manuscripta math. 81, 183 - 202 (1993)

manuscripta mathematica

ON IRRATIONALITY MEASURES OF THE VALUES OF GAUSS HYPERGEOMETRIC FUNCTION

ARI HEIMONEN, TAPANI MATALA-AHO AND KEIJO VÄÄNÄNEN

The paper gives irrationality measures for the values of some Gauss hypergeometric functions both in the archimedean and p-adic case. Further, an improvement of general results is obtained in the case of logarithmic function.

Introduction

We shall consider the irrationality measures of the values of Gauss hypergeometric function

$$F(z) = {}_2F_1\left(\begin{matrix}1\ ,\ b\\ c\end{matrix}\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(b)_n}{(c)_n} z^n, \tag{1}$$

where $b, c \neq 0, -1, -2, \ldots$ are rational parameters, and $(b)_0 = 1$, $(b)_n = b(b+1)\ldots(b+n-1)$, $n = 1, 2, \ldots$. The irrationality and linear independence measures of the values of F are considered in many works both in the general case and in some interesting special cases, see [1] [2], [4], [6], [7], [8], [9], [10], [11], [13], [14], [15], [16], [17], [18], [19], [20] and [23]. Also the transcendence of the values of F at algebraic points is considered in the important papers [3], [5] and [24].

In the present work we first give using Padé type approximations an irrationality measure for $F(r/s)$ with certain values $r/s \in \mathbf{Q}$, both in the archimedean and p-adic case. In many special values of b and c these general results can be sharpened by the careful consideration of the arithmetic properties of the coefficients of the approximation polynomials. This idea was first realised for the binomial function by Chudnovsky [8], and then in some other cases in [10], [13], [14] and [20]. Here we shall deduce a general criterion to find a common factor for the coefficients of our approximation polynomials and then apply this criterion to the logarithmic function to obtain a generalisation of the nice work of Rukhadze [20].

Results and notations

We shall denote by $\mathbb{Q}_v$ the v-adic completion of $\mathbb{Q}$, where $v \in \{\infty, \text{primes } p\}$, in particular $\mathbb{Q}_\infty = \mathbb{R}$. For an irrational number $\theta \in \mathbb{Q}_v$, we shall call an irrationality measure $m_v(\theta)$ of θ the infimum of m satisfying the following condition: for any $\varepsilon > 0$ there exists an $H_0 = H_0(\varepsilon)$ such that

$$\left|\theta - \frac{P}{Q}\right|_v > H^{-m-\varepsilon}$$

for all rationals P/Q satisfying $H = \max\{|P|, |Q|\} > H_0$. In the following we denote $m_\infty(\theta) = m(\theta)$. All our measures are effective in the sense that H_0 can be effectively determined.

Throughout this paper we shall assume that $c > b > 0$, $b = a/f$, $c = g/h$, where a, f, g, h are natural numbers such that $(a, f) = (g, h) = 1$. Let us denote $B = b - 1 = E/F$, $C = c - b - 1 = G/H$ with $E, G \in \mathbb{Z}$, $F, H \in \mathbb{N}$, $(E, F) = (G, H) = 1$. Further, let $L = \text{l.c.m.}(F, H)$, and use H^* to denote the denominator of h/H (therefore $H^* \mid H$). We shall also need the notations

$$\mu_F = \prod_{p \mid F} p^{\frac{1}{p-1}}, \quad \lambda(h) = \frac{h}{\phi(h)} \sum_{\substack{i=1 \\ (i,h)=1}}^{h} \frac{1}{i}$$

to state the following result.

Theorem 1. *If $r/s \in (-1, 1)$ is a non-zero rational number satisfying*

$$(r, s) = 1, \quad LH^* \mu_L \mu_{H^*} e^{\lambda(h)} \left(\sqrt{s} - \sqrt{s-r}\right)^2 < 1,$$

then

$$m\left(F\left(\frac{r}{s}\right)\right) \le 1 - \frac{2\ln\left(\sqrt{s} + \sqrt{s-r}\right) + \lambda(h) + \ln(LH^*\mu_L\mu_{H^*})}{2\ln\left|\sqrt{s} - \sqrt{s-r}\right| + \lambda(h) + \ln(LH^*\mu_L\mu_{H^*})}.$$

As a p-adic analogue of this result we state the following sharpening of [17].

Theorem 1p. *Suppose that p is a prime such that $p \nmid fh$. If $r/s > 1$ is a rational number satisfying*

$$|r/s|_p < 1, \quad (r, s) = 1, \quad LH^* \mu_L \mu_{H^*} e^{\lambda(h)} r |r|_p^2 < 1,$$

then

$$m_p\left(F\left(\frac{r}{s}\right)\right) \le \frac{2\ln|r|_p}{2\ln|r|_p + \ln r + \lambda(h) + \ln(LH^*\mu_L\mu_{H^*})}$$

(in writing $m_p(f(\dots))$ we always think of f as a corresponding p-adic series).

If $b = 1$, $c = 2$, then Theorem 1p implies for the p-adic logarithm the following

Corollary 1p. *If $r/s > 1$ is a rational number satisfying*

$$|r/s|_p < 1, \quad (r,s) = 1, \quad er|r|_p^2 < 1,$$

then

$$m_p\left(\log\left(1 - \frac{r}{s}\right)\right) \leq \frac{2\ln|r|_p}{2\ln|r|_p + \ln r + 1}.$$

In particular, for all $p^l > e$ we have

$$m_p\left(\log\left(1 - p^l\right)\right) \leq \frac{2l\ln p}{l\ln p - 1}.$$

For the real logarithm we obtain, by Theorem 1, the well-known result

$$m\left(\log\left(1 - \frac{r}{s}\right)\right) \leq 1 - \frac{2\ln\left(\sqrt{s} + \sqrt{s-r}\right) + 1}{2\ln\left|\sqrt{s} - \sqrt{s-r}\right| + 1},$$

if $r/s \in [-1, 1)$ is a rational number satisfying

$$e\left(\sqrt{s} - \sqrt{s-r}\right)^2 < 1.$$

To get a sharpening of this result we define, for a rational $\alpha = u/v \in (0,1]$, $u, v \in \mathbb{N}$, $(u,v) = 1$, the subsets I_1 and I_2 of $\{1, \ldots, v-1\}$ such that

$$i \in I_1 \quad \text{iff} \quad [\alpha i] + 1 = \left[\alpha i + \frac{\alpha}{2}\right], \quad i \in I_2 \quad \text{iff} \quad [\alpha i] = \left[\alpha i + \frac{\alpha}{2}\right].$$

Let then

$$\tau_1(\alpha) = \frac{1}{v}\left(\sum_{i\in I_1}\left(\Psi\left(\frac{1+[\alpha i]}{u}\right) - \Psi\left(\frac{i}{v}\right)\right) + \sum_{i\in I_2}\left(\Psi\left(\frac{2i-[\alpha i]}{2v-u}\right) - \Psi\left(\frac{i}{v}\right)\right)\right),$$

where Ψ is the digamma function (see e.g. [12], pp. 15–20). Further with a given rational $\beta \geq \alpha$ we define

$$A(\alpha,\beta,z) = \min_{0<\rho<|z|+\frac{1}{2}(1-\operatorname{sgn} z)}\left(\frac{(\rho+|z|)(\rho+|z|-\operatorname{sgn} z)^\beta}{\rho^\alpha}\right)$$

for all $z \geq 1$ or $z < 0$, and

$$R(\alpha,\beta,z) = \max_{0\leq t\leq 1}\frac{t(1-t)^\beta}{(1-zt)^\alpha}$$

for all $z \in [-1, 1)$.

Theorem 2. *If*

$$Q(\alpha)=e^{2-\alpha-r_1(\alpha)}|r|^{2-\alpha}A\left(\alpha,1,\frac{s}{r}\right),\quad R(\alpha)=e^{2-\alpha-r_1(\alpha)}|r|^{2}s^{-\alpha}R\left(\alpha,1,\frac{r}{s}\right),$$

then

$$m\left(\log\left(1-\frac{r}{s}\right)\right)\le \inf_{\alpha}{}^{*}\left\{1-\frac{\ln Q(\alpha)}{\ln R(\alpha)}\right\},$$

where $\inf^*_{\alpha}$ *means that for a given non-zero rational* $r/s\in[-1,1)$ *the infimum is taken over all rationals* $\alpha\in(0,1]$ *satisfying* $R(\alpha)<1$.

As numerical examples we give the following list, where u.b. means the obtained upper bound for $m(\log(1-r/s))$.

$\frac{r}{s}$	α	u.b.	u.b.$(\alpha=1)$
-1	$\frac{6}{7}$	3.891399 ...	4.6221 ...
$-\frac{2}{3}$	$\frac{18}{19}$	9.7551 ...	11.1449 ...
$-\frac{3}{5}$	$\frac{30}{31}$	53.8149 ...	90.7656 ...
$-\frac{1}{2}$	$\frac{12}{13}$	3.3317 ...	3.5474 ...
$-\frac{1}{3}$	$\frac{16}{17}$	3.1105 ...	3.2240 ...
$-\frac{7}{30}$	$\frac{160}{161}$	619.5803 ...	1798.6314 ...
$-\frac{1}{120}$	$\frac{578}{579}$	2.3854 ...	2.3862 ...
$\frac{1}{15}$	$\frac{68}{69}$	2.6411 ...	2.6535 ...
$\frac{3}{20}$	$\frac{88}{89}$	5.7392 ...	5.7977 ...

In the first row of this list we have Rukhadze's [20] measure for log 2. However we note that in some other cases, e.g. if $r/s=-1/2,\ -1/3$, we are not able to reach the measures announced in [20].

Padé type approximations

We use l,m and n to denote positive integer parameters satisfying $l\le\min\{m,n\}$. In the proof of our theorems 1 and 1p we shall use only the choice $l=m=n$, but in some interesting cases like in Theorem 2 some other choices are better. Therefore we give our next lemmas in the general form.

Let us define the polynomial $A_{l,m,n}(z)$ by

$$\begin{aligned}(2)\qquad A_{l,m,n}(z)&=\frac{1}{z^B(1-z)^C}\frac{1}{l!}\left(\frac{d}{dz}\right)^l(z^{n+B}(1-z)^{m+C}),\\ &=(-1)^l z^n(1-z)^{m-l}\sum_{k=0}^{l}\binom{n+B}{k}\binom{m+C}{l-k}\left(\frac{z-1}{z}\right)^k\\ &=\frac{(-m-C)_l}{l!}z^n(1-z)^{m-l}{}_2F_1\left(\begin{matrix}-n-B\ ,\ -l\\ 1+m+C-l\end{matrix}\middle|\,\frac{z-1}{z}\right).\end{aligned}$$

Thus the polynomial $A_{l,m,n}$ is of degree $\leq n+m-l$ and has a zero of order $\geq n-l$. By defining

$$
\begin{aligned}
Q_{l,m,n}(z) &= z^{n+m-l} A_{l,m,n}\left(\frac{1}{z}\right) \\
&= \frac{(-m-C)_l}{l!}(z-1)^{m-l} {}_2F_1\left(\begin{matrix} -n-B\ ,\ -l \\ 1+m+C-l \end{matrix}\middle| 1-z\right) \\
&= \frac{(-m-n-B-C)_l}{l!}(z-1)^{m-l} {}_2F_1\left(\begin{matrix} -n-B\ ,\ -l \\ -n-m-B-C \end{matrix}\middle| z\right)
\end{aligned}
$$

we get a polynomial of degree $\leq m$, where the last equality is obtained using the formula 2.10 (1) of [12].

The function $F(z)$ has for all $|z| < 1$ an integral representation

$$
F(z) = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 \frac{\omega(t)}{1-zt}\, dt, \quad \omega(t) = t^{b-1}(1-t)^{c-b-1}. \tag{3}
$$

Therefore, for all $0 < |z| < 1$,

$$
\begin{aligned}
Q_{l,m,n}(z)F(z) &= \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 \frac{Q_{l,m,n}(z)\omega(t)}{1-zt}\, dt \\
= \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} &\left(z^{n+m-l-1} \int_0^1 \frac{A_{l,m,n}(1/z) - A_{l,m,n}(t)}{\frac{1}{z} - t} \omega(t)\, dt \right. \\
+z^{n+m-l} &\left. \int_0^1 \frac{A_{l,m,n}(t)\omega(t)}{1-zt}\, dt \right) = z^{n+m-l-1} B_{l,m,n}(1/z) + R_{l,m,n}(z)
\end{aligned}
$$

with obvious definitions of the polynomial $B_{l,m,n}$ and the function $R_{l,m,n}$.

We next consider more closely the remainder function $R_{l,m,n}$. If

$$
f(z) = z^{n+B}(1-z)^{m+C},
$$

then, by partial integration and our assumption $l \leq \min\{m,n\}$,

$$
\begin{aligned}
R_{l,m,n}(z) &= \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \frac{z^{n+m-l}}{l!} \int_0^1 \frac{f^{(l)}(t)}{1-zt}\, dt \\
&= \cdots = \frac{(-1)^l \Gamma(c)}{\Gamma(c-b)\Gamma(b)} z^{n+m} \int_0^1 \frac{f(t)}{(1-zt)^{l+1}}\, dt \\
&= (-1)^l z^{n+m} \frac{\Gamma(c)\Gamma(m+c-b)\Gamma(n+b)}{\Gamma(c-b)\Gamma(b)\Gamma(n+m+c)} {}_2F_1\left(\begin{matrix} l+1\ ,\ n+b \\ n+m+c \end{matrix}\middle| z\right). \\
&= (-1)^l z^{n+m} \frac{(b)_n(c-b)_m}{(c)_{n+m}} {}_2F_1\left(\begin{matrix} l+1\ ,\ n+b \\ n+m+c \end{matrix}\middle| z\right).
\end{aligned} \tag{4}
$$

Therefore the Taylor expansion of $R_{l,m,n}$ has rational coefficients and vanishes at $z = 0$ at least to the order $n+m$.

By the above considerations we now have the approximation formula

$$
R_{l,m,n}(z) = Q_{l,m,n}(z)F(z) - P_{l,m,n}(z), \tag{5}
$$

where

$$P_{l,m,n}(z) = z^{n+m-l-1} B_{l,m,n}(\frac{1}{z}) \tag{6}$$

is a polynomial of degree $n + m - l - 1$. Thus (5) is an identity with rational coefficients, and therefore we can use it also in other metrics if the series converge.

The estimation of the polynomials and the remainder term

Let us suppose that $l = [\alpha n]$, $m = [\beta n]$, where α and β are rationals satisfying $0 < \alpha \leq \min\{1, \beta\}$, and let us denote

$$P_n(z) = P_{l,m,n}(z), \quad Q_n(z) = Q_{l,m,n}(z), \quad R_n(z) = R_{l,m,n}(z).$$

We shall first estimate the remainder term $R_n(z)$. Let $\delta = \delta(v)$ be 1, if $v = \infty$, and 0, if $v = p$. By $c_1, c_2, \ldots$ we shall denote positive constants independent of n. We now obtain the following

Lemma 1. *If $|z|_v < 1$ and in the finite case $v \not| fh$, then we have*

$$|R_n(z)|_v \leq c_1 n^{1-\delta} \left(|z|_v^{1+\beta} R(\alpha, \beta, z)^\delta\right)^n$$

for all $n \geq c_2$. In the archimedean case the bound on the right-hand side of this inequality is an asymptotic for $|R_n(z)|$ $(n \to \infty)$.

Remark 1. In the archimedean case the bound holds at the point $z = -1$, too.

Proof. In the archimedean case the result follows immediately from the integral representation (4) of $R_{l,m,n}(z)$.

To prove the finite case we denote

$$Q_n(z) = \sum_{j=0}^{m} q_j z^j, \qquad F(z) = \sum_{j=0}^{\infty} f_j z^j.$$

By (5) we then have

$$R_n(z) = \sum_{k=m+n}^{\infty} \left(\sum_{j=0}^{m} q_j f_{k-j} \right) z^k = z^{m+n} \sum_{k=0}^{\infty} e_k z^k,$$

where

$$e_k = \sum_{j=0}^{m} q_j f_{k+m+n-j}, \qquad k = 0, 1, \ldots.$$

Because q_j are v-integers (i.e.$|q_j|_v \leq 1$), it follows that

$$|e_k|_v \leq \max_{0 \leq j \leq m} \{|f_{k+m+n-j}|_v\}.$$

Here

$$f_{k+m+n-j} = \frac{h^{k+m+n-j} a(a+f) \ldots (a + (k+m+n-j-1)f)}{f^{k+m+n-j} g(g+h) \ldots (g + (k+m+n-j-1)h)},$$

and therefore $|e_k|_v \le p^{r(k)}$ $(v = p)$, where

$$r(k) \le \max_{0 \le j \le m} \sum_{\mu \le \frac{\ln(|g|+(k+m+n)h)}{\ln p}} \left(\left[\frac{k+m+n-j}{p^\mu} \right] + 1 - \left[\frac{k+m+n-j}{p^\mu} \right] \right)$$
$$\le \frac{\ln h(|c| + k + (1+\beta)n)}{\ln p}, \quad k = 0, 1, \ldots .$$

Thus

$$|e_k|_v \le h(|c| + k + (1+\beta)n), \quad k = 0, 1, \ldots,$$

which implies the estimate

$$\left| e_k z^k \right|_v \le h(|c| + k + (1+\beta)n)|z|_v^k \le c_1 n$$

for all $n \ge c_2$. This proves our lemma. $\square$

The function $f(t) = t^{n+B}(1-t)^{m+C}$ is analytic in a complex domain D obtained by cutting the plane from 0 to infinity and from 1 to infinity. We choose these cuts in such a way that they avoid the point z. To estimate the polynomials $P_n(z)$ and $Q_n(z)$ we first consider the polynomial $A_n(z) = A_{l,m,n}(z)$ by using Cauchy's integral formula to get

$$A_n(z) = \frac{1}{z^B(1-z)^C} \frac{1}{l!} \left(\frac{d}{dz} \right)^l f(z) = \frac{1}{2\pi i} \frac{1}{z^B(1-z)^C} \oint_\Gamma \frac{f(t)}{(t-z)^{l+1}}\, dt, \tag{7}$$

where Γ is a simple closed curve in D.

Lemma 2. *If* $|z| > 1$ *or* $z = -1$, *then*

$$|A_n(z)| \le c_3 \left(A(\alpha, \beta, z)^n + |z|^{-\alpha n} R\left(\alpha, \beta, \frac{1}{z} \right)^n \right),$$

and if $-1 < z < 0$, *then*

$$|A_n(z)| \le c_4 \left(A(\alpha, \beta, z)^n + (1-z)^{-\alpha n} R\left(\frac{\alpha}{\beta}, \frac{1}{\beta}, \frac{1}{1-z} \right)^{\beta n} \right).$$

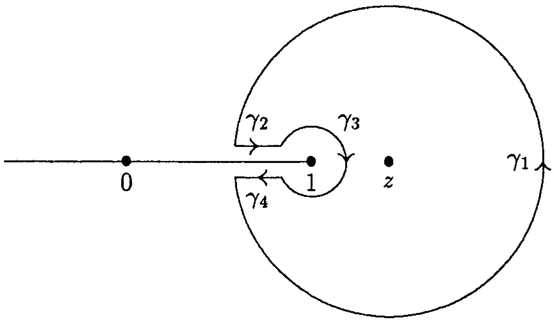
In the case $\alpha = \beta = 1$ *we have*

$$|A_n(z)| \le c_5 n^q$$

for all $0 \le z \le 1$, *if* $q = \max\{B, C\} \ge -\frac{1}{2}$.

Proof. We divide our proof into four cases. Let first $z > 1$. Then we cut the plane along the real line from 1 to $-\infty$, and take $\Gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, where $\gamma_1 : |t - z| = \rho < z$ and $\gamma_3 : |t - 1| = \varepsilon$ with some $\varepsilon > 0$ (see Picture 1). Then

$$\left| \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(t)}{(t-z)^{l+1}}\, dt \right| \le \frac{(\rho+z)^{n+B}(\rho+z-1)^{m+C}}{\rho^l}$$
$$\le \frac{(\rho+z)^B(\rho+z-1)^{C-\{\beta n\}}}{\rho^{-\{\alpha n\}}} \left(\frac{(\rho+z)(\rho+z-1)^\beta}{\rho^\alpha} \right)^n$$



Picture 1.

(this is all we need if $\rho < z-1$). In the case $z-1<\rho<z$ we have (with small ε)

$$\left|\frac{1}{2\pi i}\oint_{\gamma_3}\frac{f(t)}{(t-z)^{l+1}}\,dt\right| \le \left|\frac{1}{2\pi i}\int_{\pi}^{-\pi}\frac{\left(1+\varepsilon e^{i\phi}\right)^{n+B}\left(\varepsilon e^{i\phi}\right)^{m+C}}{(1+\varepsilon e^{i\phi}-z)^{l+1}}\varepsilon\, d\phi\right|$$

$$\le \frac{(1+\varepsilon)^{n+B}\varepsilon^{m+1+C}}{(z-1-\varepsilon)^{l+1}} \to 0, \text{ when } \varepsilon\to 0.$$

Further it follows that

$$\left|\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(t)}{(t-z)^{l+1}}\,dt\right| = \frac{1}{2\pi}\left|\int_{z-\rho}^{1}\frac{f(t)}{(t-z)^{l+1}}\,dt\right|$$

$$\le \frac{1}{2\pi|z|^{l+1}}\int_0^1\frac{f(t)}{(1-\frac{t}{z})^{l+1}}\,dt \le \frac{c_6}{|z|^{\alpha n}}R\left(\alpha,\beta,\frac{1}{z}\right)^n.$$

These estimates give the truth of our lemma in this case.

The cases $z\le -1$ and $-1<z<0$ are analogous.

In the case $\alpha=\beta=1$ our polynomial is connected with the Jacobi polynomial $P_n^{(B,C)}(z)$ by the formula

$$A_n(z) = P_n^{(B,C)}(1-2z), \quad 0\le z\le 1.$$

Therefore our result follows immediately from Theorem 7.32.1 of [22]. □

Lemma 3. *If* $|z|<1$ *and* $R(\alpha,\beta,z)\le |z|^{-\alpha}A\left(\alpha,\beta,\frac{1}{z}\right)$, *then*

$$\max\{|Q_n(z)|,|P_n(z)|\}\le c_7\left(|z|^{1+\beta-\alpha}A\left(\alpha,\beta,\frac{1}{z}\right)\right)^n.$$

If $\alpha=\beta=1$ *and* $z>1$, *then*

$$\max\{|Q_n(z)|,|P_n(z)|\}\le c_8 n^{q+2}|z|^n.$$

If $\alpha=\beta=1$, $z<-1$, *and* $\frac{z}{z-1}R\left(1,1,\frac{z}{z-1}\right)<A\left(1,1,\frac{1}{z}\right)$, *then*

$$\max\{|Q_n(z)|,|P_n(z)|\}\le c_9\left(|z|A\left(1,1,\frac{1}{z}\right)\right)^n.$$

Remark 2. Since (5) holds in the archimedean case at $z = -1$, the first part of our lemma is true at $z = -1$.

Proof. Since

$$Q_n(z) = z^{n+m-l} A_n\left(\frac{1}{z}\right),$$

the bounds for $Q_n(z)$ follow from Lemma 2. By (5)

$$Q_n(z)F(z) - P_n(z) = R_n(z)$$

for all $|z| < 1$. If $|P_n(z)| > c_{10}\left(|z|^{1+\beta-\alpha} A(\alpha, \beta, \frac{1}{z})\right)^n$ with a suitable c_{10} we have a contradiction with our hypothesis $R(\alpha, \beta, z) < |z|^{-\alpha} A\left(\alpha, \beta, \frac{1}{z}\right)$. This proves Lemma 3 in the case $|z| < 1$.

Next we assume that $\alpha = \beta = 1$, $z > 1$. From the definition of $P_n(z)$ it follows that

$$P_n(z) = z^{n-1} B_n\left(\frac{1}{z}\right), \quad B_n(u) = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 \frac{A_n(u) - A_n(t)}{u - t} \omega(t)\, dt.$$

If $0 < u < 1$, we give the integral in the form ($\frac{u}{2} < \gamma < \frac{1-u}{2}$)

$$B_n(u) = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \left(\int_0^{u-\gamma} + \int_{u-\gamma}^{u+\gamma} + \int_{u-\gamma}^{1}\right) \frac{A_n(u) - A_n(t)}{u - t} \omega(t)\, dt = I_1 + I_2 + I_3,$$

say. For $|I_1|$ and $|I_3|$ we have the upper bound $2c_{11}n^q/\gamma$ by Lemma 2. Further, by the mean value theorem

$$I_2 = \int_{u-\gamma}^{u+\gamma} A_n'(v)\omega(t)\, dt,$$

where $v = v(t)$ is some point between u and t. Here

$$A_n'(v) = \frac{d}{dv} P_n^{(B,C)}(1 - 2v) = -2\left(P_n^{(B,C)}\right)'(1 - 2v),$$

and from Theorem 7.32.4 of [22] we obtain

$$|A_n'(v)| \le c_{12} n^{\max\{2+B, 2+C, \frac{1}{2}\}} \le c_{12} n^{q+2}.$$

The case $z < -1$ can be considered in an analogous way. Thus Lemma 3 is true. □

On the properties of the coefficients of P_n and Q_n

Let p be a prime and $r \in \mathbf{Q}$, $r \ne 0$. As usual we define $v_p(r)$ by $r = p^{v_p(r)} R/S$, where $(R, S) = (R, p) = (S, p) = 1$. In the following we shall also need the notation

$$\mu_F(j) = \prod_{p \mid F} p^{v_p(j!)}.$$

Using this notation we see that the coefficients

$$a_j = \binom{n+B}{j}\binom{m+C}{l-j}$$

of the polynomial

$$A_n(z) = \sum_{j=0}^{l}(-1)^m a_j z^{n-j}(z-1)^{m-l+j}$$

satisfy

$$\text{(8)} \qquad a_j \in \frac{1}{F^j H^{l-j}\mu_F(j)\mu_H(l-j)}\mathbb{Z} \quad \text{and} \quad a_j \in \frac{1}{L^l \mu_L(l)}\mathbb{Z}, \quad j = 0, 1, \ldots, l,$$

where $L = \text{l.c.m.}(F, H)$. Since

$$Q_n(z) = z^{n+m-l} A_n\left(\frac{1}{z}\right) = \sum_{j=0}^{l}(-1)^m a_j (1-z)^{m-l+j},$$

it follows that

$$\text{(9)} \qquad Q_n\left(\frac{r}{s}\right) \in \frac{(r-s)^{m-l}}{s^m L^l \mu_L(l)}\mathbb{Z}.$$

The polynomial $A_n(z)$ can also be given in the form

$$\text{(10)} \qquad A_n(z) = \sum_{j=m-l}^{n+m-l} c_j (1-z)^j,$$

where

$$c_j = \sum_{i=\min\{0,j-m\}}^{j-m+l} (-1)^{m-j} a_{j-m+l-i}\binom{n+m-l-j+i}{i}.$$

By (8) we have

$$\text{(11)} \qquad c_j \in \frac{1}{F^{j-m+l}H^l \mu_F(j-m+l)\mu_H(l)}\mathbb{Z} \quad \text{and} \quad c_j \in \frac{1}{L^l \mu_L(l)}\mathbb{Z},$$
$$j = m-l, \ldots, n+m-l.$$

Next we investigate the polynomial

$$P_n(z) = z^{n+m-l-1} B_n\left(\frac{1}{z}\right) = z^{n+m-l-1}\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \frac{A_n\left(\frac{1}{z}\right) - A_n(t)}{\frac{1}{z} - t}\omega(t)\,dt.$$

By (10)

$$\int_0^1 \frac{A_n\left(\frac{1}{z}\right) - A_n(t)}{\frac{1}{z} - t}\omega(t)\,dt = -\sum_{j=m-l}^{n+m-l} c_j \sum_{i=0}^{j-1} \hat{z}^{j-1-i}\frac{\Gamma(b)\Gamma(i+c-b)}{\Gamma(i+c)},$$

where we have used the notation $\hat{z} = 1 - 1/z$. Therefore we immediately obtain

$$P_n(z) = -z^{n+m-l-1} \sum_{j=m-l}^{n+m-l} c_j \sum_{i=0}^{j-1} \hat{z}^{j-1-i} \frac{(C+1)\dots(C+i)}{c(c+1)\dots(c+i-1)}.$$

By the Gauss formula (see [12], p. 104, and [15])

$${}_2F_1\left(\begin{matrix} -i\,,\ a \\ b \end{matrix} \middle| 1 \right) = \frac{(b-a)_i}{(b)_i}$$

we get

$$\begin{aligned} \frac{h^i i!}{g(g+h)\dots(g+(i-1)h)} &= \frac{(g/h-(g/h-1))_i}{(g/h)_i} \\ &= 1 + \sum_{j=1}^{i} (-1)^j \binom{i}{j} \frac{g-h}{g+(j-1)h} \in \frac{1}{d_i(g,h)} \mathbb{Z}, \end{aligned}$$

where $d_i(g,h) = \text{l.c.m.}\{g, g+h, \dots, g+(i-1)h\}$. Since h and $d_i(g,h)$ have no common prime factors this implies

$$\frac{(C+1)\dots(C+i)}{c(c+1)\dots(c+i-1)} \in \frac{h^i \mu_h(i)}{H^i \mu_H(i) d_i(g,h)} \mathbb{Z}.$$

Combining these facts we are led to the result

$$P_n\left(\frac{r}{s}\right) \in \frac{1}{s^{n+m-l} L^l H^{*\,n+m-l} \mu_L(l) \mu_{H^*}(n+m-l) d_{n+m-l}(g,h)} \mathbb{Z}, \tag{12}$$

where H^* denotes the denominator of h/H.

We now use (9) and (12) to obtain the following

Lemma 4. *If*

$$\Omega_n = s^{n+m-l} L^l H^{*\,n+m-l} \mu_L(l) \mu_{H^*}(n+m-l) d_{n+m-l}(g,h),$$

then

$$\Omega_n Q_n\left(\frac{r}{s}\right), \quad \Omega_n P_n\left(\frac{r}{s}\right) \in \mathbb{Z}.$$

Approximation sequences

The above considerations are performed to find good approximation sequences (q_n, p_n, r_n) for $F(r/s)$, i.e. to find integers q_n, p_n such that

$$q_n F\left(\frac{r}{s}\right) - p_n = r_n,$$

where r_n tends to zero as $n \to \infty$.

In considering the general case we note that

$$\mu_F^{n-p\frac{\ln n}{\ln p}} \le \mu_F(n) \le \mu_F^n$$

and, by [1], Lemma 1,

$$\lim_{n\to\infty} \frac{1}{n} \ln d_n(g,h) = \frac{h}{\phi(h)} \sum_{\substack{i=1 \\ (i,h)=1}}^{h} \frac{1}{i} = \lambda(h).$$

We now use Lemma 4 to obtain the integers

$$q_n = \Omega_n Q_n\left(\frac{r}{s}\right), \quad p_n = \Omega_n P_n\left(\frac{r}{s}\right).$$

By denoting

$$(13) \qquad \begin{cases} \omega(\alpha,\beta) = L^\alpha H^{*1+\beta-\alpha} \mu_L \mu_{H^*}^{1+\beta-\alpha} e^{(1+\beta-\alpha)\lambda(h)}, \\ \nu(\alpha,\beta) = |r|^{1+\beta-\alpha} A\left(\alpha,\beta,\frac{s}{r}\right), \\ \mu(\alpha,\beta) = |r|^{1+\beta} s^{-\alpha} R\left(\alpha,\beta,\frac{r}{s}\right), \\ Q(\alpha,\beta) = \omega(\alpha,\beta)\nu(\alpha,\beta), \quad R(\alpha,\beta) = \omega(\alpha,\beta)\mu(\alpha,\beta), \end{cases}$$

we get, by Lemmas 1, 3 and 4, the following

Lemma 5. *Let $\varepsilon > 0$ be given.*

(i) If $|r/s| < 1$ and $R(\alpha,\beta) < \min\{1, Q(\alpha,\beta)\}$, then the above q_n, p_n and $r_n = q_n F(r/s) - p_n$ satisfy

$$\max\{|p_n|,|q_n|\} \le Q(\alpha,\beta)^{(1+\varepsilon)n},$$
$$R(\alpha,\beta)^{(1+\varepsilon)n} \le |r_n| \le R(\alpha,\beta)^{(1-\varepsilon)n}$$

for all $n \ge c_{13}$.

(ii) If p is a prime such that $p \nmid fh$ and $|r/s|_p < 1$, then

$$|r_n|_p \le c_1 |r|_p^{(1+\beta-\varepsilon)n}$$

for all $n \ge c_{14}$.

(iii) Suppose that $\alpha = \beta = 1$. If $r/s > 1$ then

$$\max\{|p_n|,|q_n|\} \le (\omega(1,1)|r|)^{(1+\varepsilon)n},$$

and if $r/s < -1$ and $\frac{r}{r-s} R(1,1,r/(r-s)) < A(1,1,s/r)$, then

$$\max\{|p_n|,|q_n|\} \le Q(1,1)^{(1+\varepsilon)n},$$

for all $n \ge c_{15}$.

Some determinants

In the archimedean case we have an asymptotic formula for the remainder term r_n in the lemmas above. On the other hand it seems difficult to obtain such a result in the p-adic case. Therefore we need the nonvanishing of the determinant

$$\Delta_n(z) = \begin{vmatrix} Q_n(z) & P_n(z) \\ Q_{n+1}(z) & P_{n+1}(z) \end{vmatrix}$$

in the p-adic considerations.

Lemma 6. *If $\alpha = \beta = 1$ or $l = m = n$, then we have*

$$\Delta_n(z) = (-1)^n \frac{(b)_n (c-b)_n}{(c)_{2n}} \binom{-n-c}{n+1} z^{2n}.$$

Proof. Clearly $\Delta_n(z)$ is a polynomial in z of $\deg \Delta_n(z) \leq 2n$. Since $Q_n(z)F(z) - P_n(z) = R_n(z)$, we have

$$\Delta_n(z) = Q_{n+1}(z)R_n(z) - Q_n(z)R_{n+1}(z).$$

Thus $\operatorname{ord}_{z=0}\Delta_n(z) \geq 2n$ and our lemma follows from (2) and (4). $\square$

Proof of Theorem 1 and 1p

In the archimedean case we may use following well-known result (see e.g. [8], Corollary 3.3). Let $x > 0$ and $y < 0$ be given. Suppose that for each $\varepsilon > 0$ there exists a constant c_{16} and rational integers p_n, q_n satisfying for all $n \geq c_{16}$ the inequalities

$$\frac{1}{n} \ln \max\{|q_n|, |p_n|\} < x + \varepsilon,$$
$$y - \varepsilon < \frac{1}{n} \ln |r_n| < y + \varepsilon,$$

where $r_n = q_n F(r/s) - p_n$. Then the number $F(r/s)$ has an irrationality measure $m(F(r/s))$ not greater than $1 - x/y$.

If $|z| < 1$, then we have

$$R(1,1,z) = (1+\sqrt{1-z})^{-2}, \quad A\left(1,1,\frac{1}{z}\right) = \frac{(1+\sqrt{1-z})^2}{|z|}.$$

Therefore, if $z = r/s$, then (13) implies

$$Q(1,1) = \omega(1,1)\left(\sqrt{s} + \sqrt{s-r}\right)^2,$$
$$R(1,1) = \omega(1,1)\left(\frac{r}{\sqrt{s}+\sqrt{s-r}}\right)^2 = \omega(1,1)\left(\sqrt{s} - \sqrt{s-r}\right)^2.$$

The assumption $LH^*\mu_L\mu_{H^*}e^{\lambda(h)}\left(\sqrt{s} - \sqrt{s-r}\right)^2 < 1$ means that $R(1,1) < 1$. Thus the use of Lemma 5 gives us an upper bound

$$1 - \frac{\ln Q(1,1)}{\ln R(1,1)}$$

for the irrationality measure of $F(r/s)$. This proves our Theorem 1.

To give our p-adic results we prove the following simple lemma.

Lemma 7. *Let $\theta \in \mathbb{Q}_p$ be such that there exists a sequence (q_n, p_n) of integers satisfying for all $n \geq c_{17}$*

$$\max\{|q_n|, |p_n|\} \leq Q(p)^n, \quad p_n q_{n+1} - q_n p_{n+1} \neq 0, \quad |r_n|_p \leq c_1 R(p)^n,$$

where $r_n = q_n\theta - p_n$. If $Q(p)R(p) < 1$, then θ has an irrationality measure

$$m_p(\theta) \leq \frac{\ln R(p)}{\ln R(p) + \ln Q(p)}.$$

Proof. We shall find a lower bound for $|L|_p = |Q\theta - P|_p$, where (Q, P) is a non-trivial pair of integers with $H = \max\{|Q|, |P|\}$. Since $Q(p)R(p) < 1$, the inequality

$$\frac{1}{2c_1 H} \leq (Q(p)R(p))^n \tag{14}$$

has only a finite number of solutions $n \in \mathbb{N}$. Let $\overline{n}$ denote the greatest of these. We choose H large enough, say $H \geq H_0$, to satisfy $\overline{n} \geq c_{17}$. From the assumption $p_n q_{n+1} - q_{n+1} p_n \neq 0$ it follows that there exists a natural number N either $= \overline{n}+1$ or $= \overline{n}+2$ such that

$$\Delta = \begin{vmatrix} q_N & -p_N \\ Q & -P \end{vmatrix} = \begin{vmatrix} q_N & r_N \\ Q & L \end{vmatrix}$$

is a non-zero integer. Hence

$$1 \leq |\Delta||\Delta|_p \leq 2HQ(p)^N \left|q_N L - Q r_N\right|_p.$$

By our choice of N we have

$$2HQ(p)^N \left|Q r_N\right|_p \leq 2c_1 H (Q(p)R(p))^N < 1,$$

and therefore, by (14),

$$|L|_p \geq |q_N L|_p \geq \frac{1}{2HQ(p)^N} \geq c_{18} H^{-1+\ln Q(p)/\ln(Q(p)R(p))}.$$

This proves our lemma. □

By (ii) and (iii) of Lemma 5 we may use Lemma 7, where

$$Q(p) = (\omega(1,1)r)^{1+\varepsilon} = \left(LH^* \mu_L \mu_{H^*} e^{\lambda(h)} r\right)^{1+\varepsilon}, \quad R(p) = |r|_p^{2-\varepsilon}.$$

Since $\varepsilon > 0$ may be chosen arbitrarily small, our Theorem 1p follows immediately.

The assumption $r/s > 1$ is of course not necessary. To consider other cases we only have to use part (i) or the second part of (iii) of Lemma 5.

A common factor of the coefficients of P_n and Q_n

It turns out that in many cases the coefficients of the polynomials P_n and Q_n have a big common factor which must be eliminated to get sharp irrationality measures. This kind of idea appears already in Siegel's [21] paper, and it was used in an ingenious way by Chudnovsky [8] to consider certain binomial series, see also [11]. Later this idea combined with Padé-type approximations is used e.g. in [13], [14] and [20]. We shall now introduce a general criterion (see Lemma 10 below) to find a common factor of the coefficients of P_n and Q_n, and then this criterion will be applied to the consideration of the logarithms. Using (2) and the definition of $Q_n(z)$ we see that each common factor of

$$\binom{n+B}{i}\binom{m+C}{l-i}, \qquad i = 0, 1, \ldots, l, \tag{15}$$

is also a common factor for all the coefficients of Q_n and P_n. Therefore we shall find out which primes $p > c_{19}\sqrt{n}$ divide the numbers (15). It was Chudnovsky's [8] observation that only these big primes are really important here. To find a criterion for primes dividing the numbers (15) we first give two lemmas.

To state our lemmas we use for a rational number r the notations $p\,|\,r$ or $r \equiv 0 \pmod{p}$, if $v_p(r) \geq 1$. Further, if $v_p(r) \geq 0$, then there exists a unique $\overline{r} \in \{0, 1, \ldots, p-1\}$ satisfying $\overline{r} \equiv r \pmod{p}$.

Lemma 8. *Let $r = R/S \in \mathbb{Q}$, $(R, S) = 1$, $S > 0$, $i \in \mathbb{N}$, and let p be a prime satisfying $p \not| S$, $p^2 > \max\{i, \max_{1\leq j\leq i}\{|R + (j-1)S|\}\}$. Let $i = Ap + \overline{i}$. Then*

$$v_p\left((r)_i\right) = A + 1 \quad \textit{if and only if} \quad \overline{-r} < \overline{i}.$$

Further

$$p \left| \binom{r}{i} \right.$$

if and only if $\overline{r} < \overline{i}$.

Proof. First we suppose that $0 \leq \overline{i} \leq \overline{-r}$. Then

$$\begin{aligned} v_p\left((r)_i)\right) &= v_p\left(R(R+S)\ldots(R+(i-1)S)\right) \\ &= v_p\left(R(R+S)\ldots(R+(Ap-1)S)\right) \\ &\quad + v_p\left((R+ApS)\ldots(R+(Ap+\overline{i}-1)S)\right) = A + 0 = A \end{aligned}$$

because $R + \overline{(-r)}S \equiv 0 \pmod{p}$. On the other hand we have

$$v_p\left((R+ApS)\ldots(R+(Ap+\overline{i}-1)S)\right) = 1,$$

if $\overline{-r} < \overline{i}$. Thus we have $v_p\left((r)_i)\right) = A + 1$ in this case. This proves the first part of our lemma.

To prove the second part we note that

$$\binom{r}{i} = (-1)^i \frac{(-r)_i}{i!}.$$

Then $v_p\left((-r)_i)\right) = A + 1$ if and only if $\overline{r} < \overline{i}$ by the above consideration. Moreover $v_p(i!) = A$, which completes the proof. $\square$

Lemma 9. *Let $r_1 = R_1/S_1$, $r_2 = R_2/S_2$ denote rationals satisfying $(R_1, S_1) = (R_2, S_2) = 1$, $S_1 > 0$ and $S_2 > 0$, and let p be a prime satisfying $v_p(r_1) \geq 0$, $v_p(r_2) \geq 0$ and*

$$p^2 > \max\left\{l, \max_{1\leq j\leq l}\{|R_1 + (j-1)S_1|, |R_2 + (j-1)S_2|\}\right\}.$$

If

$$\overline{r}_1 + \overline{r}_2 + 1 \leq \overline{l}, \tag{16}$$

then

$$p \,\Big|\, \binom{r_1}{i}\binom{r_2}{l-i}, \quad i = 0, 1, \ldots, l.$$

Proof. Let us suppose that $\overline{l} \leq \overline{i}$. Then we have, by our assumption (16),

$$\overline{r_1} + 1 \leq \overline{r}_1 + \overline{r}_2 + 1 \leq \overline{l} \leq \overline{i}.$$

By Lemma 8 it follows that $p \,\big|\, \binom{r_1}{i}$.

We now consider the case $\overline{l} \geq \overline{i} + 1$. If $\overline{r}_1 + 1 \leq \overline{i}$, then $p \,\big|\, \binom{r_1}{i}$, again by Lemma 8. If $\overline{r}_1 \geq \overline{i}$, then (16) implies

$$\overline{r}_2 + 1 \leq \overline{l} - \overline{r}_1 \leq \overline{l} - \overline{i} = \overline{l - i}.$$

We use once again Lemma 8 and obtain $p \,\Big|\, \binom{r_2}{l-i}$. This proves Lemma 9. □

Lemma 9 gives immediately the following.

Lemma 10 (Divisibility criterion for the coefficients of Q_n). *Let $P(n, \alpha, \beta)$ denote the set of all primes satisfying $p \nmid FH$,*

$$p^2 > \max\left\{l, \max_{1\leq j\leq l}\{|E + (n-j)F|, |G + (n-j)H|\}\right\},$$

$$\overline{n + B} + \overline{m + C} + 1 \leq \overline{l}.$$

Then

$$\left(\prod_{p\in P(n,\alpha,\beta)} p\right) \,\Bigg|\, \binom{n+B}{i}\binom{m+C}{l-i}, \quad i = 0, 1, \ldots, l.$$

We now apply this criterion to the logarithmic function. In this case $B = C = 0$, and we further choose $\beta = 1$, i.e. $m = n$. To use Lemma 10 we have to characterize the primes $p \geq c_{20}\sqrt{n}$ satisfying $2\overline{n} + 1 \leq \overline{l}$. By denoting $\overline{n} = n - Np, \overline{l} = l - Lp$ this condition becomes

$$0 \leq 2(n - Np) \leq l - Lp - 1 \leq p - 2 \tag{17}$$

or

$$\max\left\{\frac{2n - l + 1}{2N - L}, \frac{l+1}{L+1}\right\} \leq p \leq \frac{n}{N}. \tag{18}$$

Conversely, if p is in this interval for some L and $N(\geq L)$, then p satisfies (17).

We now consider carefully the inequalities (18) assuming $n \geq c_{21}$. If

$$\frac{2n-l+1}{2N-L} \leq \frac{l+1}{L+1} < \frac{n}{N}$$

or

$$\text{(i)} \qquad \alpha N - 1 < L \leq \alpha N - 1 + \frac{\alpha}{2},$$

then all primes in the interval

$$\left(\frac{l+1}{L+1}, \frac{n}{N}\right)$$

satisfy (17). Further, if

$$\frac{l+1}{L+1} < \frac{2n-l+1}{2N-L} < \frac{n}{N}$$

or

$$\text{(ii)} \qquad \alpha N - 1 + \frac{\alpha}{2} < L < \alpha N,$$

then all the primes in

$$\left(\frac{2n-l+1}{2N-L}, \frac{n}{N}\right)$$

also satisfy (17).

We assume that $\alpha = u/v$, $u, v \in \mathbb{N}$, $(u,v) = 1$, and set $N = vK + i$, where $i \in \{0, 1, \ldots, v-1\}$. Then (i) is of the form

$$uK + \alpha i - 1 < L \leq uK + \alpha i + \frac{\alpha}{2} - 1.$$

Therefore, if

$$\text{(19)} \qquad [\alpha i + \alpha/2] = [\alpha i] + 1,$$

then $L = uK + [\alpha i]$ satisfies (i) and all the primes p in the interval

$$\left(\frac{[\alpha n] + 1}{uK + [\alpha i] + 1}, \frac{n}{vK + i}\right)$$

satisfy our condition (17).

By the prime number theorem it follows (see [8]) that the product of all the primes in above intervals is asymptotically equal to $e^{n\Sigma_1}$, where Σ_1 is equal to

$$\sum_{i \in (19)} \sum_{K=0}^{\infty} \left(\frac{1}{vK+i} - \frac{u/v}{uK + [\alpha i] + 1}\right) = \frac{1}{v} \sum_{i \in (19)} \sum_{K=0}^{\infty} \left(\frac{1}{K + \frac{i}{v}} - \frac{1}{K + \frac{1+[\alpha i]}{u}}\right),$$

and here $i \in (19)$ means that i satisfies (19). By the well-known properties of the digamma function Ψ (see [12], 1.7, (3)) we obtain

$$\Sigma_1 = \frac{1}{v} \sum_{i \in (19)} \left(\Psi\left(\frac{1+[\alpha i]}{u}\right) - \Psi\left(\frac{i}{v}\right)\right).$$

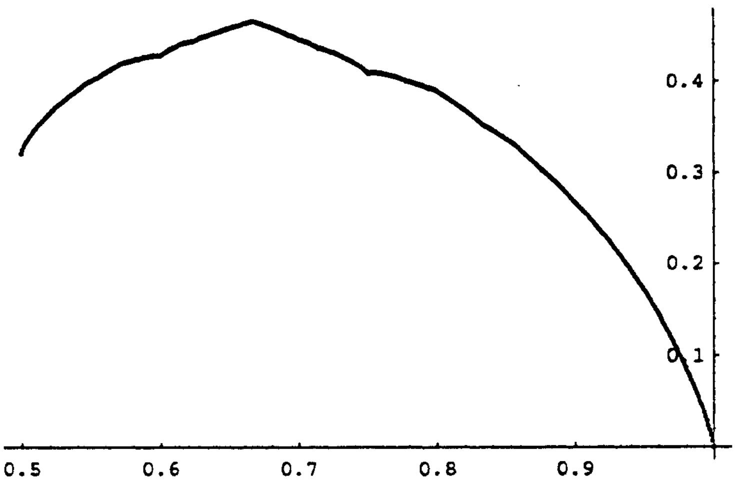
In the same way, if

$$\alpha i \neq [\alpha i] = [\alpha i + \alpha/2], \tag{20}$$

then $L = uK + [\alpha i]$ satisfies (ii) and this case gives an asymptotic $e^{n\Sigma_2}$, where

$$\Sigma_2 = \frac{1}{v} \sum_{i \in (20)} \left(\Psi\left(\frac{2i-[\alpha i]}{2v-u}\right) - \Psi\left(\frac{i}{v}\right)\right).$$

Combining the above considerations we obtain an asymptotic $e^{n\tau_1(\alpha)}$, where $\tau_1(\alpha) = \Sigma_1 + \Sigma_2$. The values of $\tau_1(\alpha)$ are given in the following graph (the interval of the subsequent arguments in the graph is of length 1/1000):



Picture 2.

Proof of Theorem 2

Let us assume that $B = C = 0$, $\beta = 1$. From the above considerations it follows that for a given rational $\alpha \in (0,1]$ there exists a common factor D_n of the coefficients of P_n and Q_n asymptotically equal to $e^{n\tau_1(\alpha)}$. Thus the use of (13) and Lemma 5 immediately gives us the following result concerning the integers

$$q_n = \frac{\Omega_n Q_n(r/s)}{D_n}, \quad p_n = \frac{\Omega_n P_n(r/s)}{D_n},$$

and the remainder term

$$r_n = q_n\, {}_2F_1\left(\begin{matrix}1,\ 1\\ 2\end{matrix}\,\middle|\,\frac{r}{s}\right) - p_n.$$

Lemma 11. *Let $\varepsilon > 0$ be given, and let*

$$\omega_1 = \omega_1(\alpha) = e^{2-\alpha-r_1(\alpha)}, \quad Q(\alpha) = \omega_1 \nu(\alpha, 1), \quad R(\alpha) = \omega_1 \mu(\alpha, 1).$$

If $|r/s| < 1$ and $R(\alpha) < 1$, then we have

$$\max\{|p_n|, |q_n|\} \leq Q(\alpha)^{(1+\varepsilon)n},$$
$$R(\alpha)^{(1+\varepsilon)n} \leq |r_n| \leq R(\alpha)^{(1-\varepsilon)n}$$

for all $n \geq c_{22}$.

By using this lemma we now get the truth of Theorem 2 analogously to the proof Theorem 1.

We note that Lemma 10 may be used to obtain improvements of Theorem 1 in some other special cases, too. These will be considered in another work.

Remark 3. All the numerical computations including Picture 2 are made using MATHEMATICA programs.

References

1. Alladi K. & Robinson M.L., *Legendre polynomials and irrationality*, J. Reine Angew. Math. **318**, 137–155 (1980)
2. Baker A., *Rational approximation to $\sqrt[3]{2}$ and other algebraic numbers*, Q. J. Math. Oxf. II Ser. **15**, 375–383 (1964)
3. Beukers F., *Algebraic values of G-functions*, J. Reine Angew. Math. **434**, 45–65 (1993)
4. Beukers F. & Matala-aho T. & Väänänen K., *Remarks on the arithmetic properties of the values of hypergeometric functions*, Acta Arith. **XLII**, 281–289 (1983)
5. Beukers F. & Wolfart J., *Algebraic values of hypergeometric functions*, in: New advances in transcendence theory, Cambridge 1988
6. Bundschuh P., *Zur Approximation gewisser p-adischer algebraischer Zahlen durch rationalen Zahlen*, J. Reine Angew. Math. **265**, 154–159 (1974)
7. Chudnovsky G.V., *Number theoretic applications of polynomials with rational coefficients defined by extremality conditions*, in: Arithmetic and geometry, Birkhäuser, Boston 1983, 61–105
8. Chudnovsky G.V., *On the method of Thue-Siegel*, Ann. of Math., II. Ser. **117**, 325–382 (1983)
9. Danilov L. V., *Rational approximations of some functions at rational points*, (Russian), Mat. Zametki **24, no. 4**, 449–458 (1978)
10. Dubitskas A. K., *Approximation on $\pi/\sqrt{3}$ by rational fractions*, (Russian), Vestn. Mosk. Univ., Ser. I **no. 6**, 73–76 (1987)
11. Easton D., *Effective Irrationality Measures for Certain Algebraic Numbers*, Math. Comput. **46, no 174**, 613–622 (1986)
12. Erdélyi A. (ed.), *Higher transcendental functions*, vol. 1, McGraw-Hill Book Company, Inc., New York, 1953

13. Hata M., *Legendre type polynomials and irrationality measures*, J. Reine Angew. Math. **407**, 99–125 (1990)

14. Hata M., *Irrationality measures of the values of hypergeometric functions*, Acta Arith. **LX**, 335–347 (1992)

15. Huttner M., *Problème de Riemann et irrationalité d'un quotient de deux fonctions hypergéométriques de Gauss*, C.R. Acad. Sci., Paris Sér. I **302**, 603–606 (1986)

16. Huttner M., *Irrationalité de certaines integrales hypergéométriques*, J. Number Theory **26**, 166–178 (1987)

17. Kuukasjärvi M. & Matala-aho T. & Väänänen K., *On irrationality measures of certain P-adic numbers*, Acta Univ. Oulu., Ser A., Sci. Rerum Nat. **207**, 1–16 (1989)

18. Nikisin E.M., *Arithmetic properties of the Markov function for the Jacobi weight*, Anal. Math. **8**, 39–46 (1982)

19. Rhin G., *Approximants de Padé et mesures effectives d'irrationalité*, Prog. Math. **71**, 155-164 (1987)

20. Rukhadze E. A., *Lower estimate for rational approximations of* ln 2, (Russian), Vestn. Mosk. Univ., Ser. I **no. 6**, 25–29 (1987)

21. Siegel C. L., *Über einige Anwendungen diophantischer Approximationen*, Abh. d. Preuss. Akad. d. Wiss., Phys.-mat. **Kl. 1 No. 1** (1929)

22. Szegö G., *Orthogonal Polynomials*, American Mathematical Society, New York, 1939

23. Vasilenko O.N., *On the irrationality of the values of the Gaussian hypergeometric function*, (Russian), Vestn. Mosk. Univ., Ser. I **no 3**, 15–18 (1985)

24. Wolfart J., *Werte hypergeometrischer Funktionen*, Invent. Math. **92**, 187–216 (1988)

Department of Mathematics
University of Oulu
90570 Oulu
Finland

(Received May 24, 1993)