceding Proposition, we get

Corollary 4 *For any non-decreasing sequence $\{T_j, j \geq 1\}$ of positive reals,*

$$\sum_{j=1}^{\infty} \big|M_{T_{j+1}} - M_{T_j}\big|^2 \leq 3.2^{2k+3}\Big(\sum_{n=1}^{N} \frac{1}{n^\sigma}\Big)^{2k}. \tag{22}$$

Now let $\lambda_1, \ldots, \lambda_N$ be distinct real numbers and consider the simplest mean integral

$$\int_0^T \Big|\sum_{n=1}^{N} d(n)e^{it\lambda_n}\Big|^2 \mathrm{d}t.$$

The following form of Hilbert's inequality due to Montgomery and Vaughan yields precise estimates of this integral.

Lemma 5 *Let $\delta > 0$ be a real number such that $|\lambda_m - \lambda_n| \geq \delta$ whenever $m \neq n$. Then*

$$\Big|\sum_{\substack{1\leq m,n\leq N\\ m\neq n}} \frac{x_m y_n}{\lambda_m - \lambda_n}\Big| \leq \frac{\pi}{\delta}\Big(\sum_{m=1}^{N}|x_m|^2\Big)^{1/2}\Big(\sum_{n=1}^{N}|y_n|^2\Big)^{1/2}. \tag{23}$$

By squaring out and integrating term-by-term, we get

$$\int_0^T \Big|\sum_{n=1}^{N} d(n)e^{it\lambda_n}\Big|^2 \mathrm{d}t = T\sum_{n=1}^{N} d^2(n) + 2\Re\Big\{\sum_{1\leq m<n\leq N} d(m)\overline{d(n)}\,\frac{e^{i(\lambda_m-\lambda_n)T}-1}{i(\lambda_m-\lambda_n)}\Big\}.$$

Since the sum in the parenthesis is the difference

$$\sum_{1\leq m<n\leq N} d(m)e^{i\lambda_m T}\overline{d(n)}\,e^{-i\lambda_n T}\frac{1}{i(\lambda_m-\lambda_n)} - \sum_{1\leq m<n\leq N}\frac{d(m)\overline{d(n)}}{i(\lambda_m-\lambda_n)},$$

by applying (23) to each part, we get

$$\Big|\sum_{1\leq m<n\leq N} d(m)\overline{d(n)}\frac{e^{i(\lambda_m-\lambda_n)T}-1}{i(\lambda_m-\lambda_n)}\Big| \leq \frac{2\pi}{\delta}\sum_{n=1}^{N}|d(n)|^2.$$

Consequently, we have

Proposition 6

$$\Big|\int_0^T \Big|\sum_{n=1}^{N} d(n)e^{it\lambda_n}\Big|^2 \mathrm{d}t - T\sum_{n=1}^{N}|d(n)|^2\Big| \leq \frac{4\pi}{\delta}\sum_{n=1}^{N}|d(n)|^2. \tag{24}$$

Choose $\lambda_n = \log n$ and observe that for $m < n \leq N$, $\lambda_n - \lambda_m = \log\frac{n}{m} \geq \log\frac{n}{n-1} \geq cN^{-1}$. We get in this case

$$\Big|\frac{1}{T}\int_0^T \Big|\sum_{n=1}^{N}\frac{d(n)}{n^{it}}\Big|^2 \mathrm{d}t - \sum_{n=1}^{N} d^2(n)\Big| \leq C\frac{N}{T}\sum_{n=1}^{N}|d(n)|^2. \tag{25}$$

This inequality remains true without change when replacing the interval of integration by any other of same length.

Consider now higher moments. Let q be some positive integer and denote

$$E_q = \Big\{ \underline{k} = (k_1, \ldots, k_N); k_i \in \mathbf{N} : k_1 + \ldots + k_N = q \Big\}.$$

We assume that $\lambda_1, \ldots, \lambda_N$ are linearly independent reals. The typical example is $\lambda_j = \log p_j$, $j = 1, \ldots, N$, where $p_1, p_2, \ldots, p_N$ are different primes, see (68).

Introduce a *coefficient of linear spacing* of order q by putting

$$\xi = \xi_\lambda(N, q) := \inf_{\substack{\underline{h}, \underline{k} \in E_q \\ \underline{h} \neq \underline{k}}} \big| (h_1 - k_1)\lambda_1 + \ldots + (h_N - k_N)\lambda_N \big|.$$

By assumption $\xi_\lambda(N, q) > 0$ and $\xi_\lambda(N, 1) = \inf_{\substack{1 \le i,j \le N \\ i \neq j}} \big| \lambda_i - \lambda_j \big|$. If we expand the integrand, next integrate, we shall get similarly

Proposition 7 *For any interval J, denoting $|J|$ its length,*

$$\frac{1}{|J|} \int_J \Big| \sum_{n=1}^N d(n) e^{it\lambda_n} \Big|^{2q} \mathrm{d}t \le \big(\sum_{n=1}^N |d(n)|^2 \big)^q \Big(q! + \frac{2 \min(N^q, \pi q!)}{|J| \xi} \Big). \qquad (26)$$

Proof. Let $J = [d, d+T]$. Put $P(t) = \sum_{n=1}^N d(n) e^{it\lambda_n}$. Plainly

$$\begin{aligned} \big| P(t) \big|^{2q} &= \sum_{\underline{k}, \underline{h} \in E_q} \frac{(q!)^2}{k_1! h_1! \ldots k_N! h_N!} \prod_{n=1}^N d(n)^{k_n} \overline{d(n)}^{h_n} e^{it(k_n - h_n)\lambda_n} \\ &= \sum_{\underline{k} \in E_q} \Big(\frac{q!}{k_1! \ldots k_N!} \Big)^2 \prod_{n=1}^N |d(n)|^{2k_n} + R(t) \qquad (27) \end{aligned}$$

where

$$R(t) = \sum_{\substack{\underline{k}, \underline{h} \in E_q \\ \underline{k} \neq \underline{h}}} \Big(\frac{(q!)^2}{k_1! h_1! \ldots k_N! h_N!} \Big) \prod_{n=1}^N d(n)^{k_n} \overline{d(n)}^{h_n} e^{it(k_n - h_n)\lambda_n}. \qquad (28)$$

By integrating and using linear independence

$$\begin{aligned} \frac{1}{T} \int_J \big| P(t) \big|^{2q} \mathrm{d}t &= \sum_{\underline{k} \in E_q} \Big(\frac{q!}{k_1! \ldots k_N!} \Big)^2 \prod_{n=1}^N |d(n)|^{2k_n} \\ &\quad + \sum_{\substack{\underline{k}, \underline{h} \in E_q \\ \underline{k} \neq \underline{h}}} \frac{(q!)^2}{k_1! h_1! \ldots k_N! h_N!} \prod_{n=1}^N d(n)^{k_n} \overline{d(n)}^{h_n} \\ &\quad \times \Big[\frac{e^{i(d+T) \sum_{n=1}^N (k_n - h_n)\lambda_n} - e^{id \sum_{n=1}^N (k_n - h_n)\lambda_n}}{iT (\sum_{n=1}^N (k_n - h_n)\lambda_n)} \Big]. \qquad (29) \end{aligned}$$

Put

$$\mathbf{c}_{\underline{k}} = \prod_{n=1}^N \frac{(d(n) e^{i(d+T)\lambda_n})^{k_n}}{k_n!}, \quad \mathbf{d}_{\underline{k}} = \prod_{n=1}^N \frac{(d(n) e^{id\lambda_n})^{k_n}}{k_n!}, \quad \mathbf{l}_{\underline{k}} = \sum_{n=1}^N k_n \lambda_n.$$

Then

$$\frac{1}{T}\int_J \big|P(t)\big|^{2q}\mathrm{d}t = (q!)^2\sum_{\underline{k}\in E_q}|\mathbf{d}_{\underline{k}}|^2 + \frac{(q!)^2}{iT}\Big\{\sum_{\substack{\underline{k},\underline{h}\in E_q\\ \underline{k}\neq\underline{h}}}\frac{\mathbf{c}_{\underline{k}}\overline{\mathbf{c}}_{\underline{h}}}{\mathbf{l}_{\underline{k}}-\mathbf{l}_{\underline{h}}} - \sum_{\substack{\underline{k},\underline{h}\in E_q\\ \underline{k}\neq\underline{h}}}\frac{\mathbf{d}_{\underline{k}}\overline{\mathbf{d}}_{\underline{h}}}{\mathbf{l}_{\underline{k}}-\mathbf{l}_{\underline{h}}}\Big\}. \tag{30}$$

We shall apply Hilbert's inequality under the following form: let $\{x_{\underline{k}}, y_{\underline{k}}, \underline{k}\in E_q\}$. Let also $\{\lambda_{\underline{k}}, \underline{k}\in E_q\}$ be distinct real numbers such that $\min\{|\lambda_{\underline{k}}-\lambda_{\underline{h}}|, \underline{k}\neq\underline{h}\}\geq\delta$. Let $\nu=\#\{E_q\}$ and consider a bijection $i:\{1,\ldots,\nu\}\to E_q$. By using Lemma 5

$$\begin{aligned}\Big|\sum_{\substack{\underline{k},\underline{h}\in E_q\\ \underline{k}\neq\underline{h}}}\frac{x_{\underline{k}}y_{\underline{h}}}{\lambda_{\underline{k}}-\lambda_{\underline{h}}}\Big| &= \Big|\sum_{\substack{1\leq u,v\leq\nu\\ u\neq v}}\frac{x_{i(u)}y_{i(v)}}{\lambda_{i(u)}-\lambda_{i(v)}}\Big| \\ &\leq \frac{\pi}{\delta}\Big(\sum_{1\leq u\leq\nu}|x_{i(u)}|^2\Big)^{1/2}\Big(\sum_{1\leq v\leq\nu}|y_{i(v)}|^2\Big)^{1/2} \\ &= \frac{\pi}{\delta}\Big(\sum_{\underline{k}\in E_q}|x_{\underline{k}}|^2\Big)^{1/2}\Big(\sum_{\underline{h}\in E_q}|y_{\underline{h}}|^2\Big)^{1/2}. \end{aligned} \tag{31}$$

Apply it to each of the two sums in parenthesis of the right-term in (30), we find

$$\frac{(q!)^2}{T}\Big|\sum_{\substack{\underline{k},\underline{h}\in E_q\\ \underline{k}\neq\underline{h}}}\frac{\mathbf{c}_{\underline{k}}\overline{\mathbf{c}}_{\underline{h}}}{\mathbf{l}_{\underline{k}}-\mathbf{l}_{\underline{h}}} - \sum_{\substack{\underline{k},\underline{h}\in E_q\\ \underline{k}\neq\underline{h}}}\frac{\mathbf{d}_{\underline{k}}\overline{\mathbf{d}}_{\underline{h}}}{\mathbf{l}_{\underline{k}}-\mathbf{l}_{\underline{h}}}\Big| \leq \frac{2\pi(q!)^2}{T\xi}\sum_{\underline{k}\in E_q}|\mathbf{d}_{\underline{k}}|^2 \leq \frac{2\pi q!}{T\xi}\Big[\sum_{n=1}^N|d(n)|^2\Big]^q, \tag{32}$$

since

$$\begin{aligned}(q!)^2\sum_{\underline{k}\in E_q}|\mathbf{d}_{\underline{k}}|^2 &= \sum_{k_1+\ldots+k_N=q}\Big[\frac{q!}{k_1!\ldots k_N!}\Big]^2\prod_{n=1}^N|d(n)|^{2k_n} \\ &\leq q!\sum_{k_1+\ldots+k_N=q}\frac{q!}{k_1!\ldots k_N!}\prod_{n=1}^N|d(n)|^{2k_n} = q!\Big[\sum_{n=1}^N|d(n)|^2\Big]^q. \end{aligned} \tag{33}$$

The way to bound in (33), in turn, already appeared in [38].

By substituting in (30), we therefore obtain

$$\frac{1}{T}\int_J\big|P(t)\big|^{2q}\mathrm{d}t \leq q!\Big(1+\frac{2\pi}{T\xi}\Big)\Big[\sum_{n=1}^N|d(n)|^2\Big]^q. \tag{34}$$

Further, from (29) we also get by using Cauchy-Schwarz inequality

$$\begin{aligned}\frac{1}{T}\int_J\big|P(t)\big|^{2q}\mathrm{d}t &= \sum_{\underline{k}\in E_q}\Big(\frac{q!}{k_1!\ldots k_N!}\Big)^2\prod_{n=1}^N|d(n)|^{2k_n} \\ &\quad+\sum_{\substack{\underline{k},\underline{h}\in E_q\\ \underline{k}\neq\underline{h}}}\frac{(q!)^2}{k_1!h_1!\ldots k_N!h_N!}\prod_{n=1}^N d(n)^{k_n}\overline{d(n)}^{h_n} \\ &\quad\times\Big[\frac{e^{i(d+T)\sum_{n=1}^N(k_n-h_n)\lambda_n}-e^{id\sum_{n=1}^N(k_n-h_n)\lambda_n}}{iT(\sum_{n=1}^N(k_n-h_n)\lambda_n)}\Big]\end{aligned}$$

$$\leq \quad q!\Big[\sum_{n=1}^{N}|d(n)|^2\Big]^q + \frac{2}{T\xi}\Big(\sum_{n=1}^{N}|d(n)|\Big)^{2q}$$

$$\leq \quad \Big(q! + \frac{2N^q}{T\xi}\Big)\Big[\sum_{n=1}^{N}|d(n)|^2\Big]^q. \tag{35}$$

Combining the two last estimates gives

$$\frac{1}{T}\int_J \big|P(t)\big|^{2q}\mathrm{d}t \leq \Big(q! + \frac{2\min(N^q, \pi q!)}{T\xi}\Big)\Big[\sum_{n=1}^{N}|d(n)|^2\Big]^q. \tag{36}$$

■

Now we pass to high moments of Dirichlet approximating polynomials

$$\frac{1}{T}\int_0^T \Big|\sum_{n=1}^{N}\frac{1}{n^{\frac{1}{2}+it}}\Big|^{2\nu}\mathrm{d}t.$$

Apply Proposition 6 to

$$\Big(\sum_{n=1}^{N}\frac{1}{n^{\frac{1}{2}+it}}\Big)^{\nu} := \sum_{m=1}^{N^\nu}\frac{b_m}{m^{\frac{1}{2}+it}}. \tag{37}$$

Since $\delta \geq \min\big\{\log(1+\frac{m-n}{n}) : 1 \leq n < m \leq N^\nu\big\} \geq \frac{1}{2N^\nu}$, we get

$$\Big|\int_0^T \Big|\sum_{n=1}^{N}\frac{1}{n^{\frac{1}{2}+it}}\Big|^{2\nu}\mathrm{d}t - T\sum_{m=1}^{N^\nu}\frac{b_m^2}{m}\Big| \leq CN^\nu\sum_{m=1}^{N^\nu}\frac{b_m^2}{m}. \tag{38}$$

Recall that $d_\nu(n)$ denotes the number of representations of the integer n as a product of ν factors. As $b_m = \#\{(n_1,\ldots,n_\nu); n_j \leq N : m = n_1\ldots n_\nu\} \leq d_\nu(m)$, and ([21], Section 9.5)

$$\sum_{m\leq N}\frac{d_\nu^2(m)}{m} = (C_\nu + o(1))\log^{\nu^2} N, \tag{39}$$

it follows that

$$\frac{1}{T}\int_0^T \Big|\sum_{n=1}^{N}\frac{1}{n^{\frac{1}{2}+it}}\Big|^{2\nu}\mathrm{d}t \leq C_\nu(1+\frac{N^\nu}{T})\log^{\nu^2} N. \tag{40}$$

Hence if $T \geq N^\nu$

$$\frac{1}{T}\int_0^T \Big|\sum_{n=1}^{N}\frac{1}{n^{\frac{1}{2}+it}}\Big|^{2\nu}\mathrm{d}t \leq C_\nu\log^{\nu^2} N. \tag{41}$$

The latter estimate is in fact two-sided, see Corollary 10. It can also be reformulated as

$$c_\nu\sum_{m=1}^{N^\nu}\frac{b_m^2}{m} \leq \frac{1}{T}\int_0^T \Big|\sum_{n=1}^{N}\frac{1}{n^{\frac{1}{2}+it}}\Big|^{2\nu}\mathrm{d}t \leq C_\nu\sum_{m=1}^{N^\nu}\frac{b_m^2}{m}. \tag{42}$$

Now, by using approximation formula (2), it follows that the reformulation (5) of the LH is also equivalent to

$$\frac{1}{N}\int_0^N \Big|\sum_{n=1}^N \frac{1}{n^{\frac12+it}} - \frac{N^{\frac12-it}}{\frac12-it}\Big|^{2\nu}\mathrm{d}t = \mathcal{O}_\varepsilon(N^\varepsilon) \qquad \nu=1,2,\dots. \tag{43}$$

The critical range of values of T in (41) is thus $T\sim N$. But it is a simple matter to observe that in this case, estimate (41) can no longer be true, unless the LH is false. Indeed by the Minkowski inequality, if (41) and (43) were simultaneously true, we would have

$$\frac{1}{N}\int_0^N \Big|\frac{N^{\frac12-it}}{\frac12-it}\Big|^{2\nu}\mathrm{d}t = \frac{N^{\nu-1}}{2}\int_0^N \frac{\mathrm{d}t}{(\frac14+t^2)^\nu} \sim CN^{\nu-1} = \mathcal{O}_\varepsilon(N^\varepsilon), \tag{44}$$

which is absurd as soon as $\nu>1$. It also follows from these observations that the order of

$$\frac{1}{T}\int_0^T \Big|\sum_{n=1}^N \frac{1}{n^{\frac12+it}}\Big|^{2\nu}\mathrm{d}t$$

is necessarily much bigger for $T\le N^\nu$ than for $T\ge N^\nu$.

Concerning upper bounds, there is a useful argument ([29], p.131) which can be applied for arbitrary even powers. Consider the kernel

$$K_T(t) = \big(1-|t|/T\big)\chi_{\{|t|\le T\}}, \qquad \widehat{K}_T(u) = T^{-1}\big(\frac{\sin Tu}{u}\big)^2.$$

Then

$$\chi_{\{|t-H|\le T\}} \le K_T(t-H) + K_T(t-H+T) + K_T(t-H-T). \tag{45}$$

This can be used to prove

Proposition 8 *Let q be any positive integer. Let $c_1,\dots,c_N$ be complex numbers and nonnegative reals $a_1,\dots,a_N$ such that $|c_n|\le a_n$, $n=1,\dots,N$. Then for any reals $\varphi_1,\dots,\varphi_N$ and any reals T,T_0 with $T>0$*

$$\int_{|t-T_0|\le T}\Big|\sum_{n=1}^N c_n e^{it\varphi_n}\Big|^{2q}dt \le 3\int_{|t|\le T}\Big|\sum_{n=1}^N a_n e^{it\varphi_n}\Big|^{2q}dt. \tag{46}$$

From (46) one can derive the following lower bound [49].

Theorem 9 *For any positive integer q, there exists a constant c_q, such that for any reals $\varphi_1,\dots,\varphi_N$, any non-negative reals $a_1,\dots,a_N$, and any $T>0$,*

$$c_q\Big(\sum_{n=1}^N a_n^2\Big)^q \le \frac{1}{2T}\int_{|t|\le T}\Big|\sum_{n=1}^N a_n e^{it\varphi_n}\Big|^{2q}dt. \tag{47}$$

The L^1-case is related to Ingham's inequality. Recall the sharp form due to Mordell [31]: let $0<\varphi_1<\ldots<\varphi_N$ and let γ be such that $\min_{1<n\le N}\varphi_n-\varphi_{n-1}\ge \gamma>0$. Then

$$\sup_{n=1}^N |a_n| \le \frac{K}{T}\int_{-T}^T\Big|\sum_{n=1}^N a_n e^{it\varphi_n}\Big|dt \qquad \text{with } T=\frac{\pi}{\gamma}, \tag{48}$$

where $K \le 1$. Further with no restriction, one has a very similar inequality in the theory of uniformly almost periodic functions:

$$\sup_{n=1}^{N} |a_n| \le \limsup_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \Big|\sum_{n=1}^{N} a_n e^{it\varphi_n}\Big| dt \le \sup_{t\in\mathbf{R}} \Big|\sum_{n=1}^{N} a_n e^{it\varphi_n}\Big|, \tag{49}$$

Inequality (47) is obtained by choosing $c_n = \varepsilon_n a_n$ in (46), where $\underline{\varepsilon} = \{\varepsilon_n, n \ge 1\}$ is a Rademacher sequence, next taking expectation and using Khintchine-Kahane inequalities for Rademacher sums. We shall deduce from it the following lower bound.

Corollary 10 *For every N, T and ν*

$$c_\nu \log^{\nu^2} N \le \frac{1}{2T}\int_{|t|\le T} \Big|\sum_{n=1}^{N} \frac{1}{n^{\frac12+it}}\Big|^{2\nu} dt.$$

Indeed, apply (47) with $q = 2$ to the sum

$$\Big(\sum_{n=1}^{N} \frac{1}{n^{\frac12+it}}\Big)^{\nu} := \sum_{m=1}^{N^\nu} \frac{b_m}{m^{\frac12+it}}.$$

Then for all N and T

$$c_\nu \sum_{m=1}^{N^\nu} \frac{b_m^2}{m} \le \frac{1}{2T}\int_{|t|\le T} \Big|\sum_{n=1}^{N} \frac{1}{n^{\frac12+it}}\Big|^{2\nu} dt.$$

Notice that if $m \le N$, $b_m = d_\nu(m)$ and we know that

$$\sum_{m\le x} \frac{d_\nu^2(m)}{m} = (C_\nu + o(1)) \log^{\nu^2} x.$$

See [21] section 9.5. Thus

$$\sum_{m=1}^{N^\nu} \frac{b_m^2}{m} \ge \sum_{m=1}^{N} \frac{b_m^2}{m} \ge c_\nu \log^{\nu^2} N$$

Henceforth

$$c_\nu \log^{\nu^2} N \le \frac{1}{2T}\int_{|t|\le T} \Big|\sum_{n=1}^{N} \frac{1}{n^{\frac12+it}}\Big|^{2\nu} dt.$$

■

Put

$$M_\nu(T) = \int_0^T \big|\zeta(\frac12 + it)\big|^{2\nu} dt$$

The corresponding inequality for the Riemann Zeta-function is Ramachandra's well-known lower bound and we recall ([40] p.180, see also [21] section 9.5 and [36]) that

$$c_\nu T(\log T)^{\nu^2} \le M_\nu(T). \tag{50}$$

Assuming RH, Soundararajan recently proved in [39] that for every positive real number ν, and every$\varepsilon > 0$, we have

$$M_\nu(T) \le C_{\nu,\varepsilon}\ T(\log T)^{\nu^2+\varepsilon}. \tag{51}$$

We conclude this section by mentionning and briefly discussing some related results for the Riemann ζ-function.

Remark 11 (Mean value results for the ζ-function) For the critical value $\sigma = 1/2$, the best known results are

$$\int_0^T |\zeta(\frac{1}{2}+it)|^2 dt = T\log\Big(\frac{T}{2\pi}\Big) + (2\gamma-1)T + E(T), \tag{52}$$

where γ is Euler's constant and the error term $E(T)$ satisfies $E(T) \ll_\varepsilon T^{1/3+\varepsilon}$, see [40] p.176. And

$$\int_0^T |\zeta(\frac{1}{2}+it)|^4 dt = \frac{T\log^4 T}{2\pi^2} + \mathcal{O}(T\log^3 T), \tag{53}$$

see [40] p.148. The approximation (2) already suffices to show

$$\int_0^T |\zeta(\frac{1}{2}+it)|^2 dt = \mathcal{O}(T\log T). \tag{54}$$

The very formulation of (2) yields for the fourth moment that it is equivalent to work with

$$\Big|\sum_{n\le x}\frac{1}{n^s} - \frac{x^{1-s}}{1-s}\Big|^4, \qquad x\sim T,$$

instead of $|\zeta(\frac12+it)|^4$, when $0\le t\le T$. However there is apparently no known proof of $\int_0^T|\zeta(\frac12+it)|^4dt = \mathcal{O}(T\log^4 T)$ based on (2), which is a bit frustrating. In place of this bound, one has to use the following more elaborated approximate equation

$$\zeta(s) = \sum_{n\le x}\frac{1}{n^s} + \chi(s)\sum_{n\le y}\frac{1}{n^{1-s}} + \mathcal{O}(x^{-\sigma}\log|t|) + \mathcal{O}(|t|^{\frac12-\sigma}y^{\sigma-1}), \tag{55}$$

in which h is a positive constant, $0<\sigma<1$, $2\pi xy = t$, $x>h>0$, $y>h>0$ and

$$\chi(s) = \frac{\zeta(s)}{\zeta(1-s)} = 2^{s-1}\pi^2\sec\Big(\frac{s\pi}{2\Gamma(s)}\Big).$$

This function verifies in any fixed strip $\alpha\le\sigma\le\beta$, $|\chi(s)|\sim\big(t/2\pi\big)$, as $t\to\infty$.

The knewledge concerning moments

$$\int_0^T |\zeta(\frac{1}{2}+it)|^k \mathrm{d}t$$

beyond $k=4$ is, at least, very sparse. For the case $k=12$, we may quote the beautiful result due to Heath-Brown $\int_0^T|\zeta(\frac12+it)|^{12}dt \ll T^2\log^{17}T$. See [40] p. 79, 95 and 178 for the aforementionned facts. See also [21] Section 8.3.

There are also alternative mean-value theorems involving integrals of the form

$$J(\delta) = \int_0^\infty |\zeta(\frac{1}{2} + it)|^{2k} e^{-\delta t} \mathrm{d}t, \qquad \delta \to 0.$$

The behavior of these integrals is similar to the one of

$$I(T) = \int_0^T |\zeta(\frac{1}{2} + it)|^{2k} \mathrm{d}t, \qquad T \to \infty,$$

and this follows notably from integral versions of a well-known Tauberian result of Hardy and Littlewood. More precisely, if $f \geq 0$, then ([40], (7.12.1) and (7.12.2))

$$\int_0^\infty f(t) e^{-\delta t} \mathrm{d}t \overset{\delta \to 0}{\sim} \frac{1}{\delta} \qquad \Rightarrow \qquad \int_0^T f(t) \mathrm{d}t \overset{T \to \infty}{\sim} T. \tag{56}$$

When $1/2 < \sigma < 1$, we have

$$\int_1^T |\zeta(\sigma + it)|^2 \mathrm{d}t = T \sum_{n=1}^\infty \frac{1}{n^{2\sigma}} + \mathcal{O}\big(T^{2-2\sigma}\big). \tag{57}$$

$$\int_1^T |\zeta(\sigma + it)|^4 \mathrm{d}t = T \sum_{n=1}^\infty \frac{d_2^2(n)}{n^{2\sigma}} + \mathcal{O}_\varepsilon\big(T^{3/2-\sigma+\varepsilon}\big). \tag{58}$$

There are also results for $k = 1/2$. For $k > 2$ integer, it is known ([40], p.125) that

$$\lim_{T\to\infty} \frac{1}{T} \int_1^T \big|\zeta(\sigma + it)\big|^{2k} \mathrm{d}t = \sum_{n=1}^\infty \frac{d_k^2(n)}{n^{2\sigma}}, \qquad (\sigma > 1 - 1/k). \tag{59}$$

We refer to [21], Chapter 8 and notably Theorem 8.5 for improvments of this, up to power 12, under weaker conditions on σ.

Remark 12 (Square function of the Riemann-zeta function) Let $\theta = \{T_j, j \geq 1\}$ be such that $T_j \uparrow \infty$. Given any fixed positive integer k, we define for $1/2 < \sigma < 1$ the ζ-square function $\mathcal{S}_\theta(k, \sigma)$ associated to θ as follows

$$\mathcal{S}_\theta(k, \sigma) := \Bigg(\sum_{j=1}^\infty \Big| \frac{1}{T_{j+1}} \int_0^{T_{j+1}} |\zeta(\sigma + it)|^{2k} dt - \frac{1}{T_j} \int_0^{T_j} |\zeta(\sigma + it)|^{2k} \Big|^2 \Bigg)^{1/2}.$$

The finiteness in (21) of the square function linked to the Fourier inversion formula and the analogy described at the beginning of Section 2 between Lindelöf Hypothesis and Fourier inversion formula suggest to investigate properties of the ζ-square function $\mathcal{S}_\theta(k, \sigma)$.

Problem. For which sequences θ is $\mathcal{S}_\theta(k, \sigma)$ finite for all $1/2 < \sigma < 1$? When is the same also true independently of the value of $k \geq 1$?

In the case $k = 1$, $k = 2$, it follows trivially from (57), (58) that any geometrically increasing sequence θ, $T_{j+1}/T_j \geq M > 1$, is suitable. One may wonder whether this condition is also necessary for the finiteness of $\mathcal{S}_\theta(k, \sigma)$ for every $1/2 < \sigma < 1$. The case $k = 1/2$ is also of interest.

It may be worth noticing that for the approximating term in (2), we have for any integers $n \geq m \geq 1$,

$$\Big|\sum_{k=1}^{n} \frac{1}{k^{\frac12+it}} - \frac{n^{\frac12-it}}{\frac12-it}\Big|^2 - \Big|\sum_{k=1}^{m} \frac{1}{k^{\frac12+it}} - \frac{m^{\frac12-it}}{\frac12-it}\Big|^2$$
$$= \sum_{\ell=m+1}^{n} \frac{1}{\ell} + 2 \sum_{\substack{m+1\leq k\leq n\\ 1\leq \ell<k}} \Re\Big\{ \frac{1}{\sqrt{k\ell}} e^{it\log\frac{k}{\ell}} - \int_{\log\frac{k-1}{\ell}}^{\log\frac{k}{\ell}} e^{(\frac12+it)x}\mathrm{d}x\Big\}. \qquad (60)$$

This follows from the more homogeneous reformulation

$$\Big|\sum_{k=1}^{n} \frac{1}{k^{\frac12+it}} - \frac{n^{\frac12-it}}{\frac12-it}\Big|^2 = \sum_{\ell=1}^{n} \frac{1}{\ell} + 2 \sum_{1\leq \ell<k\leq n} \Re\Big\{ \frac{1}{\sqrt{k\ell}} e^{it\log\frac{k}{\ell}} - \int_{\log\frac{k-1}{\ell}}^{\log\frac{k}{\ell}} e^{(\frac12+it)x}\mathrm{d}x\Big\}. \qquad (61)$$

Indeed,

$$\begin{aligned}
& \sum_{\ell=1}^{n} \frac{1}{\ell} + 2 \sum_{1\leq \ell<k\leq n} \Re\Big\{ \frac{1}{\sqrt{k\ell}} e^{it\log\frac{k}{\ell}} - \int_{\log\frac{k-1}{\ell}}^{\log\frac{k}{\ell}} e^{(\frac12+it)x}\mathrm{d}x\Big\} \\
= \; & \sum_{\ell=1}^{n} \frac{1}{\ell} + 2 \sum_{1\leq \ell<k\leq n} \Re\Big\{ \frac{1}{\sqrt{k\ell}} e^{it\log\frac{k}{\ell}}\Big\} - 2 \sum_{1\leq \ell<k\leq n} \Re\Big\{ \int_{\log\frac{k-1}{\ell}}^{\log\frac{k}{\ell}} e^{(\frac12+it)x}\mathrm{d}x\Big\} \\
= \; & \Big|\sum_{k=1}^{n} \frac{1}{k^{\frac12+it}}\Big|^2 - 2\sum_{\ell=1}^{n-1} \Re\Big\{ \int_{0}^{\log\frac{n}{\ell}} e^{(\frac12+it)x}\mathrm{d}x\Big\} \\
= \; & \Big|\sum_{k=1}^{n} \frac{1}{k^{\frac12+it}}\Big|^2 - 2\sum_{\ell=1}^{n-1} \Re\Big\{ \frac{1}{\frac12+it}\Big[\Big(\frac{n}{\ell}\Big)^{\frac12+it} - 1\Big]\Big\} \\
= \; & \Big|\sum_{k=1}^{n} \frac{1}{k^{\frac12+it}}\Big|^2 - 2\Re\Big\{ \frac{n^{\frac12+it}}{(\frac12+it)} \sum_{\ell=1}^{n-1} \frac{1}{\ell^{\frac12+it}}\Big\} + \frac{n}{\frac14+t^2}.
\end{aligned}$$

If $1/2 < \sigma < 1$, there is a similar formula ([48], Corollary 5)

$$\begin{aligned}
\Big|\sum_{k=1}^{n} \frac{1}{k^{\sigma+it}} - \frac{n^{\sigma-it}}{\sigma-it}\Big|^2 \; = \; & \sum_{\ell=1}^{n} \frac{1}{\ell^{2\sigma}} + 2 \sum_{1\leq \ell<k\leq n} \Re\Big\{ \frac{e^{it\log\frac{k}{\ell}}}{(k\ell)^\sigma} \\
& - \ell^{1-2\sigma} \int_{\log\frac{k-1}{\ell}}^{\log\frac{k}{\ell}} e^{(1-\sigma+it)x}\mathrm{d}x\Big\} - \frac{\Psi_\sigma}{(1-\sigma)^2+t^2}, \qquad (62)
\end{aligned}$$

where $\Psi_\sigma = \sigma + (1-2\sigma)2\sigma \sum_{k=1}^{\infty} \int_0^1 \frac{t-t^2}{2}(k+t)^{-2\sigma-1}dt + \mathcal{O}(n^{1-2\sigma})$.

4 Supremum of Dirichlet polynomials

We begin with some general considerations. Let $d : \mathbf{N} \to \mathbf{R}$. The supremum of the Dirichlet polynomials $P(s) = \sum_{n=1}^{N} d(n)n^{-s}$ over lines $\{s = \sigma + it,\ t \in \mathbf{R}\}$ is naturally related to that of corresponding Dirichlet series, via the abscissa of uniform convergence

$$\sigma_u = \inf \Big\{ \sigma : \sum_{n=1}^{\infty} d(n)n^{-\sigma-it} \text{ converges uniformly over } t \in \mathbf{R}\Big\},$$

through the relation

$$\sigma_u = \limsup_{N\to\infty} \frac{\log \sup_{t\in\mathbf{R}} |\sum_{n=1}^{N} d(n)n^{-it}|}{\log N} \quad . \tag{63}$$

We refer to [4],[18] or [15] for background and related results. This naturally justifies the investigation of the supremum of Dirichlet polynomials.

It is natural to start by comparing the behavior of the suprema of Dirichlet polynomials with the one of trigonometric polynomials, which we shall do by investigating Rudin-Shapiro polynomials. We refer to [29] Chapter 7 where a comparative study is presented.

Rudin-Shapiro polynomials. Recall the classical setting. For any trigonometric polynomial we have

$$\frac{\sum_{n=1}^{N} |d(n)|}{\sqrt{N}} \le \sup_{t\in\mathbf{R}} |\sum_{n=1}^{N} d(n)e^{int}| \le \sum_{n=1}^{N} |d(n)|. \tag{64}$$

The arguments for getting the lower bound are the inequality between the sup-norm and L_2-norm, the orthogonality of $(e^{int})_n$ and Hölder inequality.

Rudin and Shapiro constructed a fairly simple sequence $d(n) \in \{-1,+1\}$ such that the right order of the lower bound is attained:

$$\sup_{t\in\mathbf{R}} |\sum_{n=1}^{N} d(n)e^{int}| \le (2+\sqrt{2})\sqrt{N+1} \sim (2+\sqrt{2})\ \frac{\sum_{n=1}^{N} |d(n)|}{\sqrt{N}}. \tag{65}$$

Consider now the Dirichlet polynomials instead of the trigonometric ones. It is known from [23] and [35] that

Theorem 13 *For any* $(d(n))$

$$\sup_{t\in\mathbf{R}} |\sum_{n=1}^{N} d(n)n^{it}| \ge \alpha_1 \frac{\sum_{n=1}^{N} |d(n)|}{\sqrt{N}} \exp\{\beta_1 \sqrt{\log N \log\log N}\}. \tag{66}$$

and for some $(d(n))$

$$\sup_{t\in\mathbf{R}} |\sum_{n=1}^{N} d(n)n^{it}| \le \alpha_2 \frac{\sum_{n=1}^{N} |d(n)|}{\sqrt{N}} \exp\{\beta_2 \sqrt{\log N \log\log N}\}, \tag{67}$$

with some universal constants $\alpha_1, \alpha_2, \beta_1, \beta_2$*.*

A finer result with explicit constants was recently obtained by de la Bretèche in [5]. Therefore the lower bound for Dirichlet polynomials is necessarily worse than in the classical case. Notice also that the construction in [35] is a probabilistic one; no explicit example of Rudin-Shapiro type is known for Dirichlet polynomials.

There is a basic reduction step in the study of the suprema. Introduce a useful notion. A set of numbers $\varphi_1, \varphi_2, \ldots, \varphi_k$ is linearly independent if no linear relation $a_1\varphi_1 + a_2\varphi_2 + \ldots + a_k\varphi_k = 0$, with integer coefficients, not all zero, holds. For a proof of the classical result below, we refer to [16].

Kronecker's theorem. If $\varphi_1, \varphi_2, \ldots, \varphi_k, 1$ are linearly independent, $\vartheta_1, \vartheta_2, \ldots, \vartheta_k$ are arbitrary, and N, ε are positive, then there are integers $n > N$, $n_1, n_2, \ldots, n_k$ such that

$$\max_{1\le m\le k} |n\varphi_m - n_m - \vartheta_m| < \varepsilon.$$

Consequently, the set of points $\{n\varphi_1\}, \{n\varphi_1\}, \ldots, \{n\varphi_k\}$ is dense in $\mathbf{T}^k$.

Let $p_1, p_2, \ldots, p_k$ be different primes. By the fundamental theorem of arithmetic

$$\log p_1, \log p_2, \ldots, \log p_k \qquad \text{are linearly independent.} \tag{68}$$

This will enable to replace the Dirichlet polynomial by some relevant *trigonometric* polynomial. Introduce the necessary notation. Let $2 = p_1 < p_2 < \ldots$ be the sequence of consecutive primes. If $n = \prod_{j=1}^{\tau} p_j^{a_j(n)}$, we write $\underline{a}(n) = \{a_j(n), 1 \le j \le \tau\}$. According to the standard notation we also denote $\Omega(n) = a_1(n) + \ldots + a_\tau(n)$ and by $\pi(N)$ the number of prime numbers less or equal to N. Let us fix N. We put in what follows $\mu = \pi(N)$ and define for $\underline{z} = (z_1, \ldots, z_\mu) \in \mathbf{T}^\mu$,

$$Q(\underline{z}) = \sum_{n=2}^{N} d(n) n^{-\sigma} e^{2i\pi\langle \underline{a}(n), \underline{z}\rangle},$$

H. Bohr's observation states that

$$\sup_{t\in\mathbf{R}} \Big|\sum_{n=2}^{N} d(n) n^{-(\sigma+it)}\Big| = \sup_{\underline{z}\in\mathbf{T}^\mu} \big|Q(\underline{z})\big| \,. \tag{69}$$

Remark 14 Naturally no similar reduction occurs when considering the supremum over a given bounded interval I. However, when the length of I is of exponential size with respect to N, precisely when

$$|I| \ge e^{(1+\varepsilon)\omega N(\log N\omega)\log N},$$

the related supremum becomes comparable, for ω large, to the one taken over the real line, with an error term of order $\mathcal{O}(\omega^{-1})$. This is in turn a rather general phenomenon due to existence of "localized" versions of Kronecker's theorem; and in the present case to Turán's estimate (see [46] for a slighly improved form of it using a probabilistic approach, and references therein). When the length is of sub-exponential order, the study however still belong to the field of application of the general theory of regularity of stochastic processes.

Before going further notice, as an immediate consequence of Kronecker's Theorem, that if $\varphi_1, \varphi_2, \ldots, \varphi_N$ are linearly independent then

$$\sup_{t\in\mathbf{R}} \Big|\sum_{n=1}^{N} d(n) e^{-it\varphi_n}\Big| = \sum_{n=1}^{N} |d(n)|. \tag{70}$$

Let us first consider lower bounds. Subsets $A \subseteq \{1, \ldots, N\}$ such that

$$\forall \{\delta_n, n\in A\} \in \big\{0, \frac{1}{2}\big\}^A, \quad \exists \underline{z} \in \mathbf{T}^\tau : \sum_{j=1}^{\tau} a_j(n) z_j = \delta_n \bmod(1), \quad \forall n \in A$$

are of particular interest, since $e^{2i\pi\langle \underline{a}(n),\underline{z}\rangle} = 1$ or -1 according to $\delta_n = 0$ or $1/2$. As $\langle \underline{a}(p_j), \underline{z}\rangle = z_j$, by choosing $\underline{z}$ so that $z_j = 0$ or $1/2$, we deduce with (69)

Bohr's lower bound ([3])

$$\mathbf{E}\sup_{t\in\mathbf{R}}\Big|\sum_{n=2}^{N} d(n)n^{-it}\Big| \ge \sum_{p\le N} |d(p)|p^{-\sigma}.$$

This was generalized in [34] by Queffélec who proved

Proposition 15 *For any integer* $m\ge 1$

$$C_m\,\mathbf{E}\sup_{t\in\mathbf{R}}\Big|\sum_{n=2}^{N} d(n)n^{-it}\Big| \ge \Big(\sum_{\substack{n\le N\\ \Omega(n)=m}} |d(n)|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}},$$

where $C_m = \big(\frac{2}{\sqrt{\pi}}\big)^{m-1}\frac{m^{\frac{m}{2}}(m+1)^{\frac{m+1}{2}}}{2^m (m!)^{\frac{2}{m+1}}}$, $(C_1 = 1)$. *Further* $C_m \le m^{\frac{m}{2}}$.

These estimates are crucial ([23], see section 4) in the proof of Theorem 13.

Local suprema of Dirichlet polynomials. Let $\varphi_1,\dots,\varphi_N$ be linearly independent reals. In [50], the local suprema of the Dirichlet polynomials $P(t) = \sum_{n=1}^{N} c_n e^{it\varphi_n}$ is investigated. Let q be some positive integer. Then ([50], Theorem 4),

There exists a constant C_q *depending on q only, such that for any intervals* J, L

$$\Big(\frac{1}{|J|}\int_J \Big|\sup_{t\in L}|P(\vartheta+t)|d\vartheta\Big)^{1/2q} \le C_q\,\mathcal{B}\max\Big\{1, |L|\tilde\varphi_N\Big\}^{1/2q}\Big\{\Big[\sum_{n=1}^{N}|c_n|^2\Big]^{1/2} + \min\Big(|L|, \frac{1}{\tilde\varphi_N}\Big)\Big[\sum_{n=1}^{N}|c_n|^2\varphi_n^2\Big]^{1/2}\Big\}, \qquad (71)$$

where $\mathcal{B} = \big[q!\big(1+\frac{2\pi}{|J|\xi_\varphi(N,q)}\big)\big]^{1/2q}$, $\tilde\varphi_N = \sup_{n\le N}|\varphi_n|$.

This result is used in the same paper to investigate by means of Turán's result (14), zerofree regions of the Riemann-zeta function.

5 Random Dirichlet polynomials

Studies for random Dirichlet polynomials and random Dirichlet series were developed in [13] and [33],[34],[35] notably, see also [26],[27] and references therein. Such investigations concerning random Dirichlet series and random power series go back to earlier works of Hartman [14], Clarke [7] and Dvoretzky-Erdös [8],[9].

Let us first quote some general results. For instance let $\underline{\xi} = \{\xi, \xi_n, n\ge 1\}$ be a sequence of i.i.d. random variables and let σ_c and σ_a be, respectively, the almost sure abscissa of convergence and of absolute convergence of the Dirichlet series $\sum_{n=1}^{\infty}\xi_n n^{-s}$. If $\xi\ne 0$ holds with positive probability, let $k_\xi := \sup\{\gamma : \mathbf{E}\,|\xi|^\gamma < \infty\}$. The connection between the abscissas σ_c and σ_a and integrability of ξ has been clarified by Clarke in [7].

Proposition 16 *We have the implications:*

$$\begin{array}{ccc} k_\xi = 0 & \Rightarrow & \sigma_a = \sigma_c = \infty \\ 0 < k_\xi \le 1 & \Rightarrow & \sigma_a = \sigma_c = 1/k_\xi \\ (k_\xi > 1 \text{ and } \mathbf{E}\,\xi \neq 0) & \Rightarrow & \sigma_a = \sigma_c = 1 \\ (k_\xi > 1 \text{ and } \mathbf{E}\,\xi = 0) & \Rightarrow & \sigma_a = 1 \text{ and } \sigma_c = \max(1/k_\xi, 1/2). \end{array} \tag{72}$$

Now let here and throughout the remaining part of the paper $\varepsilon = \{\varepsilon_i, i \ge 1\}$ be a sequence of independent Rademacher random variables ($\mathbf{P}\{\varepsilon_i = \pm 1\} = 1/2$) with basic probability space $(\Omega, \mathcal{A}, \mathbf{P})$. The following result is due to Bayart, Konyagin and Quéffelec [1].

Theorem 17 *Let $\{d(n), n \ge 1\}$ be a sequence of complex numbers. If*

$$\limsup_{N\to\infty} \frac{1}{\log\log N} \sum_{n=0}^{N} |d(n)|^2 = \gamma > 0,$$

then for almost all ω the series $\sum_{n=0}^{\infty} \varepsilon_n(\omega) d(n) n^{it}$ diverges for each $t \in \mathbf{R}$.

The result is nearly optimal: if $0 < \delta_n \to 0$, there exists a sequence $\{d(n), n \ge 1\}$ such that $\limsup_{N\to\infty} \frac{1}{\delta_N \log\log N} \sum_{n=0}^{N} |d(n)|^2 > 0$, but for each ω, the series $\sum_{n=0}^{\infty} \varepsilon_n(\omega) d(n) n^{it}$ converges for at least one $t \in \mathbf{R}$.

In relation with the above, we may quote Hedenmalm and Saksman's extension [17] of Carleson's result:

Theorem 18 *Under the assumption $\sum_{n=0}^{\infty} |d(n)|^2 < \infty$ the Dirichlet series*

$$\sum_{n=0}^{\infty} \varepsilon_n d(n) n^{-1/2+it}$$

converges for almost all t.

A simple and elegant proof is given in Konyagin and Quéffelec [23] p.158/159.

6 Supremum of random Dirichlet polynomials

Now consider the random Dirichlet polynomials

$$\mathcal{D}(s) = \sum_{n=1}^{N} \varepsilon_n d(n) n^{-s}, \qquad s = \sigma + it, \tag{73}$$

and examine their supremum properties. When $d(n) \equiv 1$, there are optimal results. If $\sigma = 0$, then for some absolute constant C, and all integers $N \ge 2$

$$C^{-1} \frac{N}{\log N} \le \mathbf{E} \sup_{t\in\mathbf{R}} \Big| \sum_{n=2}^{N} \varepsilon_n n^{-it} \Big| \le C \frac{N}{\log N} \ . \tag{74}$$

This has been proved by Halász and was later extended by Queffélec to the range of values $0 \le \sigma < 1/2$. Queffélec gave a probabilistic proof of the original one, using Bernstein's inequality for polynomials.

Theorem 19 *There exists a constant C_σ depending on σ only, such that for all integers $N \geq 2$ we have,*

$$C_\sigma^{-1}\frac{N^{1-\sigma}}{\log N} \leq \mathbf{E}\ \sup_{t\in\mathbf{R}} |\sum_{n=2}^{N} \varepsilon_n n^{-\sigma-it}| \leq C_\sigma \frac{N^{1-\sigma}}{\log N}. \tag{75}$$

Extensions of (75) were obtained in the recent works [26],[27]. The approach used does not appeal to Bernstein's inequality, and is completely based on stochastic process method, notably the metric entropy method. Further a new lower bound is obtained, which is of a completely different nature than Bohr's deterministic lower bound used in Queffélec's proof. For random Dirichlet polynomials defined in (73), a new approach is developed in [26]. Define

$$\mathcal{L}_j = \Big\{ n = p_j\,\tilde n \ :\ \tilde n \leq \frac{N}{p_j} \text{ and } P^+(\tilde n) \leq p_{\mu/2}\Big\}, \qquad\qquad j \in (\mu/2, \mu].$$

Theorem 20 (Lower bound)

$$\mathbf{E}\sup_{t\in\mathbf{R}} |\mathcal{D}(\sigma+it)| \geq c \sum_{\mu/2<j\leq\mu} \Big(\sum_{n\in\mathcal{L}_j} d(n)^2\, n^{-2\sigma}\Big)^{1/2}.$$

Now we turn to upper bounds. We assume that d is sub-multiplicative:

$$d(nm) \leq d(n)d(m) \qquad \text{provided } (n,m) = 1. \tag{76}$$

A typical example is for instance function $d_K(n) = \chi\{(n,K) = 1\}$. Naturally all multiplicative functions are sub-multiplicative, and so is the case of $d(n) = \lambda^{\omega(n)}$, where $\lambda > 1$ and $\omega(n) = \#\{p : p \mid n\}$.

In [47] a general upper bound is obtained, containing and strictly improving the main results in [26],[27]. Further the proof is entirely based on Gaussian comparison properties, all suprema of auxiliary Gaussian processes used being computable exactly. Introduce a basic decomposition. Denote by $P^+(n)$ the largest prime divisor of n. Then

$$\{2,\ldots,N\} = \sum_{1\leq j\leq\pi(N)} E_j, \qquad\qquad E_j = \big\{2 \leq n \leq N : P^+(n) = p_j\big\}$$

It is natural to disregard cells E_j such that $d(n) \equiv 0$, $n \in E_j$. We thus set $\mathcal{H}_d = \big\{1 \leq j \leq \pi(N) : d_{|E_j} \not\equiv 0\big\}$, $\tau_d = \max(H_d)$. The relevant assumption is the following:

$$p|n \implies d(n) \leq C\, d(\frac{n}{p}), \qquad \text{and} \qquad d(p^j) \leq C_1\lambda^j, \tag{77}$$

for some positive C, C_1, λ with $\lambda < \sqrt{2}$, any prime number p, any integers n, j.

Clearly, if $C < \sqrt{2}$, the second property is implied by the first, although this is not always so as the following example yields. Fix some prime number P_1 as well as some reals $1 < \lambda_1 < \sqrt{2}$, $C_1 \geq 1$, and put

$$d(n) = \begin{cases} C_1\lambda^j, & \text{if } P_1^j || n, \\ 1, & \text{if } (n, P_1) = 1. \end{cases} \tag{78}$$

Then d is sub-multiplicative, and satisfies condition (77) with a constant C which has to be larger than $C_1\lambda$. Now put

$$\begin{aligned} D_1(M) &= \sum_{m=1}^M d(m), \qquad \widetilde{D}_1(M) = \max_{1\le m\le M} \frac{D_1(m)}{m}, \\ D_2(M) &= \sum_{m=1}^M d(m)^2, \qquad \widetilde{D}_2^2(M) = \max_{1\le m\le M} \frac{D_2(m)}{m}. \end{aligned} \tag{79}$$

Theorem 21 (Upper bound) *Let d be a non-negative sub-multiplicative function. Assume that condition (77) is realized. Let $0 \le \sigma < 1/2$. Then there exists a constant $C_{\sigma,d}$ depending on σ and d only, such that for any integer $N \ge 2$,*

$$\mathbf{E} \sup_{t\in\mathbf{R}} |\mathcal{D}(\sigma+it)| \le C_{\sigma,d}\, \widetilde{D}_2(N)\, B,$$

where

$$B = \begin{cases} \frac{N^{1/2-\sigma}\tau_d^{1/2}}{(\log N)^{1/2}} & \text{, if } \left(\frac{N\log\log N}{\log N}\right)^{1/2} \le \tau_d \le \pi(N), \\ \frac{N^{3/4-\sigma}(\log\log N)^{1/4}}{(\log N)^{3/4}} & \text{, if } \left(\frac{N}{(\log N)\log\log N}\right)^{1/2} \le \tau_d \le \left(\frac{N\log\log N}{\log N}\right)^{1/2}, \\ N^{1/2-\sigma}\left(\frac{\tau_d\log\log\tau_d}{\log\tau_d}\right)^{1/2} & \text{, if } 1 \le \tau_d \le \left(\frac{N}{(\log N)\log\log N}\right)^{1/2}. \end{cases}$$

This yields, when combined with Theorem 20, sharp estimates. Consider the following example.

Example 1. Take some positive integer K, and let $d_K(n) = \chi\{(n,K)=1\}$. Then d_K is sub-multiplicative and condition (77) is satisfied with $C = 1 = \lambda$. By (73), this defines the remarkable class of random Dirichlet polynomials,

$$\mathcal{D}(s) = \sum_{\substack{(n,K)=1\\ 1\le n\le N}} \frac{\varepsilon_n}{n^s}, \tag{80}$$

containing the one of $\mathcal{E}_\tau$-based Dirichlet polynomials considered in [35] and [26], where $\mathcal{E}_\tau = \{2 \le n \le N : P^+(n) \le p_\tau\}$. Here $\mathcal{H}_{d_{K_\tau}} = \sum_{j\le\tau} E_j$. We therefore neglect cells E_j, $j > \tau$. Further, we have $\widetilde{D}_1(N) = \widetilde{D}_2(N) \le 1$. As a consequence of Corollary 3 of [47] and Theorem 20, we have in particular

Theorem 22 *Let $0 \le \sigma < 1/2$.*
a) If $\left(\frac{N\log\log N}{\log N}\right)^{1/2} \le \tau \le \pi(N)$,

$$C_1(\sigma)\frac{N^{1/2-\sigma}\tau^{1/2}}{(\log N)^{1/2}} \le \mathbf{E}\sup_{t\in\mathbf{R}}\Big|\sum_{n\in\mathcal{E}_\tau}\varepsilon_n n^{-\sigma-it}\Big| \le C_2(\sigma)\frac{N^{1/2-\sigma}\tau^{1/2}}{(\log N)^{1/2}}.$$

b) If $\left(\frac{N}{(\log N)\log\log N}\right)^{1/2} \le \tau \le \left(\frac{N\log\log N}{\log N}\right)^{1/2}$.

$$C_1(\sigma)\,\frac{N^{1/2-\sigma}\tau^{1/2}}{(\log N)^{1/2}} \le \mathbf{E}\sup_{t\in\mathbf{R}}\Big|\sum_{n\in\mathcal{E}_\tau}\varepsilon_n n^{-\sigma-it}\Big| \le C_2(\sigma)\frac{N^{3/4-\sigma}(\log\log N)^{1/4}}{(\log N)^{3/4}}.$$

c) If $1 \le \tau \le \big(\frac{N}{(\log N)\log\log N}\big)^{1/2}$. *Assume that* $\tau \ge N^\varepsilon$ *for some fixed* $0<\varepsilon<1/2$. *Then,*

$$C_1(\sigma,\varepsilon)\,\frac{N^{1/2-\sigma}\tau^{1/2}}{(\log\tau)^{1/2}} \ \le \mathbf{E}\ \sup_{t\in\mathbf{R}}\Big|\sum_{n\in\mathcal{E}_\tau}\varepsilon_n n^{-\sigma-it}\Big| \le C_2(\sigma)N^{1/2-\sigma}\big(\frac{\tau\log\log\tau}{\log\tau}\big)^{1/2}.$$

We notice that the gap is always less that $(\log\log N)^{1/2}$. Theorem 21 also applies (see [47]) to the case $\tau \ll_\varepsilon N^\varepsilon$, as well as to other classes of examples, for instance

Example 2. Consider multiplicative functions satisfying the following condition:

$$\frac{d(p^a)}{d(p^{a-1})} \le \lambda, \qquad a = 1,2,\dots \tag{81}$$

Clearly (81) implies (77) and further $M_d := \sup_p d(p) < \infty$, with $M_d \le \lambda d(1)$. By theorem 2 of [15], any non-negative multiplicative function d satisfying a Wirsing type condition $d(p^m) \le \lambda_1\lambda_2^m$, for some constants $\lambda_1>0$ and $0<\lambda_2<2$ and all prime powers $p^m \le x$, also satisfies

$$\frac{1}{x}\sum_{n\le x} d(n) \le C(\lambda_1,\lambda_2)\exp\Big\{\sum_{p\le x}\frac{d(p)-1}{p}\Big\}, \tag{82}$$

where $C(\lambda_1,\lambda_2)$ depends on λ_1,λ_2 only. This and the fact that d^2 is multiplicative and satisfies (81) with $\lambda^2<2$, yield that

$$\widetilde{D}_1(N) \le C(\lambda)(\log N)^{M_d}, \qquad \widetilde{D}_2(N) \le C(\lambda)(\log N)^{M_d^2}. \tag{83}$$

Proof of Theorem 21 *(Sketch).* The proof is long and technically delicate. We only outline the main steps and will avoid calculation details. Let $M \le N$ and $0<\sigma<1/2$. Fix some integer ν in $[1,\tau]$ and let $F_\nu = \sum_{1\le j\le\nu} E_j$, $F^\nu = \sum_{\nu<j\le\tau} E_j$. The basic principle of the proof consists of a decomposition of Q in (69) into a sum of two trigonometric polynomials $Q = Q_1^\varepsilon + Q_2^\varepsilon$, where

$$Q_1^\varepsilon(\underline{z}) = \sum_{n\in F_\nu}\varepsilon_n d(n) n^{-\sigma} e^{2i\pi\langle\underline{a}(n),\underline{z}\rangle}, \qquad Q_2^\varepsilon(\underline{z}) = \sum_{n\in F^\nu}\varepsilon_n d(n) n^{-\sigma} e^{2i\pi\langle\underline{a}(n),\underline{z}\rangle}.$$

By the contraction principle $\mathbf{E}\ \sup_{\underline{z}\in\mathbf{T}^\tau}\big|Q_i^\varepsilon(\underline{z})\big| \le C\mathbf{E}\ \sup_{\underline{z}\in\mathbf{T}^\tau}\big|Q_i(\underline{z})\big|$, $i=1,2$ where Q_i is the same process as Q_i^ε except that the Rademacher random variables ε_n are replaced by independent $\mathcal{N}(0,1)$ random variables μ_n. Consequently, both the suprema of Q_1 and of Q_2 can be estimated, via their associated L^2-metric. First evaluate the supremum of Q_2. We have

$$Q_2(\underline{z}) \quad = \quad \sum_{\nu<j\le\tau} e^{2i\pi z_j}\sum_{n\in E_j}\mu_n d(n) n^{-\sigma} e^{2i\pi\{\sum_k a_k(\frac{n}{p_j})z_k\}}.$$

And so

$$\sup_{\underline{z}\in\mathbf{T}^\tau}\big|Q_2(\underline{z})\big| \le 4\sup_{\gamma\in\Gamma}\big|X(\gamma)\big|, \tag{84}$$

where the random process X is defined by

$$X(\gamma) = \sum_{\nu<j\le\tau}\alpha_j\sum_{n\in E_j}\mu_n\frac{d(n)}{n^\sigma}\beta_{\frac{n}{p_j}}, \qquad \gamma\in\Gamma, \tag{85}$$

with $\gamma = \big((\alpha_j)_{\nu<j\le\tau}, (\beta_m)_{1\le m\le N/2}\big)$ and $\Gamma = \big\{\gamma : |\alpha_j| \vee |\beta_m| \le 1, \nu < j \le \tau, 1 \le m \le N/2\big\}$. The problem now reduces to estimating the supremum over Γ of the real valued Gaussian process X. Plainly

$$\|X_\gamma - X_{\gamma'}\|_2^2 \quad \le \quad 2 \sum_{\nu<j\le\tau} \sum_{n\in E_j} d(n)^2 n^{-2\sigma} \big[(\alpha_j - \alpha'_j)^2 + (\beta_{\frac{n}{p_j}} - \beta'_{\frac{n}{p_j}})^2\big].$$

Condition (77) and Abel summation yield

$$\begin{aligned} \|X_\gamma - X_{\gamma'}\|_2^2 &\le 2 \sum_{\nu<j\le\tau} \sum_{n\in E_j} \frac{d(n)^2}{n^{2\sigma}} \big[(\alpha_j - \alpha'_j)^2 + (\beta_{\frac{n}{p_j}} - \beta'_{\frac{n}{p_j}})^2\big] \\ \le \quad &\lambda^2 \sum_{\nu<j\le\tau} (\alpha_j - \alpha'_j)^2 \frac{N^{1-2\sigma} \widetilde{D}_2^2(N/p_j)}{p_j} + C\lambda^2 \sum_{m\le N/p_\nu} K_m^2 (\beta_m - \beta'_m)^2, \end{aligned}$$

where $K_m = \sum_{\substack{\nu<j\le\tau \\ mp_j\le N}} \frac{d(m)^2}{(mp_j)^{2\sigma}}$ and $\sum_{m\le N/p_\nu} K_m \le CN^{1-\sigma}\widetilde{D}_1(\frac{N}{p_\nu})/\sqrt{\nu}\log\nu$, by Abel summation. Define a second Gaussian process by putting for all $\gamma \in \Gamma$

$$Y(\gamma) = \sum_{\nu<j\le\tau} \big(\frac{\widetilde{D}_2^2(N/p_j) N^{1-2\sigma}}{p_j}\big)^{1/2} \alpha_j \xi'_j + \sum_{m\le N/p_\nu} K_m\, \beta_m \xi''_m := Y'_\gamma + Y''_\gamma,$$

where ξ'_i, ξ''_j are independent $\mathcal{N}(0,1)$ random variables. Thus for some suitable constant C, one has $\|X_\gamma - X_{\gamma'}\|_2 \le C\|Y_\gamma - Y_{\gamma'}\|_2$ for all $\gamma, \gamma' \in \Gamma$. By the Slepian lemma ([51], Lemma 10.2.3), and (84)

$$\mathbf{E} \sup_{\underline{z}\in\mathbf{T}^\tau} \big|Q_2(\underline{z})\big| \le C\mathbf{E} \sup_{\gamma\in\Gamma} \big|Y(\gamma)\big|. \tag{86}$$

As

$$\begin{aligned} \mathbf{E} \sup_{\gamma\in\Gamma} |Y'(\gamma)| \quad &\le \quad C\ N^{\frac{1}{2}-\sigma} \widetilde{D}_2(N/p_\nu)\ \frac{\tau^{1/2}}{(\log\tau)^{1/2}} \\ \mathbf{E} \sup_{\gamma\in\Gamma} |Y''(\gamma)| \quad &\le \quad \frac{CN^{1-\sigma}\widetilde{D}_1(N/p_\nu)}{\nu^{1/2}\log\nu}, \end{aligned} \tag{87}$$

by reporting (87) into (86), we get

$$\mathbf{E} \sup_{\underline{z}\in\mathbf{T}^\tau} \big|Q_2(\underline{z})\big| \le C\Big(N^{1/2-\sigma}\widetilde{D}_2(N/p_\nu)\ \frac{\tau^{1/2}}{(\log\tau)^{1/2}} + \frac{N^{1-\sigma}\widetilde{D}_1(N/p_\nu)}{\nu^{1/2}\log\nu}\Big). \tag{88}$$

For estimating the supremum of Q_1, we introduce the auxiliary Gaussian process

$$\Upsilon(\underline{z}) = \sum_{n\in F_\nu} d(n) n^{-\sigma} \big\{\vartheta_n \cos 2\pi\langle \underline{a}(n), \underline{z}\rangle + \vartheta'_n \sin 2\pi\langle \underline{a}(n), \underline{z}\rangle\big\}, \qquad \underline{z} \in \mathbf{T}^\nu,$$

where ϑ_i, ϑ'_j are independent $\mathcal{N}(0,1)$ random variables. By symmetrization

$$\mathbf{E} \sup_{\underline{z}\in\mathbf{T}^\nu} \big|Q_1(\underline{z})\big| \le \sqrt{8\pi}\mathbf{E} \sup_{\underline{z}\in\mathbf{T}^\nu} \big|\Upsilon(\underline{z})\big|. \tag{89}$$

Plainly $\|\Upsilon(\underline{z})-\Upsilon(\underline{z})\|_2^2 \le 4\pi^2 \ \sum_{n\in F_\nu} \frac{d(n)^2}{n^{2\sigma}}\big[\sum_{j=1}^\nu a_j(n)|z_j-z_j'|\big]^2$. Now,

$$\sum_{n\in F_\nu}\frac{d(n)^2}{n^{2\sigma}}\Big[\sum_{j=1}^\nu a_j(n)|z_j-z_j'|\Big]^2 = \sum_{j=1}^\nu |z_j-z_j'|^2\sum_{n\in F_\nu}\frac{a_j(n)^2d(n)^2}{n^{2\sigma}}$$

$$+\sum_{\substack{1\le j_1,j_2\le\nu\\ j_1\neq j_2}} |z_{j_1}-z_{j_1}'|\ |z_{j_2}-z_{j_2}'|\sum_{n\in F_\nu}\frac{a_{j_1}(n)a_{j_2}(n)d(n)^2}{n^{2\sigma}} := S+R.$$

By sub-multiplicativity, we have for R

$$R\le C\sum_{j_1\neq j_2}|z_{j_1}-z_{j_1}'||z_{j_2}-z_{j_2}'|\sum_{b_1,b_2=1}^\infty\frac{b_1d(p_{j_1}^{b_1})^2}{p_{j_1}^{2b_1\sigma}}\frac{b_2d(p_{j_2}^{b_2})^2}{p_{j_2}^{2b_2\sigma}}\sum_{k\le N/p_{j_1}^{b_1}p_{j_2}^{b_2}}\frac{d(k)^2}{k^{2\sigma}}. \tag{90}$$

By using Abel summation, one deduces that

$$R\le C_\lambda\widetilde{D}_2(N)^2N^{1-2\sigma}\Big[\sum_{j=1}^\nu\frac{|z_j-z_j'|}{p_j}\Big]^2,$$

As to S, we have similarly

$$\begin{aligned} S &\le \sum_{j=1}^\nu|z_j-z_j'|^2\sum_{b=1}^\infty\frac{b^2d(p_j^b)^2}{p_j^{2b\sigma}}\sum_{m\le\frac{N}{p_j^b}}\frac{d(m)^2}{m^{2\sigma}}\\ &\le C\widetilde{D}_2(N)^2N^{1-2\sigma}\Big[\sum_{j=1}^\nu\frac{|z_j-z_j'|^2}{p_j}\Big]. \end{aligned}\tag{91}$$

Consequently

$$\begin{aligned} \|\Upsilon(\underline{z})-\Upsilon(\underline{z})\|_2 &\le C_\lambda N^{1/2-\sigma}\widetilde{D}_2(N)\max\Big[\sum_{j=1}^\nu\frac{|z_j-z_j'|}{p_j},\Big[\sum_{j=1}^\nu\frac{|z_j-z_j'|^2}{p_j}\Big]^{\frac12}\Big]\\ &\le C_\lambda N^{1/2-\sigma}\widetilde{D}_2(N)(\log\log\nu)^{1/2}\Big(\sum_{j=1}^\nu\frac{|z_j-z_j'|^2}{p_j}\Big)^{1/2}, \end{aligned}\tag{92}$$

where we used Cauchy-Schwarz's inequality in the last inequality. Let $g_1,\dots,g_\nu$ be independent standard Gaussian r.v.'s. Then $U(z):=\sum_{j=1}^\nu g_jp_j^{-1/2}z_j$ satisfies $\|U(z)-U(z')\|_2=\big(\sum_{j=1}^\nu\frac{|z_j-z_j'|^2}{p_j}\big)^{1/2}$. And so

$$\big\|\Upsilon(\underline{z})-\Upsilon(\underline{z})\big\|_2\le C_\lambda N^{1/2-\sigma}\widetilde{D}_2(N)(\log\log\nu)^{1/2}\|U(z)-U(z')\|_2. \tag{93}$$

Henceforth

$$\mathbf{E}\sup_{\underline{z},\underline{z}'\in T^\nu}|\Upsilon(\underline{z}')-\Upsilon(\underline{z})|\le C_\lambda N^{1/2-\sigma}\widetilde{D}_2(N)(\log\log\nu)^{1/2}\mathbf{E}\sup_{\underline{z},\underline{z}'\in T^\nu}|U(\underline{z}')-U(\underline{z})|.$$

But obviously $\sup_{\underline{z}\in T^\nu}|U(z)|=\sum_{j=1}^\nu|g_j|p_j^{-1/2}$, and so $\mathbf{E}\sup_{\underline{z}'\in T^\nu}|U(\underline{z}')-U(\underline{z})|\le C(\nu/\log\nu)^{1/2}$. By reporting, and since $\|\Upsilon(\underline{z})\|_2\le CN^{1/2-\sigma}\widetilde{D}_2(N)$, for any $\underline{z}\in\mathbf{T}^\nu$, we get

$$\mathbf{E}\sup_{\underline{z}'\in T^\nu}|\Upsilon(\underline{z}')|\le CN^{1/2-\sigma}\widetilde{D}_2(N)\big(\frac{\nu\log\log\nu}{\log\nu}\big)^{1/2}. \tag{94}$$

By substituting in (89) and combining with (88) consequently get

$$\mathbf{E}\sup_{t\in\mathbf{R}}|\mathcal{D}(\sigma+it)| \le C_{\sigma,\lambda}\, N^{1/2-\sigma}\widetilde{D}_2(N)\left[\Big(\frac{\nu\log\log\nu}{\log\nu}\Big)^{1/2}+\frac{\tau^{1/2}}{(\log\tau)^{1/2}}+\frac{N^{1/2}}{\nu^{1/2}\log\nu}\right]. \tag{95}$$

The proof is accomplished by estimating separately the upper bound in the three cases:

i) $(\frac{N\log\log N}{\log N})^{1/2} \le \tau \le \pi(N)$.

ii) $(\frac{N}{(\log N)\log\log N})^{1/2} \le \tau \le (\frac{N\log\log N}{\log N})^{1/2}$.

iii) $1 \le \tau \le \big(\frac{N}{(\log N)\log\log N}\big)^{1/2}$. ■

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Michel Weber, Mathématique (IRMA), Université Louis-Pasteur et C.N.R.S., 7 rue René Descartes, 67084 Strasbourg Cedex, France.
E-mail: weber@math.u-strasbg.fr