ELSEVIER

Discrete Mathematics 186 (1998) 253–268

DISCRETE MATHEMATICS

A game of cops and robbers played on products of graphs

S. Neufeld[a,*], R. Nowakowski[b]

[a] *Department of Mathematics and Statistics, University of Winnipeg, Winnipeg, Man., Canada R3C 1Y3*

[b] *Department of Mathematics, Statistics, and Computing Science, Dalhousie University, Halifax, Nova Scotia, Canada B3H 3J5*

Received 12 May 1995; received in revised form 12 March 1997; accepted 7 April 1997

Abstract

The game of cops and robbers is played with a set of 'cops' and a 'robber' who occupy some vertices of a graph. Both sides have perfect information and they move alternately to adjacent vertices. The robber is captured if at least one of the cops occupies the same vertex as the robber. The problem is to determine on a given graph, G, the least number of cops sufficient to capture the robber, called the cop-number, $c(G)$. We investigate this game on three products of graphs: the Cartesian, categorical, and strong products.

1. Introduction

Cops and robbers is a 2-player pursuit game played on an undirected graph, $G=(V,E)$; a set of cops versus the robber. Player 1 starts the game by choosing vertices (not necessarily distinct) for a set of cops after which Player 2 chooses a vertex for the robber. The two players then take turns beginning with Player 1 who slides a subset of the cops along the edges of G to adjacent vertices. Player 2 responds by moving the robber to an adjacent vertex or by keeping the robber at his current position (passing). For convenience, we often say Player 1 is the set of cops and Player 2 is the robber. Both sides always know each other's positions and we assume they play their optimal strategy at all times.

The cops win if, in a finite number of moves, one (or more) of them occupies the same vertex as the robber. The robber wins if he can perpetually avoid this situation.

By varying the constraints imposed on the two parties, many versions of the game are possible. However, we shall only consider two quite natural variations, called passive and active, which differ in the moves allowed for each side.

In the *passive game* both sides have the option of passing. We note that a pass by the cops cannot be negative for the robber since he then has an opportunity to improve

* Corresponding author.

PII S0012-365X(97)00165-9

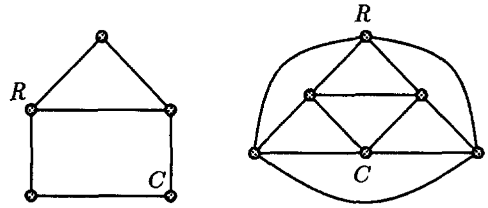

Fig. 1.

his position by moving. If no move is advantageous, then the robber passes but in neither case has his position deteriorated. We illustrate in Fig. 1 two situations where, given it is the robber's turn, he must pass in order to win.

In the *active game* both the robber and a non-empty subset of the cops must move at their respective turns. This variation was first introduced by Aigner and Fromme [1] and investigated further by Tošić [13] and Neufeld [8].

If G is a reflexive graph (i.e., with a loop at each vertex) then the passive and the active games are equivalent since passing is equivalent to moving along a loop. Thus, we consider only graphs with no loops (i.e. irreflexive).

The minimum number of cops sufficient to win on a graph, G, is called the *cop-number* and is denoted by $c(G)$ for the passive game and by $c'(G)$ for the active game. In addition begin irreflexive, the graphs we consider in this paper are simple (no multiple edges) and connected. We impose these conditions because the cop-number is unaffected by multiple edges and because $c(G)$ is just the sum of the cop-numbers of each connected component of G. Thus, unless specified otherwise, all graphs in this paper are assumed to be simple, undirected, connected, and irreflexive.

Note that the cop-number is unaffected by the initial position since from any given position the cops may migrate to their optimal initial position (since G is connected) and consider this to be the starting point of the game.

Lemma 1.1. *Let G be a graph. Then*

$$c(G) - 1 \leqslant c'(G) \leqslant c(G).$$

Proof. We observe that $c(G)$ cops must have a winning strategy in the passive game in which at least one of their number moves at each turn. Since the robber's options in the active game are a subset of those in the passive game we see that $c'(G) \leqslant c(G)$.

Let $c'(G) + 1$ cops play the passive game on G. Let one of these cops, S, move toward the robber so that the robber cannot perpetually pass. The remaining $c'(G)$ cops play their winning strategy in the active game whenever the robber moves. If the robber passes, then all cops pass except S who moves toward the robber. Hence, $c(G) \leqslant c'(G) + 1$. □

If $c'(G) = c(G) - 1$, then we refer to G as c^* win. If $c(G) = 1$, then G is simply called cop-win. Quilliot [11] and, independently, Nowakowski and Winkler [10]

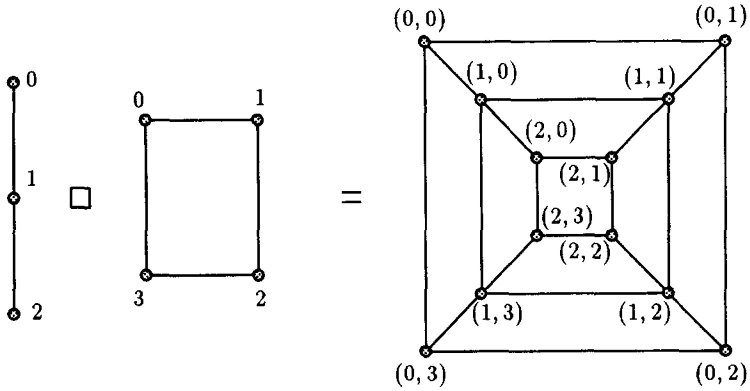

Fig. 2.

obtained a characterization of cop-win graphs. Later Aigner and Fromme [1] showed that $c(G) \leqslant 3$ for planar graphs G, and Quilliot [12] extended this to graphs of positive genus k obtaining $c(G) \leqslant 3+2k$. Andreae [2] examined graphs with excluded minors and Frankl [5,6] graphs with large girths and Cayley graphs. Bridged graphs were investigated by Anstee and Farber [3].

In this paper we consider the cop-number for three products of graphs; namely, the Cartesian, categorical, and strong products and we obtain some bounds and some exact results. We use the symbols $\square$, $\times$, and $\boxtimes$, due to Nešetřil, which represent the product of two edges in the Cartesian, categorical and strong products respectively. The n-fold products we denote by $\square_{i=1}^{n} G_i$, $\times_{i=1}^{n} G_i$, and $\boxtimes_{i=1}^{n} G_i$.

We use the symbol $\times$ to refer to the generic product, i.e. any one of the three products named above.

In all three products, the vertex set of the product graph is the Cartesian product of the vertex sets of the factors $G_i=(V_i,E_i)$, $i=1,2,\ldots,n$. Denote the vertices in the product graph by $\underline{a}=(a_1,a_2,\ldots,a_n)$ where a_i is a vertex of G_i for each i, $1 \leqslant i \leqslant n$.

Two vertices $\underline{a}=(a_1,a_2,\ldots,a_n)$ and $\underline{b}=(b_1,b_2,\ldots,b_n)$ are adjacent in the n-fold *Cartesian* product if and only if $a_i \neq b_i$ for precisely one i, $1 \leqslant i \leqslant n$ and, for this i, (a_i,b_i) is an edge in G_i.

Thus, in the Cartesian product a cop or the robber moves by changing exactly one of her/his coordinates.

An example of a Cartesian product is shown in Fig. 2. Here $c(G \square H) = c'(G \square H) = 2$, whereas the constituent graphs, G and H, have cop-numbers equal to 1 and 2, respectively. A second example is the n-cube which we may regard as the n-fold Cartesian product of n paths of length 1.

In the n-fold *categorical* product $(\underline{a},\underline{b})$ is an edge if and only if for all i, $1 \leqslant i \leqslant n$, (a_i,b_i) is an edge in G_i. Note that if each graph G_i, $1 \leqslant i \leqslant n$, is bipartite then the resulting product graph is not connected.

Fig. 3 illustrates the categorical product. Here $c(G \times H) = 4$ and $c'(G \times H) = 2$.

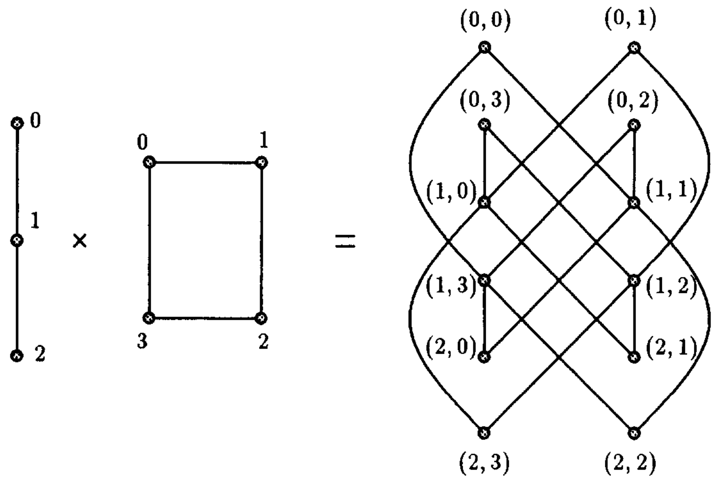

Fig. 3.

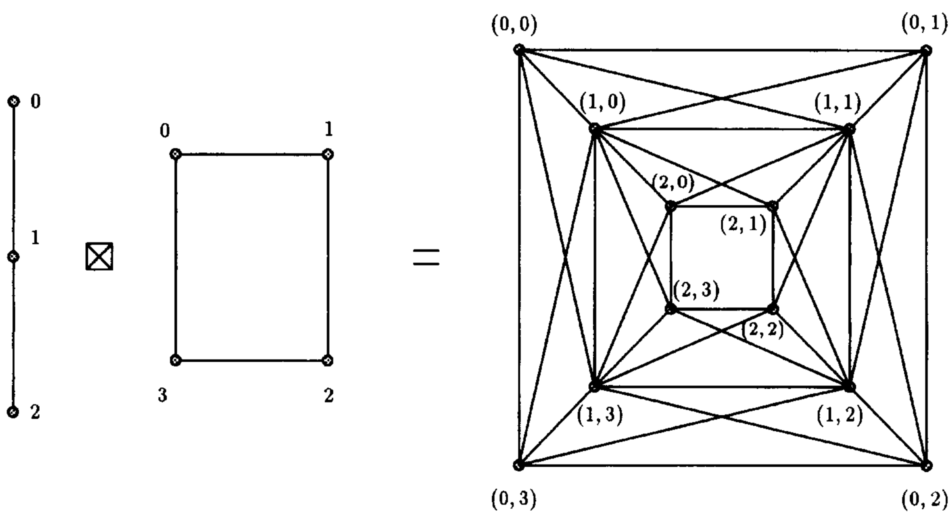

Fig. 4.

In the n-fold *strong* product $(\underline{a},\underline{b})$ is an edge if and only if for each i, $1 \leqslant i \leqslant n$, either (a_i, b_i) is an edge in G_i or $a_i = b_i$. Note that the edge set for $G_1 \boxtimes G_2$ is the disjoint union of the edge sets of $G_1 \square G_2$ and $G_1 \times G_2$ but this is no longer true if there are three or more graphs in the product.

In the example of the strong product shown in Fig. 4, $c(G \boxtimes H) = c'(G \boxtimes H) = 2$.

A *projection* of $G \times H$ onto G is a map $\pi_G : G \times H \to G$ defined as $\pi_G(x, y) = x$. Likewise a projection of $G \times H$ onto H we define as $\pi_H(x, y) = y$.

We observe that if m and n are the number of vertices of H and G, respectively, then the vertex set of any one of our three products can be thought of as consisting of n copies of vertices of H or alternatively m copies of vertices of G. We use the notation

$G.\{x\}$ to refer to the subgraph of $G \times H$ where the second coordinate is $x \in V(H)$ and the first coordinate is any vertex of G. We define $\{y\}.H$ in a similar way. Since in the Cartesian and strong products the subgraphs $G.\{x\}$ and $\{y\}.H$ are isomorphic to G and H we refer to $G.\{x\}$ and $\{y\}.H$ as *copies* of G and H in the product. In the categorical product, $G.\{x\}$ and $\{y\}.H$ are empty graphs on n and m vertices; nevertheless, to avoid cumbersome language we also speak of $G.\{x\}$ and $\{y\}.H$ as copies of G and H.

We say that a player *moves on* G if his projection onto H remains unchanged. We similarly define moving on H. A vertex v is said to be *captured* if a cop occupies v. *Capturing the projection* of a vertex $w=(u,v)$, $u \in V(G)$, $v \in V(H)$, onto G (resp. H) means capturing (u,x) for some $x \in V(H)$ (resp. (x,v) for some $x \in V(G)$). A cop is said to *shadow* the robber on G (resp. H) if after each turn his projection onto G (resp. H) is the same as the robber's.

A *walk* W from u to v is a sequence of not necessarily distinct vertices $V(W) = (u,a_1,a_2,\dots,v)$ and a set of edges $E(W)=(u,a_1),(a_1,a_2),\dots,(a_{i-1},a_i),\dots,(a_j,v)$. A *path* is a walk where all the vertices are distinct.

The *neighborhood* of a vertex, denoted $N(u)$, consists of the vertex u and all vertices adjacent to u. The neighborhood of a player is defined in a similar way. The *distance* between two vertices u,v, denoted $d(u,v)$, is the length of a shortest path between them. We let $\delta(G)$ denote the minimum vertex degree among all vertices of G. For any terms not defined in this paper please refer to Chartrand and Lesniak [4].

2. Cartesian products

The game of cops and robbers has been previously investigated on Cartesian products by Tošić [13], and by Maamoun and Meyniel [7].

Theorem 2.1 (Tošić [13]). *Let G and H be graphs with cop-numbers $c(G)$ and $c(H)$. Then*

$$c(G \square H) \leqslant c(G) + c(H).$$

Clearly, Theorem 2.1 implies $c(\square_{i=1}^{n} G_i) \leqslant \sum_{i=1}^{n} c(G_i)$. We also note the relationship given in Theorem 2.1 holds for the active game.

A trivial lower bound for $c(G \square H)$ is: $\max(c(G), c(H)) \leqslant c(G \square H)$ since this number of cops are required if the robber simply restricts his movements to $G.\{x\}$ where $x \in V(H)$ or to $\{y\}.H$, where $y \in V(G)$. However, in many cases the cop-number is much larger than this.

Thus, for the Cartesian product we can in general provide an upper bound and a (trivial) lower bound. However, for some special graphs such as paths and trees, cycles and complete graphs, we can calculate the cop-number of their Cartesian product more specifically. We begin with the following theorem.

Theorem 2.2. *Let G and H be graphs and suppose at least one of G and H is c^* win. Then,*

$$c(G \square H) \leqslant c(G) + c(H) - 1.$$

Proof. Suppose, without loss of generality, that G is c^* win. Let $c(G)+c(H)-1$ cops play on a copy of H in $G \square H$ so that $c(G)-1$ of their number occupy the projection of the robber onto H. Thus, these cops are in the same copy of G as the robber and from now on shadow the robber on H. Now the remaining $c(H)$ cops move to the same copy of H as the robber where they play their winning strategy on H. Thus, the robber must, from time to time, move on G. Whenever this occurs, the $c(G)-1$ shadowing cops play their winning active game strategy on G. Hence, the robber is eventually captured. □

An immediate consequence of Theorem 2.2 is the following:

Corollary 2.1. *Let $G_1, G_2, \ldots, G_n$ be graphs and suppose k of them, $0 \leqslant k \leqslant n$, are c^* win. Then,*

$$c\left(\mathop{\square}_{i=1}^{n} G_i\right) \leqslant \begin{cases} \sum_{i=1}^{n} c(G_i) - k, & \text{if } k<n, \\ \sum_{i=1}^{n} c(G_i) - n + 1, & \text{if } k=n. \end{cases}$$

Proof. Apply the technique in Theorem 2.2 together with induction on n. □

Thus, if we know that graphs are c^* win we can obtain a refinement of Tosic's result given in Theorem 2.1. However, being c^* win is quite a strong condition on a graph and we would like to weaken this condition somewhat and obtain a result similar to Theorem 2.2.

We note that certain graphs are in a sense 'close' to c^* win. For example, consider a cycle, C, of length 4 and, more generally, the Cartesian product, $T_1 \square T_2$ of two trees. In both cases, $c'(C)=c(C)=c'(T_1 \square T_2)=c(T_1 \square T_2)=2$. However, if we force the robber to move first, then $c'(C)=c(C)=c'(T_1 \square T_2)=1$ while $c(T_1 \square T_2)=2$. This motivates the following definition.

Definition 2.1. Let $G=\square_{i=1}^{n} G_i$, $n \geqslant 1$, and let $c(G)-1$ cops play on G. Let W be the walk taken by the robber during this game where the robber avoids the $c(G)-1$ cops. G is said to be *nearly c^* win* provided these cops can play so that for each G_i, $1 \leqslant i \leqslant n$, the projection of W onto G_i necessarily contains the vertex sequence $u_i, v_i, \ldots, v_i, u_i$ for some $u_i, v_i \in V(G_i)$.

Note that in Definition 2.1 if $n=1$, then $c(G)-1$ cops can force the robber to 'backtrack', i.e., the robber must move $x \rightarrow y \rightarrow x$ for some $x, y \in V(G)$.

Examples of nearly c^* win graphs are cycles of length at least 4, the Petersen graph, and the Cartesian product of two trees.

Theorem 2.3. *Suppose G and H are nearly c^* win graphs and let T be a tree. Then,*
(i) $c(G\square H)\leqslant c(G)\square c(H)-1$,
(ii) $c(G\square T)=c(G)$.

Proof. (i) Let $G=\square_{i=1}^{n}G_i$, $n\geqslant 1$, and let $H=\square_{i=1}^{m}H_i$, $m\geqslant 1$. Let $c(G)+c(H)-1$ cops play on $G\square H$. Let the current position of the robber be $\underline{r}=(\underline{a},\underline{b})$ where $\underline{a}=(a_1,a_2,\dots,a_n)\in V(G)$ and $\underline{b}=(b_1,b_2,\dots,b_m)\in V(H)$. Let $\underline{u}=(a_1,\dots,a_{i-1},u_i,a_{i+1},\dots,a_n)\in V(G)$ such that (u_i,a_i) is an edge in G_i.

Let a subset $P=\{S_1,S_2,\dots,S_{c(H)}\}$ of the set of cops capture u. Whenever the robber moves on H, or moves so that $a_i=u_i$ the cops in P play their winning strategy on H. If the robber moves on G but not to u_i, then the cops in P maintain their position on $\underline{u}$ relative to $\underline{a}$.

The remaining $c(G)-1$ cops migrate to $G.\{\underline{b}\}$. Since G is nearly c^* win these cops can force the robber to move on H or to move so that $a_i=u_i$. Thus, there will eventually be a cop in set P, say S_1, with coordinates $(\underline{u},\underline{v})$ where $\underline{v}=(b_1,\dots,b_{j-1},v_j,b_{j+1},\dots,b_m)$ and (b_j,v_j) is an edge in H_j. On the move this occurs, the remaining cops in P move to $\underline{a}\in V(G)$.

From now on, if $a_i=u_i$ ever occurs, then the robber is captured by S_1. Hence, the robber cannot move indefinitely on G. Whenever the robber moves on H, S_1 maintains her position on $\underline{v}$ relative to $\underline{b}$. But now, since H is nearly c^* win, the remaining cops in P can force $b_j=v_j$ from which position the robber is captured by S_1.

(ii) Let $H=T$, a tree. We use the procedure in (i). The set P then consists of a single cop S_1. The cop S_1 plays her winning strategy on T whenever the robber moves on T or when $u_i=a_i$. Since G is nearly c^* win, the $c(G)-1$ remaining cops can force one of these two situations to occur. The robber is captured when he is in a copy of G adjacent to S_1 and must move either to u_i or to the copy of G containing S_1.

Thus, $c(G\square T)\leqslant c(G)$ and since $c(G)\leqslant c(G\square T)$ we have $c(G\square T)=c(G)$. □

For several of our results on Cartesian products we need the following lemma.

Lemma 2.1. *Suppose graphs $G_1,G_2,\dots,G_n$ have the property that for any $u,v\in V(G_i)$, $u\neq v$, $|N(u)\cap N(v)|\leqslant 2$. Then, for any vertices $\underline{u},\underline{v}\in V(G)$ where $G=\square_{i=1}^{n}G_i$ and $\underline{u}\neq\underline{v}$, we have $|N(\underline{u})\cap N(\underline{v})|\leqslant 2$.*

Proof. In order that $N(\underline{u})\cap N(\underline{v})\neq\emptyset$, $\underline{u}$ must be equal to $\underline{v}$ in all but one or two coordinates since two vertices are adjacent in G if and only if they differ in precisely one coordinate.

Let $\underline{u}=(u_1,u_2,\dots,u_n)$. Suppose $\underline{v}$ differs from $\underline{u}$ in two coordinates, say i and j, i.e., $\underline{v}=(u_1\dots,u_{i-1},v_i,u_{i+1}\dots,u_{j-1},v_j,u_{j+1},\dots,u_n)$. Then the only possible vertices in $N(\underline{u})\cap N(\underline{v})$ are $(u_1\dots,u_{i-1},v_i,u_{i+1}\dots,u_n)$ and $(u_1\dots,u_{j-1},v_j,u_{j+1},\dots,u_n)$. If $\underline{u}$ differs from $\underline{v}$ in one coordinate, say i, then $\underline{v}=(u_1\dots,u_{i-1},v_i,u_{i+1}\dots,u_n)$ and so $|N(\underline{u})\cap N(\underline{v})|\leqslant 2$ since by assumption $|N(u_i)\cap N(v_i)|\leqslant 2$, which completes the proof. □

Lemma 2.2. *Suppose graphs $G_i, 1 \leqslant i \leqslant n$, have the property that for any $u, v \in V(G_i)$, $|N(u) \cap N(v)| \leqslant 2$. Then,*

$$c\left(\mathop{\Box}_{i=1}^{n} G_i\right) \geqslant \left\lceil \frac{n+1}{2} \right\rceil \quad \text{and} \quad c'\left(\mathop{\Box}_{i=1}^{n} G_i\right) \geqslant \left\lceil \frac{n}{2} \right\rceil.$$

Proof. Let the position of the robber be $\underline{r} = (r_1, r_2, \ldots, r_n)$. We note $\delta(\Box_{i=1}^{n} G_i) \geqslant n$. By Lemma 2.1 each cop in the product graph can dominate at most two neighbors of $\underline{r}$. Thus, if n is even, then at least $\lceil n/2 \rceil + 1$ cops are required in the passive game and at least $\lceil n/2 \rceil$ cops in the active game. If n is odd, then $\lceil n/2 \rceil$ cops are required in both the passive and active games. Hence, $c(\Box_{i=1}^{n} G_i) \geqslant \lceil (n+1)/2 \rceil$ and $c'(\Box_{i=1}^{n} G_i) \geqslant \lceil n/2 \rceil$. □

2.1. Active game

Tošić [13] considered the active game on the n-cube, Q_n, and found that $c'(Q_n) = \lceil n/2 \rceil$, $n \not\equiv 2 \pmod 4$, and $\lceil n/2 \rceil \leqslant c'(Q_n) \leqslant \lceil n/2 \rceil + 1$, $n \equiv 2 \pmod 4$. However, this result can be improved.

Theorem 2.4. *Let Q_n be the n-cube. Then*

$$c'(Q_n) = \left\lceil \frac{n}{2} \right\rceil \quad \text{for } n \geqslant 3.$$

Proof. It suffices to prove the theorem for $n \equiv 2 \pmod 4$ since Tošić has shown $c'(Q_n) = \lceil n/2 \rceil$ when $n \not\equiv 2 \pmod 4$. Let $n \equiv 2 \pmod 4$ and put $Q_n = Q_{n-2} \Box Q_2$. Label the vertices of Q_2 cyclically 0, 1, 2, and 3. From Tošić's result we have $c'(Q_{n-2}) = \lceil (n-2)/2 \rceil = \lceil n/2 \rceil - 1$. Suppose $\lceil n/2 \rceil$ cops are located on $Q_{n-2}.\{0\}$. Since these cops could capture the robber were he confined to $Q_{n-2}.\{0\}$, one of their number, S, can therefore capture the projection of the robber onto Q_{n-2}. Once this is accomplished S shadows the robber on Q_{n-2}. The remaining $\lceil n/2 \rceil - 1$ cops migrate to the same copy of Q_{n-2} as the robber and force the robber to move to a different copy of Q_{n-2}, say to $Q_{n-2}.\{y\}$. If $y = 0$, then the robber's position coincides with S and so he is captured. If $y = 1$ or 3 the robber will be captured on the next move by S. If $y = 2$, then $\lceil n/2 \rceil - 1$ cops move to $Q_{n-2}.\{2\}$ and force the robber to move to $Q_{n-2}.\{1\}$ or $Q_{n-2}.\{3\}$ where he will be captured by S. Hence, $c'(Q_n) \leqslant \lceil n/2 \rceil$. From Tošić's result above, $\lceil n/2 \rceil \leqslant c'(Q_n)$ and so $c'(Q_n) = \lceil n/2 \rceil$ which concludes the proof. □

We now consider the active game on a Cartesian product of n trees, which is a generalization of Theorem 2.4. We begin with the following lemma.

Lemma 2.3. *Let G be the Cartesian product of two finite trees. If the robber cannot stay indefinitely on the same vertex of G, then one cop can win on G.*

Proof. Let the robber's coordinates be $a = (a_1, a_2)$ and let the coordinates of the cop, S, be $b = (b_1, b_2)$. At her turn S moves to decrease the distance $d(a, b)$ by reducing

$\max(d(a_1,b_1),d(a_2,b_2))$ unless $d(a_1,b_1)=d(a_2,b_2)$. In the latter situation, S passes and waits for the robber to move. Because each tree is finite we must eventually have $d(a_1,b_1)=d(a_2,b_2)$ and once this occurs it re-occurs after each cop move. Since the robber must move from time to time and since each tree is finite, it is clear that after a finite number of moves the robber is captured by S. □

Theorem 2.5. *Let $T_1,T_2,\ldots,T_n$ be trees. Then*

$$c'(\mathop{\square}_{i=1}^{n} T_i)=\left\lceil\frac{n}{2}\right\rceil,\quad n\geqslant 3.$$

Proof. We note that $c'(T_i)=1$, and that $c'(\square_{i=1}^{2} T_i)\geqslant 2$ since there is at least one cycle of length 4 in $\square_{i=1}^{2} T_i$.

Let two cops, S_1 and S_2, play the active game on a product of two trees. Let S_1 use the strategy in Lemma 2.3. When the winning strategy dictates that S_1 pass S_2 moves to comply with the rules of the active game. Hence, $c'(\square_{i=1}^{2} T_i)=2$.

Now consider $\square_{i=1}^{4} T_i=G\square H$ where G and H are each the Cartesian product of two trees. Again let two cops, S_1 and S_2, play the active game on this graph. From the previous paragraph, we know that one of the cops, say S_1, can capture the projection of the robber onto G. (If the robber moves on H only, then his projection onto H can be captured by S_2.)

Let S_1 shadow the robber on G. If the robber moves on H, then S_1 plays her winning strategy on H as in Lemma 2.3. The cop, S_2, moves to the same copy of G as the robber and employs the strategy in Lemma 2.3. Therefore, the robber must from time to time move on H or he will be captured by S_2. But now, by Lemma 2.3, S_1 captures the robber. Hence, $c'(\square_{i=1}^{4} T_i)=2$. The results $c'(\square_{i=1}^{2} T_i)=2$ and $c'(\square_{i=1}^{4} T_i)=2$ implies $c'(\square_{i=1}^{3} T_i)=2=\lceil n/2\rceil$. Hence, the theorem is true for $n=3,4$.

Assume $c'(\square_{i=1}^{n} T_i)=\lceil n/2\rceil$ for all n where $4\leqslant n<k$. Let $n=k$ and consider $\square_{i=1}^{k} T_i=(\square_{i=1}^{k-2} T_i)\square(\square_{i=1}^{k} T_i)=G\square H$. Let $\lceil k/2\rceil$ be the number of cops playing on $\square_{i=1}^{k} T_i$. By hypothesis, $\lceil k/2\rceil-1$ cops can capture the projection of the robber onto G. Let S shadow the robber on G. Whenever the robber changes his $(k-1)$th or kth coordinate, S plays her winning strategy on H as in Lemma 2.3. The remaining $\lceil k/2\rceil-1$ cops move to the same copy of G as the robber and force him to move on H. Thus, the robber will eventually be captured by S and hence $c'(\square_{i=1}^{k} T_i)\leqslant\lceil(k-2)/2\rceil+1=\lceil k/2\rceil$. Therefore, $c'(\square_{i=1}^{n} T_i)\leqslant\lceil n/2\rceil$ for all $n\geqslant 3$.

Since $\delta(\square_{i=1}^{n} T_i)=n$, we have by Lemmas 2.1 and 2.2 that $\lceil n/2\rceil\leqslant c'(\square_{i=1}^{n} T_i)$. Therefore, $c'(\square_{i=1}^{n} T_i)=\lceil n/2\rceil$. □

2.2. *Passive game*

Maamoun and Meyniel [7] have shown that the cop-number on a Cartesian product of n trees, $T_1,T_2,\ldots,T_n$, is $\lceil(n+1)/2\rceil$.

We will first consider the Cartesian product of cycles and then a Cartesian product of trees and cycles combined.

Theorem 2.6. *Let $G = \Box_{i=1}^{n} C_i$ where each C_i is a cycle of length at least 4. Then,*

$$c(G) = n + 1.$$

Proof. We proceed by induction on n. If $n = 1$, then it is clear that $c(G) = 2 = n + 1$ and that G is nearly c^* win. Suppose the theorem is true for $n = k - 1$ and consider $n = k$. Suppose also that $H = \Box_{i=1}^{k-1} C_i$ is nearly c^* win. Now by Theorem 2.3, $c(G) = c(H \Box C_k) \leqslant c(H) + c(C_k) - 1 = k + 1$. It remains to be shown that H is nearly c^* win implies G is nearly c^* win.

Let k cops play on G. Since H is nearly c^* win, it suffices to show that if W is the walk taken by the robber during the game, then these cops can ensure that the projection of W onto C_k contains the vertex sequence $u, v, \ldots, v, u$ for some $u, v \in V(C_k)$. Orient C_k so that we may speak of clockwise and counterclockwise directions on C_k.

Let $\underline{r} \in V(G)$ be the robber's current position and let $\underline{u} \in V(G)$ be the predecessor of $\underline{r}$ in W. Since $c(H) = k$, one cop, S, can capture the projection of $\underline{u}$ onto H. Let $H = F \Box C_{k-1}$ and let S shadow the robber on F. We may suppose the projections of $\underline{u}$ and $\underline{r}$ onto H differ in coordinate $k - 1$; otherwise, $k - 1$ cops in the same copy of F as the robber can force the robber to move on C_k, first in one direction, then in the other, ensuring that the projection of W onto C_k contains the vertex sequence $u, v, \ldots, v, u$ for some $u, v \in V(C_k)$.

Let S reduce her distance to the robber on C_{k-1} whenever the robber moves on C_{k-1} and let $\underline{s}$ be the position of S on H. Let the remaining $k - 1$ cops migrate to the same copy of H as the robber, say to $H.\{y\}$.

Since H is nearly c^* win, the cops on $H.\{y\}$ can play so that either the robber moves on C_k or the projection of $\underline{r}$ onto H must, from time to time, be $\underline{s}$. Whenever the latter occurs, S moves toward the robber on C_k, say in the clockwise direction, which eventually forces the robber to move on C_k in the same direction. Once this occurs, S moves in the counterclockwise direction on C_k whenever the projection of $\underline{r}$ onto H is $\underline{s}$. This eventually forces the robber to also move in the counterclockwise direction on C_k. Thus, the projection of W onto C_k contains the vertex sequence $u, v, \ldots, v, u$ for some $u, v \in V(C_k)$ and hence G is nearly c^* win. Thus, $c(G) = c(\Box_{i=1}^{k} C_i) \leqslant k + 1$.

Because $\delta(\Box_{i=1}^{k} C_i) = 2k$, by Lemmas 2.1 and 2.2, we have $k + 1 \leqslant c(\Box_{i=1}^{k} C_i)$. Therefore, $c(\Box_{i=1}^{k} C_i) = k + 1$ and so $c(\Box_{i=1}^{n} C_i) = n + 1$ for all $n \geqslant 1$. $\Box$

The next two theorems concern a combined Cartesian product of trees and cycles.

Theorem 2.7. *Let $C_1, C_2, \ldots, C_n$ be cycles each of length at least 4. Let $H = \Box_{i=1}^{n} C_i$ and T be a tree. Then,*

$$c(H \Box T) = n + 1.$$

Proof. We have $c(H) = n + 1$ by Theorem 2.6. Also by Theorem 2.6, H is nearly c^* win and so $c(H \Box T) = n + 1$ by Theorem 2.3. $\Box$

Theorem 2.8. *Let $C_1, C_2, \ldots, C_k$ be cycles each of length at least 4. Let $G = \square_{i=1}^{k} C_i$. Let $T_1, T_2, \ldots, T_j$ be trees and let $H = \square_{i=1}^{j} T_i$. Then*

$$c(G \square H) = k + \left\lceil \frac{j+1}{2} \right\rceil .$$

Proof. From Theorem 2.7, if $j = 1$, then $c(G \square H) = k + 1$. Suppose that the theorem is true for arbitrary fixed k and for all j where $1 \leqslant j < m$. Let $j = m$ and consider $G \square F \square T_{m-1} \square T_m$ where $F = \square_{i=1}^{m-2} T_i$. Let the position of the robber be $(\underline{p}, x, y)$ where $p \in V(G \square F)$, $x \in V(T_{m-1})$ and $y \in V(T_m)$. By hypothesis $k + \lceil (m-1)/2 \rceil$ cops can capture $\underline{p}$, say with cop S. If the robber moves on $G \square F$, then S shadows the robber on $G \square F$. If the robber moves on $T_{m-1} \square T_m$ then S plays her winning strategy on $T_{m-1} \square T_m$ as in Lemma 2.1.

The remaining $k + \lceil (m-1)/2 \rceil$ cops move to $G \square F . \{x, y\}$ and again proceed to capture $\underline{p}$. To avoid capture the robber is forced to move on $T_{m-1} \square T_m$ and so will eventually be captured by S. Therefore, $c(G \square H) \leqslant k + \lceil (j+1)/2 \rceil$, $j, k \geqslant 1$.

To see that $k + \lceil (j+1)/2 \rceil$ cops are necessary we note that $\delta(G \square H) = 2k + j$ and so by Lemmas 2.3 and 2.2, the minimum number of cops needed are $(2k + j)/2 + 1$ if j is even and $k + (j+1)/2$ if j is odd which in both cases is $k + \lceil (j+1)/2 \rceil$. Thus, $c(G \square H) = k + \lceil (j+1)/2 \rceil$, $j, k \geqslant 1$. □

Finally, we consider the Cartesian products of complete graphs.

Theorem 2.9. *Let H be the Cartesian product of n complete graphs K_i each of size at least 3. Then,*

$$c(H) = c'(H) = n.$$

Proof. Since $c(K_i) = 1$ we have, by Theorem 2.1, that $c(H) \leqslant n$ and $c'(H) \leqslant n$. Let $u, v, \in V(H)$. Since H is $\sum_{i=1}^{n} (|K_i| - 1)$ regular and $K_i \geqslant 3$, $|N(u) \cap N(v)|$ is at most one of the terms in the sum. Thus, $n \leqslant c(H)$ and $n \leqslant c'(H)$ and so $c(H) = c'(H) = n$. □

We note that Theorem 2.9 holds if the complete graphs are infinite in cardinality.

3. Categorical product

A difficulty in determining an upper bound for the categorical product is that the product may be disconnected. However, if all the constituent graphs of the product are connected and at least one of them is not bipartite, then the categorical product is also connected.

Theorem 3.1. *Let G and H be the connected non-bipartite graphs and let $c(H) \geqslant c(G)$.*

(i) *Suppose* $c(H) \geqslant 2$. *Then,*

$$c(G \times H) \leqslant 2c(G) + c(H) - 1.$$

(ii) *Suppose* $c(G) = c(H) = 1$. *Then,*

$$c(G \times H) \leqslant 3.$$

Proof. (i) Let $n = 2c(G) + c(H) - 1$ cops play on two adjacent copies of H (i.e., on $\{x, y\}.H$ where (x, y) is an edge in G). Let $\underline{r} = (a, b)$, $a \in G$, $b \in H$ be the current position of the robber. Let a set $P = \{S_1, S_2, \ldots, S_{2c(G)}\}$ of $2c(G)$ cops capture b. After this capture each $S_i \in P$ shadows the robber on H. Let one of the remaining $c(H) - 1$ cops, say S_k, force the robber to move. Because G has an odd cycle and because of the existence of S_k, the cops in P can move so that $c(G)$ of them occupy $\{x\}.H$ and $c(G)$ of them occupy $\{y\}.H$.

Match each cop S_i on $\{x\}.H$ with a 'following' cop S_i' on $\{y\}.H$. This means that whenever S_i moves from (say) (x, u) to (z, v), S_i' moves to (x, v). Let $U \subset P$ be the set of cops occupying $\{x\}.H$ and $V \subset P$ be the set occupying $\{y\}.H$. The cops in U and V shadow the robber on H which prevents the robber from entering the same copy of H as any of the cops in P.

The cops in U play their winning strategy on G whenever the robber moves, which must occur from time to time due to the existence of S_k. Thus, the robber must eventually move either to a copy of H adjacent to one containing a cop $S_i \in U$ (whence the robber is captured by S_i) or to a copy of H containing a cop $S_i \in U$ (whence the robber is captured by $S_i' \in V$).

(ii) Since both G and H are cop-win, two cops, S_1, S_2 can each capture the projection of the robber onto G with one cop capturing on $G.\{x\}$ and the other on $G.\{y\}$ where x is adjacent to y in H. They then play their winning strategy in H whenever the robber moves maintaining their position in adjacent copies of G. This prevents the robber from entering the same copy of G as either of the two cops. A third cop is needed to force the robber to move. Thus, the robber is eventually captured by S_1 or S_2. □

We have the following corollary to Theorem 3.1.

Corollary 3.1. *Let* G_i, $1 \leqslant i \leqslant n$ *be non-bipartite graphs.*

(i) *If* $c(G_i) \geqslant 2$ *for some* i, $1 \leqslant i \leqslant n$, *then,*

$$c(X_{i=1}^{n} G_i) \leqslant 2\left[\sum_{i=1}^{n} c(G_i)\right] - \max c(G_i) - n + 1.$$

(ii) *If* $c(G_i) = 1$ *for all* i, $1 \leqslant i \leqslant n$, *then,*

$$c(X_{i=1}^{n} G_i) \leqslant n + 1.$$

Proof. Apply Theorem 3.1 and proceed by induction on n. □

We can be more specific about some special graphs such as paths and complete graphs. For example, suppose G and H are each paths of length at least 3. Note that $G \times H$ consists of two disconnected components each of which is essentially the Cartesian product of two paths. Hence, in the passive game two cops on each component are sufficient and so $c(G \times H) \leqslant 4$. If each path is of length at least 3, then each component must have at least 1 cycle of length 4. Hence, if this is the case, $c(G \times H) = 4$.

On the other hand, in the active game only one cop is needed on each of the components by Lemma 2.3 and so $c'(G \times H) = 2$

For complete graphs we have the following result.

Theorem 3.2. *Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$ be complete graphs.*
(i) *Suppose $|V_1| \geqslant 4$ and $|V_2| \geqslant 4$. Then,*

$$c(G \times H) = 3 \quad \textit{and} \quad c'(G \times H) = 2.$$

(ii) *Suppose $|V_1|$ or $|V_2| = 3$. Then*

$$c(G \times H) = c'(G \times H) = 2.$$

Proof. (i) We show $c(G \times H) \leqslant 3$ and $c'(G \times H) \leqslant 2$. Suppose that all 3 cops initially choose distinct positions in the same copy of G (i.e., $G.\{x\}$, $x \in V_2$). The robber must also choose a vertex in this same copy of G, otherwise he is caught immediately. Let the coordinates of the robber be (a, x). One cop then moves to (a, y) and a second cop moves to (a, z), $z \neq y$. The third cop moves to (b, y) directly attacking the robber. The robber has no safe move and so is captured at the cops' next turn. Hence, $c(G \times H) \leqslant 3$.

In the active game, after the first two cops move as described, the robber must move to a vertex where he can be captured. Hence $c'(G \times H) \leqslant 2$.

We show $c(G \times H) \geqslant 3$ and $c'(G \times H) \geqslant 2$. We note one cop is not sufficient in the passive or active game since an avoidance strategy for the robber is simply to choose the same copy of G (or H) as the cop. Thus, $c(G \times H) \geqslant 2$ and $c'(G \times H) \geqslant 2$ which implies $c'(G \times H) = 2$.

Let two cops, S_1, S_2, play the passive game on $G \times H$ and suppose their current coordinates are (x, y) and (u, v). If they have a coordinate in common, say $x = u$, then the robber chooses vertex (x, w) where $w \neq y, v$, which is not adjacent to S_1 or S_2. Similarly, if $y = v$ the robber may choose a safe vertex. If $x \neq u$ and $y \neq v$, then the robber may choose vertex (x, v) which is not adjacent to S_1 or S_2. Thus, the robber has a safe initial position. We now consider the situation after the cops' turn.

(a) Neither cop is adjacent to the robber. The robber then passes.
(b) At least one cop, say S_1, is adjacent to the robber. This means both coordinates of S_1 differ from the robber. If S_1 and S_2 have a coordinate in common, say $x = u$, then the robber chooses the safe vertex (x, p) where $p \neq w, y, v$. This is always

possible since $|V_1|\geqslant 4$ and $|V_2|\geqslant 4$. Similarly, if $y=v$, the robber can choose a safe vertex.

If S_1 and S_2 have no coordinate in common, then the robber chooses vertex (x,v) or vertex (u,y) one of which is a safe vertex. Again this is possible because $|V_1|\geqslant 4$ and $|V_2|\geqslant 4$. Thus, when playing against two cops the robber has an avoidance strategy. Hence, $c(G\times H)\geqslant 3$ whence we conclude $c(G\times H)=3$.

(ii) Suppose $|V_1|=3$. Clearly, one cop is not sufficient to win in either the passive or the active game since the robber can always choose to go to the same copy of G as the cop where he is not threatened. Hence, $c'(G\times H)\geqslant 2$ and $c(G\times H)\geqslant 2$. By (i) we have $c'(G\times H)=2$.

Let two cops occupy different vertices in a copy of H (i.e., (x,a) and (x,b)). Then the robber must also choose this copy of H (i.e., (x,c)) or he will be caught on the next move. The cops now move to (y,d) and (z,d), $y\neq z$, $d\neq c$, from which position the robber is attacked. The robber has no safe move since $|V_1|=3$ and so is captured on the next move. By a similar argument, if $|V_2|=3$, then two cops can capture the robber. Thus, $c(G\times H)\leqslant 2$ and we conclude $c(G\times H)=2$. □

We note that in Theorem 3.2(i), G or H may be infinite complete graphs. We now extend Theorem 3.2 to the n-fold categorical product.

Theorem 3.3. *Let H be the categorical product of n complete graphs each of size at least* 3. *Then,*

$$c(H)\leqslant \left\lceil\frac{n+1}{2}\right\rceil+1 \quad and \quad c'(H)\leqslant \left\lceil\frac{n+1}{2}\right\rceil \quad for\ n\geqslant 2.$$

Proof. Let the robber have position $\underline{r}=(r_1,r_2,\ldots,r_n)$ in H and suppose cops S_1 and S_2 have positions $\underline{u}=(r_1,r_2,\ldots,r_{i-1},u_i,r_{i+1},\ldots,r_n)$ and $\underline{v}=(r_1,r_2,\ldots,r_{i-1},v_i,r_{i+1},\ldots,r_n)$, $u_i\neq v_i$. If the robber is now forced to move, he is captured on the next move by S_1 or S_2. Thus, it suffices to show this situation can be achieved.

Assume, without loss of generality, that all of the cops initially occupy the same vertex $(a_1,a_2,\ldots,a_n)$. Let the initial position of the robber be the same as the cops in k coordinates and different in j coordinates. After re-ordering let the robber's position be $(a_1,a_2,\ldots,a_k,b_{k+1},\ldots,b_{k+j})$. If $k<n/2$, then the cops at their next move will change the j coordinates of theirs which are different from the robber's to be the same as his. They now have $k>n/2$. Therefore, we may assume from the outset that $k\geqslant n/2$ which implies $j\leqslant\lceil(n-1)/2\rceil$.

Now let one cop, S_1, move to attack the robber. It is to the robber's advantage to move to a vertex with j coordinates the same as the vertex occupied by the remaining cops $S_2,S_3,\ldots$ (i.e., to $(b_1,b_2,\ldots,b_k,a_{k+1},\ldots,a_{k+j})$). Otherwise, at their next move these cops could move to a vertex which has at least $k+1$ coordinates the same as the robber's vertex.

One cop, S_2, now moves to a vertex with k coordinates identical to the robber's vertex and j coordinates different (i.e., to $(b_1,b_2,\ldots,b_{k+j})$). The remaining cops, $S_3,S_4,\ldots,$

choose a vertex with k coordinates identical to the robber's vertex and j coordinates different from both the robber's vertex and the vertex occupied by S_2 (i.e. to $(b_1, b_2, \ldots, b_k, c_{k+1}, \ldots, c_{k+j})$). This is always possible since $|K_i| \geqslant 3$. The robber cannot on his next move choose a vertex with coordinates all different from S_2 or he will be caught by S_2 on the next move. Therefore, the robber must choose a vertex with ith coordinate b_i for some i where $k < i \leqslant k + j$. Since $b_i \neq c_i$, the cops $S_3, S_4, \ldots$ can at their turn move to a vertex with at least $k+1$ coordinates identical to the vertex occupied by the robber.

By repeating this process we find that if the robber is forced to move, then $j+1$ cops are sufficient to produce a situation where two cops have positions equal to the robber's in all coordinates but one. Thus, in the active game $j+1$ cops are sufficient to produce the winning position, whereas in the passive game an additional cop for a total of $j+2$ are sufficient. Now since $j \leqslant \lceil (n-1)/2 \rceil$, we have the result. □

4. Strong product

The final product on graphs which we will consider is the strong product. Earlier Nowakowski and Winkler [10] showed that a finite strong product of cop-win graphs is also cop-win. Here we generalize somewhat and consider the finite product of graphs with arbitrary cop-numbers. Combinations of some disjoint subsets of the strong product were investigated by Neufeld and Nowakowski [9]. We begin with the following theorem.

Theorem 4.1. *Let G and H be graphs with $c(G) \geqslant 2$ or $c(H) \geqslant 2$. Then,*

$$c(G \boxtimes H) \leqslant c(G) + c(H) - 1.$$

Proof. Suppose, without loss of generality, that $c(H) \geqslant 2$. Let a set $P = \{S_1, S_2, \ldots, S_{c(H)}\}$ cops capture the projection of the robber onto G. This requires a total of $c(G) + c(H) - 1$ cops. The cops in P then shadow the robber on G and at the same time play their winning strategy on H. □

Theorem 4.1 implies that if $c(G_i) \geqslant 2$ for some i, $1 \leqslant i \leqslant n$, then $c(\boxtimes_{i=1}^{n} G_i) \leqslant \sum_{i=1}^{n} c(G_i) - n + 1$.

Finally, we consider the cop-number of the n-fold strong product of cycles.

Theorem 4.2. *Let C_i, $1 \leqslant i \leqslant n$, be cycles which each have a length at least 5. Then,*

$$c\left(\boxtimes_{i=1}^{n} C_i \right) \leqslant n + 1.$$

Proof. We proceed by induction on n. Consider first a product of two cycles $C_1 \boxtimes C_2$. Let the position of the robber be (x, y). Let three cops play on $C_1 \boxtimes C_2$ so that two of their number, say S_1 and S_2, capture x.

On subsequent moves S_1 and S_2 each shadow the robber on C_1 and at the same time move in opposite directions towards the robber on C_2. The robber is captured when both cops are in copies of C_1 adjacent to the robber. Thus, $c(C_1 \boxtimes C_2) \leqslant 3 = n+1$.

Assume $c(\boxtimes_{i=1}^{k-1} C_i) \leqslant k$ and let $n=k$. Let $G = \boxtimes_{i=1}^{k-1} C_i$ and consider $G \boxtimes C_k$. By the induction hypothesis, $k+1$ cops can play so that two of their number capture the projection of the robber onto G. These two cops then adopt the strategy used above in the product of two cycles. Thus, $c(\boxtimes_{i=1}^{k} C_i) \leqslant k+1$ and hence $c(\boxtimes_{i=1}^{n} C_i) \leqslant n+1$ for all $n \geqslant 1$. □

References

[1] M. Aigner, M. Fromme, A game of cops and robbers, Discrete Appl. Math. 8 (1984) 1–12.
[2] T. Andreae, On a pursuit game played on graphs for which a minor is excluded, J. Combin. Theory Ser. B 41 (1986) 37–47.
[3] R. Anstee, M. Farber, On bridged graphs and cop-win graphs, J. Combin. Theory 44 (1988) 22–28.
[4] G. Chartrand, L. Lesniak, Graphs and Digraphs, Wadsworth & Brooks/Cole, CA, 1986.
[5] P. Frankl, Cops and robbers in graphs with large girth and Cayley graphs, Discrete Appl. Math. 17 (1987) 301–305.
[6] P. Frankl, On a pursuit game on Cayley graphs, Combinatorica 7 (1987) 67–70.
[7] M. Maamoun, H. Meyniel, On a game of policemen and robber, Discrete Appl. Math. 17 (1987) 307–309.
[8] S. Neufeld, A game of cops and robbers, M.Sc. Thesis, Dalhousie University, Halifax, 1991.
[9] S. Neufeld, R. Nowakowski, A vertex-to-vertex pursuit game played on disjoint sets of edges, in: N. Sauer (Ed.), Proc. NATO Advanced Study Institute on Finite and Infinite Combinatorics in Sets and Logic, Kluwer Academic Publications, Dordrecht, 1993, pp. 299–312.
[10] R. Nowakowski, P. Winkler, Vertex to vertex pursuit in a graph, Discrete Math. 43 (1983) 230–239.
[11] A. Quilliot, Thèse d'Etat, Université de Paris VI, 1983.
[12] A. Quilliot, A short note about pursuit games played on a graph with a given genus, J. Combin. Theory Ser. B 38 (1985) 89–92.
[13] R. Tošić, The search number of the Cartesian product of graphs, Univ. Novom Sabu Zb. Rad. Prirod. -Mat Fak. Ser. Mat. 17 (1987) 239–243.