A global Arnoldi method for the model reduction of second-order structural dynamical systems

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Abstract

In this paper we consider the reduction of second-order dynamical systems with multiple inputs and multiple outputs (MIMO) arising in the numerical simulation of mechanical structures. In commercial software for the kind of application considered here, modal reduction is commonly used to obtain a reduced system with good approximation abilities of the original transfer function in the lower frequency range. In recent years new methods to reduce dynamical systems based on (block) versions of Krylov subspace methods emerged. This work concentrates on the reduction of second-order MIMO systems by the global Arnoldi method, an efficient extension of the standard Arnoldi algorithm for MIMO systems. In particular, a new model reduction algorithm for second order MIMO systems is proposed which automatically generates a reduced system of given order approximating the transfer function in the lower range of frequencies. It is based on the global Arnoldi method, determines the expansion points iteratively and the number of moments matched per expansion point adaptively. Numerical examples comparing our results to modal reduction and reduction via the block version of the rational Arnoldi method are presented.

Key words: Model Order Reduction, Simulation, Krylov Subspace, Global Arnoldi Algorithm, Moment Matching
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1. Introduction

In the context of the numerical simulation of machine tools second-order dynamical systems of the form

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = Fu(t), \qquad y(t) = C_v\dot{x}(t) + C_p x(t) \tag{1}$$

arise, where $M, K, D \in \mathbb{R}^{n\times n}$, $F \in \mathbb{R}^{n\times m}$, $C_v, C_p \in \mathbb{R}^{q\times n}$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^q$.

The system matrices considered are large, sparse, and non-symmetric. The matrix K is non-singular. The mass matrix M may be singular. In that case one obtains a system of differential algebraic equations. In general, m and q will be larger than one, so that the system is multi-input multi-output (MIMO). All of this accounts for unacceptable computational and resource demands in simulation and control of these models. In order to reduce these demands to acceptable computational times, usually model order reduction techniques are employed which generate a reduced order model that captures the essential dynamics of the system and preserves its important properties. That is, one tries to find a second order system of reduced dimension $r \ll n$

$$\hat{M}\ddot{\hat{x}}(t) + \hat{D}\dot{\hat{x}}(t) + \hat{K}\hat{x}(t) = \hat{F}u(t), \qquad \hat{y}(t) = \hat{C}_v\dot{\hat{x}}(t) + \hat{C}_p\hat{x}(t), \tag{2}$$

which approximates the original system in some sense, where $\hat{M}, \hat{D}, \hat{K} \in \mathbb{R}^{r\times r}$, $\hat{F} \in \mathbb{R}^{r\times m}$, $\hat{C}_v, \hat{C}_p \in \mathbb{R}^{q\times r}$, $\hat{x}(t) \in \mathbb{R}^r$, $u(t) \in \mathbb{R}^m$, $\hat{y}(t) \in \mathbb{R}^q$.

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In the last years various methods to reduce second-order dynamical systems have been proposed, see, e.g., [1, 2, 3]. As model reduction of linear first-order systems is much further developed and understood, it is tempting to transform the original second-order system (1) to a mathematically equivalent first-order system

$$\underbrace{\begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix}}_{\dot{z}(t)} = \underbrace{\begin{bmatrix} 0 & -K \\ -K & -D \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}}_{z(t)} + \underbrace{\begin{bmatrix} 0 \\ F \end{bmatrix}}_{B} u(t), \quad y(t) = \underbrace{\begin{bmatrix} C_p & C_v \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}}_{z(t)}, \tag{3}$$

where $E, A \in \mathbb{R}^{2n\times 2n}$, $B \in \mathbb{R}^{2n\times m}$, $C \in \mathbb{R}^{q\times 2n}$, $z(t) \in \mathbb{R}^{2n}$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^q$. Various other linearizations have been proposed in the literature, see, e.g., [4, 5, 6]. The linearization (3) is usually preferred as it is symmetry preserving in case K, M, D are symmetric. The system considered here is non-symmetric, so one of the various other possible linearizations could be used instead. Note that the transformation process doubles the dimension of the system. The corresponding reduced system is of the form

$$\hat{E}\dot{\hat{z}}(t) = \hat{A}\hat{z}(t) + \hat{B}u(t) \qquad \hat{y}(t) = \hat{C}\hat{z}(t), \tag{4}$$

where $\hat{E}, \hat{A} \in \mathbb{R}^{r\times r}$, $\hat{B} \in \mathbb{R}^{r\times m}$, $\hat{C} \in \mathbb{R}^{q\times r}$, $\hat{z}(t) \in \mathbb{R}^r$, $u(t) \in \mathbb{R}^m$, $\hat{y}(t) \in \mathbb{R}^q$.

In the engineering context of our application modal reduction [7] is most common. Here we will consider projection based model reduction based on Krylov subspace methods. In the recent years various new Krylov subspace based methods to reduce first- and second-order systems have been proposed, see, e.g., [8, 9, 10] and the references therein. We will consider methods which generate matrices $V \in \mathbb{R}^{2n\times r}$ with $V^TV = I_r$ such that the reduced first-order system (4) is constructed by applying the Galerkin projection $\Pi = VV^T$ to (3)

$$\hat{E} = V^TEV, \quad \hat{A} = V^TAV, \quad \hat{B} = V^TB, \quad \text{and} \quad \hat{C} = CV. \tag{5}$$

Similarly, the reduced second-order system (2) is constructed by applying a Galerkin projection to (1) such that

$$\hat{M} = V^TMV, \quad \hat{D} = V^TDV, \quad \hat{K} = V^TKV, \quad \hat{F} = V^TF, \quad \hat{C}_p = C_pV \quad \text{and} \quad \hat{C}_v = C_vV, \tag{6}$$

where $V \in \mathbb{R}^{n\times r}$ with $V^TV = I_r$. The matrix V can be constructed iteratively by employing Krylov subspace algorithms, in particular the block Arnoldi algorithm. It is well-known that Krylov subspace based methods are not guaranteed to yield reduced order models with the best overall performance in the entire frequency domain; only local approximation around the expansion point can be expected. Therefore, multi point moment matching methods have been introduced [11, 12, 13], see Section 2 for a short review. In [14] the choice of expansion points is discussed, in [15] an algorithm choosing the expansion points iteratively, called Iterative Rational Krylov Algorithm (IRKA) and in [16, 17] adaptive multi point moment matching methods have been proposed. The global Arnoldi method [18] is similar to the standard Arnoldi method except that the standard inner product is replaced by the inner product $\langle Y, Z\rangle_F = \text{trace}(Y^TZ)$ where $Y, Z \in \mathbb{R}^{n\times s}$. The associated norm is the Frobenius norm $\|\cdot\|_F$. The global Arnoldi algorithm constructs an F-orthonormal basis $V_1, V_2, \ldots, V_k$ of the Krylov subspace $\mathcal{K}_k(\Psi, \Upsilon)$, $\Psi \in \mathbb{R}^{n\times n}$, $\Upsilon \in \mathbb{R}^{n\times s}$. Here a system of vectors (matrices in $\mathbb{R}^{n\times s}$) is said to be F-orthonormal if it is orthonormal with respect to $\langle\cdot,\cdot\rangle_F$. The global Arnoldi algorithm is computational cheaper than the block Arnoldi method. It has been used for model reduction of first-order systems (3), see [19, 20, 21, 22]. In Section 3 a short introduction of the global Arnoldi method is presented. Further its extension to model reduction of second order MIMO systems is discussed. In the context of the global Arnoldi algorithm, an adaptive-order global Arnoldi algorithm has been proposed [19, 17]. This algorithm adaptively determines the number of expansions for a fixed set of expansion points. Here we propose a combination of this algorithm and a modified version of IRKA [13] to reduce second-order MIMO systems. The algorithm is based on the global Arnoldi method, determines the expansion points iteratively and the number of moments matched per expansion point adaptively. Numerical experiments are given in Section 4.

2. Model reduction using block Arnoldi type methods

A k-th order Krylov subspace is defined by

$$\mathcal{K}_k(P, q) = \text{span}\{q, Pq, P^2q, \cdots, P^{k-1}q\}, \tag{7}$$

where $P \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$. The Arnoldi method [23, 24] applied to the pair (P, q) produces a matrix V with orthonormal columns which span the Krylov subspace $\mathcal{K}_k(P, q)$ (in case no breakdown occurs during the computation).

In order to be able to treat MIMO systems, we will need to consider block Krylov subspaces

$$\mathcal{K}_k(P, Q) = \text{span}\{Q, PQ, P^2Q, \cdots, P^{k-1}Q\}, \tag{8}$$

where $P \in \mathbb{R}^{n\times n}$ and the columns of $Q \in \mathbb{R}^{n\times \ell}$ are linearly independent. Such a block Krylov subspace with ℓ starting vectors (assembled in Q) can be considered as a union of ℓ Krylov subspaces defined for each starting vector. Usually, the computation of an orthonormal basis $V_{[k]} \in \mathbb{R}^{n\times k\cdot\ell}$ with $k \cdot \ell = r$ is achieved by employing a block Arnoldi algorithm, see Algorithm 1 [25].

Algorithm 1 Block Arnoldi method

Input: matrices P,Q
Output: transformation matrix V

```
function [V] = Block_Arnoldi(P,Q)
compute the QR factorization V_1 R = Q
V = [V_1]
for j = 1, 2, ..., k do
    W = A V_j
    for i = 1, ..., j do
        H_ij = V_i^T W
        W = W − V_i H_ij
    end for
    compute the QR factorization V_{j+1} H_{j+1,j} = W
    V = [V V_{j+1}]
end for
```

The columns of $V_{[k]} = [V_1, V_2, \ldots, V_k]$ with $V_j \in \mathbb{R}^{n\times\ell}$ are an orthogonal basis for the block Krylov subspace $\mathcal{K}_k(P, Q)$ provided none of the upper triangular matrices $H_{j+1,j}$ in Algorithm 1 are rank-deficient. As in the standard Arnoldi algorithm re-orthogonalization is necessary in order to keep the computed columns of V orthogonal. The following relation will hold

$$AV_{[k]} = V_{[k]}H_{[k]} + [0, \ldots, 0, V_{k+1}H_{k+1,k}]$$

where $H_{[k]}$ is a block upper Hessenberg matrix.

2.1. First order systems

The transfer function of a first-order system (3) is the linear mapping of the Laplace transformation of the input to the output

$$H(s) = C(sE - A)^{-1}B.$$

After expansion in a Laurent expansion series around an expansion point s_0 one obtains the moments $h_j(s_0)$, $j = 0, .., \infty$ of the transfer function

$$\begin{aligned} H(s) &= \sum_{j=0}^{\infty} h_j(s_0)(s - s_0)^j, \\ \text{where} \quad h_j(s_0) &= C[(A - s_0E)^{-1}E]^j(A - s_0E)^{-1}B. \end{aligned}$$

Consider the block Krylov subspace $\mathcal{K}_k(P, Q)$ (8) for

$$P = (A - s_0E)^{-1}E \in \mathbb{R}^{2n\times 2n} \text{ and } Q = (A - s_0E)^{-1}B \in \mathbb{R}^{2n\times m}.$$

Assume that an orthogonal basis for this block Krylov subspace is generated using the block Arnoldi method. Here, and in the rest of the paper, we will assume that no breakdown occurred during the computations so that the column-space of the resulting matrix V spans the block Krylov subspace $\mathcal{K}_k(P, Q)$. Applying the similarity transformation (5)

(with $V = V_{[k]} \in \mathbb{R}^{2n\times r}$, $V^T V = I_r$ and $r = k \cdot m$) yields a reduced system whose transfer function matches at least the first k moments of the transfer function of the original system [8]. That is, at least the first k moments $\hat{h}_j(s_0)$, of the transfer function $\hat{H}(s)$ of the reduced system (4) equal the first moments $h_j(s_0)$, of the transfer function $H(s)$ of the original system (3) at expansion point s_0

$$h_j(s_0) = \hat{h}_j(s_0), \quad j = 0, 1, \cdots, k-1.$$

An alternative is to use more than one expansion point, this leads to multi point moment matching methods called Rational Krylov methods [12]. Assume that $\hat{i}$ expansion points s_i, $i = 1, 2, \cdots, \hat{i}$ are considered. The column vectors of the matrix V are determined from the $\hat{i}$ block Krylov subspaces generated by

$$P = (A - s_i E)^{-1} E \quad \text{and} \quad Q = (A - s_i E)^{-1} B, \quad i = 1, 2, \ldots, \hat{i}. \tag{9}$$

From each of these subspaces, $r_i = k_i \cdot m$ column vectors are used to generate $V \in \mathbb{R}^{2n\times r}$ (with $r = \sum_{i=1}^{\hat{i}} r_i$). Then at least k_i moments are matched per expansion point s_i,

$$h_j(s_i) = \hat{h}_j(s_i), \quad j = 0, 1, \cdots, k_i - 1, \quad i = 1, 2, \cdots, \hat{i}, \tag{10}$$

if the reduced system is generated by (5).

In [13] the choice of expansion points $s_i, i = 1, \ldots, \hat{i}$ is discussed. Starting from an initial set of expansion points a reduced order system is determined. Then a new set of expansion points is chosen as $s_i = -\lambda_i, i = 1, \ldots, \hat{i}$ where λ_i are the eigenvalues of the matrix pencil $\hat{E} - \lambda\hat{A}$ with $\hat{E}, \hat{A}$ as in (4), ordered such that $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_r|$. This algorithm is called Iterative Rational Krylov Algorithm (IRKA) [13]. Here a modified version of IRKA is proposed: A new set of expansion points is chosen from the set of eigenvalues ordered by their imaginary part such that $|\mathrm{Im}(\lambda_1)| \leq |\mathrm{Im}(\lambda_2)| \leq \ldots \leq |\mathrm{Im}(\lambda_r)|$. Starting from $s_1 = \mathrm{Im}(\lambda_1) \cdot \imath$ ($\imath = \sqrt{-1}$) the next expansion points $s_i, i = 2, \ldots, \hat{i}$ are chosen as $s_i = \mathrm{Im}(\lambda_i) \cdot \imath$. As expansion points lying a bit apart yield better approximation results, this choice of the expansion points is refined such that in addition we require $|s_{i-1} - s_i| > \epsilon$, where ϵ is chosen by the user and defines a (minimum) distance between two adjacent expansion points. Hence, if $|s_2 - s_1| \leq \epsilon$, we do not choose $s_2 = \mathrm{Im}(\lambda_2) \cdot \imath$, but test $|s_2 - s_1|$ for $s_2 = \mathrm{Im}(\lambda_3) \cdot \imath$. If this is still small than ϵ, we next test for $s_2 = \mathrm{Im}(\lambda_4) \cdot \imath$, until we have found an λ_k such that $s_2 = \mathrm{Im}(\lambda_k) \cdot \imath$ yields $|s_2 - s_1| > \epsilon$. Next we choose s_3 is the same fashion starting from λ_{k+1} such that $|s_3 - s_2| > \epsilon$. Unlike IRKA, this method cannot be guaranteed to be $\mathcal{H}_2$-optimal but after a few iterations good approximation results of the transfer function, especially for low frequencies, are obtained. The approach described here is summarized as the Modified Iterative Rational Arnoldi algorithm (MIRA) in Algorithm 2.

In [17] a strategy for an adaptive-order model reduction method based on the Arnoldi method is discussed. Given a fixed set of expansion points $s_i, i = 1, \ldots, \hat{i}$ and the reduced dimension r, an adaptive scheme for automatically choosing r_i about each expansion point s_i is proposed, see Chapter 3.2.

2.2. Second order systems

The transfer function of a second-order system is given by the Laplace transformation of (1):

$$H(s) = (C_p + sC_v)(s^2 M + sD + K)^{-1} F.$$

After expansion in a Laurent expansion series around an expansion point s_0 one obtains the moments $h_j(s_0), \quad j = 0, .., \infty$ of the transfer function

$$H(s) = \sum_{j=0}^{\infty} h_j(s_0)(s - s_0)^j,$$

where

$$h_j(s_0) = (\tilde{C}_p + s_0 C_v)[(-s_0^2 M - s_0 \tilde{D} - \tilde{K})^{-1} M]^j (-s_0^2 M - s_0 \tilde{D} - \tilde{K})^{-1} F$$

with $\tilde{D} = 2s_0 M + D$, $\tilde{K} = s_0^2 M + s_0 D + K$ and $\tilde{C}_p = C_p + s_0 C_v$. (In an abuse of notation, we denote the transfer function (the moments) of the first- and the second-order system by H (h_j). It will be clear from the context which

Algorithm 2 Modified Iterative Rational Arnoldi (MIRA)

Input: system matrices, initial expansion points s_i, $i = 1, \ldots, \hat{i}$,
$r_i = m \cdot k_i$, tolerance tol, ϵ
Output: reduced system of order $r = m\,\hat{i}\,\hat{k}$

```
set s_i^old so that max_{i∈{1,2,...î}} |s_i^old − s_i| > tol
while max_{i∈{1,2,...î}} |s_i^old − s_i| > tol do
   V = [ ]
   for k = 1, 2, ..., k̂ do
      for i = 1, 2, ..., î do
         compute V_i using the block Arnoldi method for P and Q as in (9) resp. as in (11) (modified so that
         re-orthogonalizing against the entire matrix V is used)
         V = [V V_i]
      end for
   end for
   compute reduced system matrices with V by (5) resp. (6)
   compute the eigenvalues λ_j, j = 1..., r of the reduced system ordered such that |Im(λ_1)| ≤ |Im(λ_2)| ≤ ... ≤
   |Im(λ_r)|
   s_i^old ← s_i, for i = 1, ..., î
   choose new expansion points s_i as explained at the end of Section 2.1
end while
compute the congruence transformation with V by (5) resp. (6).
```

one is refered to.) Here we consider only the special cases of systems with no or proportional damping matrix. The block Krylov subspace $\mathcal{K}_k(P, Q)$ with

$$P = -(s_0^2 M + s_0 D + K)^{-1} M \quad \text{and} \quad Q = -(s_0^2 M + s_0 D + K)^{-1} F$$

is used to generate the transformation matrix V. In [1] it was shown, that the transfer function of the system reduced by applying the congruence transformation (6) with V matches at least the first k moments of the transfer function of the original system.

If more than one expansion point is used, the $r = \sum_{i=1}^{\hat{i}} r_i$, $r_i = k_i \cdot m$ column vectors of matrix V are determined from the block Krylov subspaces generated by

$$P = -(s_i^2 M + s_i D + K)^{-1} M \quad \text{and} \quad Q = -(s_i^2 M + s_i D + K)^{-1} F. \tag{11}$$

The transfer function of the system reduced by applying the congruence transformation (6) with V matches at least the first k_i moments of the transfer function of the original system per expansion point s_i [1]. As in the case of first-order systems, the iterative approach MIRA for the choice of the expansion points s_i can be used. The pseudo-code of MIRA is given as Algorithm 2.

3. Model Reduction using the Global Arnoldi Method

The global Krylov method was first proposed in [26, 18] for solving linear equations with multiple right hand sides and Lyapunov equations. Applications to model order reductions of first-order systems are studied in [19, 20, 21, 22]. It was also used for solving large Lyapunov matrix equations [27]. The global Krylov method is similar to the standard Krylov method except that the standard inner product is replaced by the inner product $\langle Y, Z\rangle_F = \text{trace}(Y^T Z)$, $Y, Z \in \mathbb{R}^{n\times s}$. The associated norm is the Frobenius norm $\|\cdot\|_F$. A system of vectors (matrices) in $\mathbb{R}^{n\times s}$ is said to be F-orthonormal if it is orthonormal with respect to $\langle\cdot,\cdot\rangle_F$.

The global Arnoldi algorithm [18] (see Algorithm 3) constructs an F-orthonormal basis $V_1, V_2, \ldots, V_k$ with $V_j \in \mathbb{R}^{n\times s}$ of the Krylov subspace $\mathcal{K}_k(\Psi, \Upsilon)$, $\Psi \in \mathbb{R}^{n\times n}$, $\Upsilon \in \mathbb{R}^{n\times s}$; i.e.,

$$\begin{aligned} \langle V_i, V_j\rangle_F &= 0 \qquad i \neq j,\ i, j = 1, \ldots, k, \\ \langle V_j, V_j\rangle_F &= 1. \end{aligned}$$

Algorithm 3 Global Arnoldi method

Input: matrices Ψ, Υ

Output: transformation matrix $\mathbf{V}$

1: function [$\mathbf{V}$] = Global_Arnoldi(Ψ, Υ)
2: $V_1 = \Upsilon / \|\Upsilon\|_F$
3: $V = [V_1]$
4: **for** $j = 1, 2, ..., k$ **do**
5: $W = \Psi V_j$
6: **for** $i = 1, 2, ..., j$ **do**
7: $h_{ij} = \langle V_i, W \rangle_F$
8: $W = W - h_{ij} V_i$
9: **end for**
10: $h_{j+1,j} = \|W\|_F$
11: $V_{j+1} = W / h_{j+1,j}$
12: $V = [V \; V_{j+1}]$
13: **end for**

In terms of computational time, the global Arnoldi method is faster than the block Arnoldi method. Comparing both algorithms, the block Arnoldi method requires a QR decomposition of W in every step (line 2 and 10), while the global Arnoldi method only needs the division by the Frobenius norm of W. Moreover, the block Arnoldi method requires the computation of $V^T W$ in line 7, while the global Arnoldi method only needs the trace of that matrix. Finally, in line 8 of the algorithm, the block Arnoldi method requires a matrix-matrix-product, while the global Arnoldi method only needs a scalar-matrix-product.

If $s = 1$, the global Arnoldi algorithm reduces to the standard Arnoldi algorithm. Let $V_{(k)} = [V_1 \; V_2 \; \dots V_k] \in \mathbb{R}^{n \times r}$, $r = k \cdot s$ and H_k the corresponding $k \times k$ upper Hessenberg matrix. The following relation will hold

$$\Psi V_{(k)} = V_{(k)}(H_k \otimes I_s) + h_{k+1,k}[0, \dots, 0, V_{k+1}].$$

Here $\otimes$ denotes the Kronecker product of two matrices $X \in \mathbb{R}^{u \times u}$ and $Y \in \mathbb{R}^{v \times v}$

$$X \otimes Y = \begin{pmatrix} x_{11}Y & x_{12}Y & \cdots & x_{1u}Y \\ x_{21}Y & x_{22}Y & \cdots & x_{2u}Y \\ \vdots & \vdots & \vdots & \vdots \\ x_{u1}Y & x_{u2}Y & \cdots & x_{uu}Y \end{pmatrix} = [x_{ij}Y]_{i,j=1}^{u} \in \mathbb{R}^{uv \times uv}.$$

Note that the Hessenberg matrix H_k in the global Arnoldi algorithm is of dimension $k \times k$ while for the block Arnoldi algorithm $H_{[k]}$ is a block Hessenberg matrix of dimension $\ell k \times \ell k$. Moreover, as noted in [18], linear dependence between the column vectors of the generated matrices $V_i, i = 1, \dots, k$ has no effect on the global Arnoldi algorithm. The major difference between the global and the block Arnoldi algorithm lies in the computed basis of the Krylov subspace: the global Arnoldi algorithm allows to generate the F-orthonormal basis, while the block Arnoldi algorithm constructs an orthogonal basis. The matrices constructed by the block Arnoldi algorithm have their columns mutually orthogonal. Finally, note that the block Arnoldi algorithm constructs an orthonormal basis of the block Krylov subspace $\mathcal{K}_k(\Psi, \Upsilon) \subset \mathbb{R}^n$ while the global Arnoldi algorithm generates an F-orthonormal basis of the matrix Krylov subspace $\mathcal{K}_k(\Psi, \Upsilon) \subset \mathbb{R}^{n \times s}$.

If the global Arnoldi method and the block Arnoldi method are applied to the same matrix pair (Ψ, Υ), the resulting matrices $V_{(k)}$ and $V_{[k]}$ both span the same Krylov subspace $\mathcal{K}_k(\Psi, \Upsilon)$. The orthonormalization of the bases vectors of $\mathcal{K}_k(\Psi, \Upsilon)$ is the only difference whether constructed by the the block- or the global-Arnoldi method. In [12, Chapter 3] it is shown that the moment matching property (10) does only depend on the fact that the columns of $V = V_{[k]}$ resp. $V = V_{(k)}$ span the Krylov subspace $\mathcal{K}_k(P, Q)$ with (P, Q) as in (9) resp. in (11). It does not depend on the way V is computed or whether its columns have a certain additional property. Hence, the moment matching property holds for reduction methods based on the global Arnoldi algorithm as well as for reduction methods based on the block Arnoldi algorithm.

3.1. First order systems

The global Arnoldi method is the standard Arnoldi method applied to the matrix pair $((I \otimes P^j), \mathrm{vec}\,(Q)) = (\Psi, \Upsilon)$, where the inner product is replaced by $\langle X, Y\rangle_F = \mathrm{vec}\,(X)^T \mathrm{vec}\,(Y) = \mathrm{trace}(X^T Y)$. Here $\mathrm{vec}\,(\cdot)$ denotes the usual vector stacking operation [28]

$$\mathrm{vec}\,(Z) = (Z_{*1}\, Z_{*2} \cdots Z_{*v})^T \in \mathbb{R}^{uv}, \qquad Z = [Z_{*1}\, Z_{*2} \cdots Z_{*v}] \in \mathbb{R}^{u\times v},\ Z_{*j} \in \mathbb{R}^u,\ j = 1, \ldots, v.$$

For $\mathrm{vec}\,(P^j, Q)$ we have

$$\mathrm{vec}\,(P^j, Q) = (I \otimes P^j)\,\mathrm{vec}\,(Q).$$

Therefore, with P and Q as in (9), the moments of the system (3) can be associated with a vector-function.

Assume that an F-orthogonal basis for the Krylov subspace $\mathcal{K}_k(P, Q)$ for (P, Q) as in (9) is generated using the global Arnoldi method. Applying the projection $\Pi = V_{(k)} V_{(k)}^\dagger$ to (3) leads to the reduced system with

$$\hat{E} = V_{(k)}^\dagger E V_{(k)}, \qquad \hat{A} = V_{(k)}^\dagger A V_{(k)}, \qquad \hat{B} = V_{(k)}^\dagger B, \qquad \text{and} \qquad \hat{C} = C V_{(k)}. \tag{12}$$

As $V_{(k)}$ is F-orthonormal, the pseudo-inverse $V_{(k)}^\dagger = (V_{(k)}^T V_{(k)})^{-1} V_{(k)}^T$ has to be used instead of $V_{(k)}^T$.

If the global Arnoldi method and the block Arnoldi method are applied to the matrix pair (P, Q) as in (9), the moment matching property (10) discussed for the block Arnoldi based model reduction is still valid. The first moments $\hat{h}_j(s_0)$, of the transfer function of the reduced system (4) are the same as those of the original system (3). The Iterative Rational Arnoldi algorithm as in Algorithm 2 is easily modified to make use of the global Arnoldi method instead of the block Arnoldi algorithm. Besides the change of the algorithm to be used in line 6 of Algorithm 2 the reduced systems have to be determined using (12) (resp. (13)). The resulting algorithm is called Iterative Rational Global Arnoldi algorithm (IRGA).

3.2. Second order systems

Assume that an F-orthogonal basis $V_{(k)}$ for the Krylov subspace $\mathcal{K}_k(P, Q)$ for (P, Q) as in (11) is generated using the global Arnoldi method. The reduced system is then given by applying the projection $\Pi = V_{(k)} V_{(k)}^\dagger$ to (1) such that

$$\begin{aligned} \hat{M} = V_{(k)}^\dagger M V_{(k)}, \qquad \hat{D} = V_{(k)}^\dagger D V_{(k)}, \qquad \hat{K} = V_{(k)}^\dagger K V_{(k)}, \\ \hat{F} = V_{(k)}^\dagger F, \qquad \hat{C}_p = C_p V_{(k)}, \quad \text{and} \quad \hat{C}_v = C_v V_{(k)}. \end{aligned} \tag{13}$$

If the global Arnoldi method and the block Arnoldi method are applied to the matrix pair (P, Q) as in (11) the moment matching property discussed for the block Arnoldi based model reduction is here still valid. The first moments $\hat{h}_j(s_0)$, of the transfer function of the reduced system (13) are the same as those of the original system (1).

In the previous section, the r_i were chosen as $m \cdot k_i$ so that for each expansion point at least k_i moments are matched. Here a different approach suggested in [19, 29] is used which adaptively determines the r_i. The Adaptive Order Rational Global Arnoldi (AORGA) algorithm describes an adaptive scheme for automatically choosing r_i about each expansion point s_i given a fixed set of expansion points $s_i,\ i = 1, \ldots \hat{i}$ and the reduced dimension r. In the j-th iteration of AORGA an expansion point from the set of fixed expansion points corresponding to the maximum output moment error will be chosen to compute V_j. Consequently, the corresponding reduced system will yield the greatest output moment improvement among all reduced systems of the same order and the same set of expansion points.

In [17] the exact j-th moment error of reduced first-order systems at an expansion point s_i has been determined analytically

$$\|h_j(s_i) - \hat{h}_j(s_i)\|_F = \|h_\pi(s_i)\, C\, R^{(j-1)}(s_i)\|_F, \tag{14}$$

where $h_j(s_i)$ and $\hat{h}_j(s_i)$ are the j-th moments of the original resp. the reduced first-order system, $h_\pi(s_i) = \prod_{k=1}^{j-1} \|R^{(k)}(s_i)\|_F$, and $R^k(s_i)$ is the residual at expansion point s_i defined by

$$R^k(s_i) = (s_i E - A)^{-1} B, \quad \text{for } k = 1, \quad \text{and} \quad R^k(s_i) = (s_i E - A)^{-1} E V_{k-1}, \quad \text{for } k = 2, \ldots, j-1.$$

In the j-th iteration of AORGA expression (14) is used to determine the expansion point σ_j corresponding to the maximum moment error by

$$\sigma_j = \max_{s_i} \|h_\pi(s_i)\, C\, R^{(j-1)}(s_i)\|_F.$$

For more details of the algorithm see [19].

Here an algorithm to reduce second-order systems combining this approach and IRGA is proposed. This algorithm, called Adaptive Iterative Rational Global Arnoldi algorithm (AIRGA), computes a reduced system by determining the expansion points iteratively and the number of matched moments per expansion point adaptively. The method is given in pseudo-code as Algorithm 4. Modifying (14) by changing the first-order system output matrix C to $(C_p + s_i C_v)$, where C_v and C_p are the second-order system output matrices, and $R^k(s_i)$ to

$$R^k(s_i) = -(s_i^2 M + s_i D + K)^{-1} F, \quad \text{for } k = 1, \quad \text{and} \quad R^k(s_i) = -(s_i^2 M + s_i D + K)^{-1} M V_{k-1}, \quad \text{for } k = 2, \ldots, j-1,$$

the error of the j-th moment $\hat{h}_j(s_i)$ at expansion point s_i, $i = 1, \ldots, \hat{i}$ of the second-order system is given by

$$\|h_j(s_i) - \hat{h}_j(s_i)\|_F = \|h_\pi(s_i)(C_p + s_i C_v) R^{(j-1)}(s_i)\|_F,$$

where $h_j(s_i)$ and $\hat{h}_j(s_i)$ are the j-th moments of the original resp. of the reduced second-order system. In the j-th iteration of Algorithm 4 (line 12) this approach is used to determine the expansion point σ_j corresponding to the maximum j-th moment error of the reduced second-order system by

$$\sigma_j = \max_{s_i} \|h_\pi(s_i)\, (C_p + s_i C_v)\, R^{(j-1)}(s_i)\|_F.$$

4. Numerical Results

Our test model is a simplified, abstract mechanical structure of a machine tool modeled using the FEM environment MSC.PATRAN/MSC.NASTRAN© (see Figure 1, here TCP denotes the tool center point). The test model is of

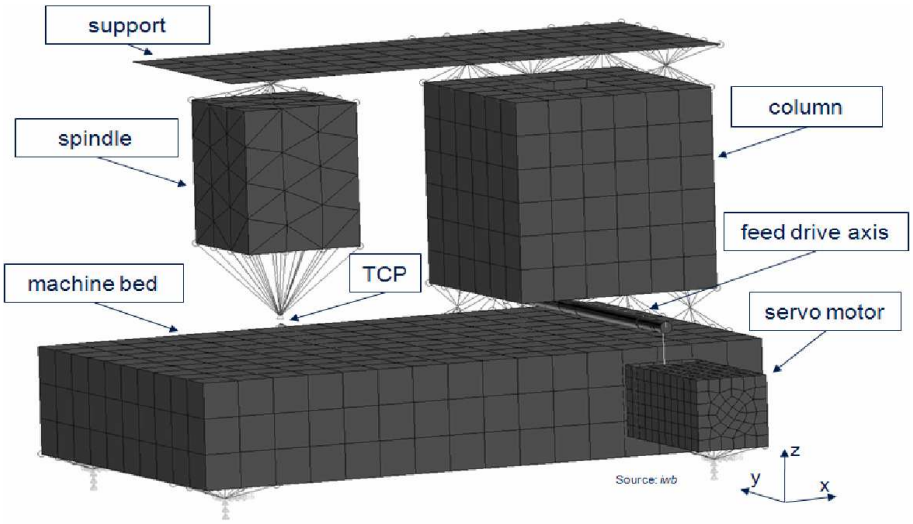

Figure 1: FE model of a simplified, abstract mechanical structure.

order $n = 51.816$, it has four inputs ($m = 4$) and eight outputs ($q = 8$). The damping matrix D was chosen as Rayleigh damping

$$D = \alpha \cdot M + \beta \cdot K,$$

i.e. D is proportional to the mass matrix M and the stiffness matrix K. The parameters for the proportional damping matrix were chosen as $\alpha = 0.02$ and $\beta = \alpha/1500$.

Algorithm 4 Adaptive Iterative Rational Global Arnoldi (AIRGA)

Input: matrices M, D, K, F, C_p, C_v, number of columns of matrix F m, initial expansion points s_i, $i = 1, \ldots, \hat{i}$, reduced dimension r, tolerance tol, ϵ

Output: reduced matrices $\hat{M}, \hat{D}, \hat{K}, \hat{F}, \hat{C}_p, \hat{C}_v$, number of expansions r_i, $i = 1, \ldots, \hat{i}$, sequence of used expansion points σ_j, $j = 1, \ldots, \lceil r/m \rceil$

set s_i^{old} so that $\max_{i \in \{1,2,\ldots \hat{i}\}} |s_i^{old} - s_i| > tol$
$J := \lceil r/m \rceil$
while $\max_{i \in \{1,2,\ldots \hat{i}\}} |s_i^{old} - s_i| > tol$ **do**
  $V = [\,]$
  for each s_i **do**
    $R^{(0)}(s_i) := -(s_i^2 M + s_i D + K)^{-1} F$
    $h_\pi(s_i) := 1$
  end for
  /* compute transformation matrix */
  for $j = 1, 2, \ldots, J$ **do**
    /* Select the expansion point with maximum output moment error */
    Choose s_i with $\max_{s_i} ||h_\pi(s_i)(C_p + s_i C_v) R^{(j-1)}(s_i)||_F$
    Let $\sigma_j = s_i$ be the expansion point in the j-th iteration
    /* Generate m new F-orthonormal vectors at σ_j */
    $h_{j,j-1}(\sigma_j) := ||R^{(j-1)}(\sigma_j)||_F$
    $V_j := R^{(j-1)}(\sigma_j)/h_{j,j-1}(\sigma_j)$
    $h_\pi(\sigma_j) := h_\pi(\sigma_j) \cdot h_{j,j-1}(\sigma_j)$
    /* Update $R^{(j)}(s_i)$ for the next iteration */
    for $i = 1, \ldots, \hat{i}$ **do**
      if $(s_i == \sigma_j)$ **then**
        $R^{(j)}(s_i) := -(s_i^2 M + s_i D + K)^{-1} M V_j$
      else
        $R^{(j)}(s_i) := R^{(j-1)}(s_i)$
      end if
      for $t = 1, 2, \ldots, j$ **do**
        $h_{t,j}(s_i) := \mathrm{trace}(V_t R^{(j)}(s_i))$
        $R^{(j)}(s_i) := R^{(j)}(s_i) - h_{t,j}(s_i) V_t$
      end for
    end for
  end for
  $V = [V_1 V_2 \cdots V_J]$
  $V = V(:, 1 : r)$
  /* transform matrices M, D and K */
  compute $\hat{M} = V^\dagger M V$, $\hat{D} = V^\dagger D V$ and $\hat{K} = V^\dagger K V$ as in (13)
  /* determine a new set of expansion points */
  compute the eigenvalues λ_j, $j = 1 \ldots, r$ of the reduced system ordered such that $|\mathrm{Im}(\lambda_1)| \leq |\mathrm{Im}(\lambda_2)| \leq \ldots \leq |\mathrm{Im}(\lambda_r)|$
  $s_i^{old} \leftarrow s_i$, for $i = 1, \ldots, \hat{i}$
  choose new expansion points s_i as explained at the end of Section 2.1
end while
/* Yield reduced system matrices by congruence transformation */
compute $\hat{M}, \hat{D}, \hat{K}, \hat{F}, \hat{C}_p, \hat{C}_v$ as in (13).

The algorithms were implemented in MATLAB[1] version 7.1 (R14) and the computations were performed on a

[1]MATLAB is a trademark of The MathWorks, Inc.

AMD Athlon(tm) 64 X2 Dual Core Processor 4400+ and 2 GB RAM. The second-order system was reduced by the following methods:

1. MIRA for systems without damping matrix which generates a Galerkin projection (6) from P and Q as in (11) by the block Arnoldi method (RA).
2. IRGA for systems without damping matrix which generates a projection (13) from P and Q as in (11) by the global Arnoldi method (GA).
3. AIRGA for systems without damping matrix which generates a projection (13) from P and Q as in (11) by the adaptive global Arnoldi method (AGA).
4. MIRA for systems with proportional damping which generates a Galerkin projection (6) from P and Q as in (11) by the block Arnoldi method (RA_PD).
5. IRGA for systems with proportional damping which generates a projection (13) from P and Q as in (11) by the global Arnoldi method (GA_PD).
6. AIRGA for systems with proportional damping which generates a projection (13) from P and Q as in (11) by the adaptive global Arnoldi method (AGA_PD).

The first three methods are modified versions of the Rational-Arnoldi method for first-order systems. They reduce second-order systems without damping matrix [12, 30]. That is, they assume $D = 0$ in (11) and compute a reduced system (2) with $\tilde{D} = 0$. A damped reduced system is obtained by adding the proportional damping matrix $\hat{D} = \alpha\hat{M} + \beta\hat{K}$. The last three methods exploit the special structure of the proportional damping matrix [1]. In case complex valued expansion points are used, the last three of the above algorithms generate complex valued matrices V. The algorithms RA, GA and AGA generate real matrices even in case of complex valued expansion points.

Once a complex expansion point is used, in the algorithms to reduce systems with proportional damping matrix, all further computations involve complex arithmetic. As the reduced systems are commonly used for further simulation in NASTRAN or SIMULINK which requires real systems, the following considerations had to be taken into account. Before computing the congruence transformation the transformation matrix V has to be transformed back to a real matrix. This can be done, e.g., as follows

$$[V, R] = \mathrm{qr}(\,[\mathrm{Re}(V(:, 1 : \lceil r/2 \rceil))\ \mathrm{Im}(V(:, 1 : \lfloor r/2 \rfloor))]\,). \tag{15}$$

In our implementation a rank-revealing QR decomposition was used to compute (15). By this process the number of columns of V doubles. Therefore, in setting up the real transformation matrices only the first $r/2$ columns of V were used, so that the resulting system is of the desired order r. Note that this process halves the number of matched moments.

The methods were started with the four initial expansion points $2\pi\imath$, $500\pi\imath$, $1000\pi\imath$ and $1500\pi\imath$. To match at least the first two moments (methods RA, GA and AGA) resp. at least the first moment (methods RA_PD, GA_PD and AGA_PD) of the original systems in our computations, the reduction methods where used with $\hat{k}_i = 2$, $\hat{i} = 4$, $tol = 0.1$ and $\epsilon = 750$. With these parameters the reduced dimension was $r = k_i \cdot \hat{i} \cdot m = 2 \cdot 4 \cdot 4 = 32$. With the procedure to choose the expansion points s_i as explained at the end of section 2.1 the last expansion point is located at frequencies higher than $\hat{i} \cdot \epsilon/(2\pi) = 358$ Hz. Hence, we expect a good approximation of the frequency range from 0 Hz to 358 Hz at least. Besides the methods already mentioned, a second-order modal reduced system of dimension 32 (modal_2o) was generated by NASTRAN in order to compare the approximation results of the various reduction methods. In Table 1 essential information about the results obtained with the various methods is summarized. The sequence of expansion points adaptively determined in the last iteration by the method AGA was $s_1, s_2, s_1, s_3, s_4, s_2, s_3, s_4$ resp. $s_1, s_2, s_3, s_4, s_1, s_2, s_3, s_4$ by the method AGA_PD. The numerical results demonstrate the applicability of the proposed methods based on the global Arnoldi method. The approximation of the transfer function and the time response by reduced systems obtained with global Arnoldi methods are comparable to those obtained with block Arnoldi reduced systems. All Krylov based reduced systems approximate the original system transfer function more accurate than the modal reduced system of the same order.

In order to compare the different methods the approximation of the original transfer function and the time response of the reduced systems were analyzed. To assess the quality of the reduced systems the following errors were used:

methods for systems without damping matrix				
	$V, W \in$	expansion points s_i in the last iteration	time to reduce the system [s]	number of iterations
RA	$\mathbb{R}^{32\times32}$	$2\pi(74.28\imath, 250.05\imath, 435.59\imath, 588, 76\imath)$	173,70	2
GA	$\mathbb{R}^{32\times32}$	$2\pi(74.28\imath, 250.05\imath, 435.59\imath, 588, 76\imath)$	159,96	2
AGA	$\mathbb{R}^{32\times32}$	$2\pi(74.28\imath, 250.05\imath, 435.59\imath, 588, 76\imath)$	174,26	2

methods for systems with proportional damping				
	$V, W \in$	expansion points s_i in the last iteration	time to reduce the system [s]	number of iterations
RA_PD	$\mathbb{C}^{32\times32}$	$2\pi(74.28\imath, 250.05\imath, 435.59\imath, 588, 76\imath)$	421,10	2
GA_PD	$\mathbb{C}^{32\times32}$	$2\pi(74.28\imath, 250.05\imath, 435.59\imath, 588, 76\imath)$	396,52	2
AGA_PD	$\mathbb{C}^{32\times32}$	$2\pi(74.28\imath, 250.05\imath, 435.59\imath, 588, 76\imath)$	440,72	2

Table 1: Information about the results obtained with the various Krylov subspace methods.

- The relative error of the transfer function at frequency f from the k-th input to the l-th output of a reduced system was computed by

$$\epsilon_{rel}(f) = \frac{|H_{k,l}(f) - \hat{H}_{k,l}(f)|}{|H_{k,l}(f)|}.$$

  Here $\epsilon_{rel}(f)$ is the relative error, $H_{k,l}$ and $\hat{H}_{k,l}$ are the transfer functions from the k-th input to the l-th output of the original resp. of the reduced system.

- The absolute time response error at time t from the k-th input to the l-th output of a reduced system was computed by

$$\epsilon_{abs}(t) = |y_{k,l}(t) - \hat{y}_{k,l}(t)|.$$

  Here $\epsilon_{abs}(t)$ is the absolute error, $y_{k,l}$ and $\hat{y}_{k,l}$ are the time responses from the k-th input to the l-th output of the original resp. of the reduced system.

4.1. Approximation of the transfer function

In Figure 2 the transfer function and the time response of the TCP's relative motion (5'th output) against the motor torque (1'st input) and in Figure 3 the relative approximation errors $\epsilon_{rel}(f)$ of the reduced systems are displayed. The relevant frequency interval is from 0 to 750 Hz because this frequency range is most important to simulate the behavior of mechanical structures. The results of the proposed methods for the system reduced without damping matrix D are displayed on the left in each of the figures, while the results for the system reduced with proportional damping matrix are given on the right hand side. Besides the relative error for the different Krylov subspace reduced models all figures also include the relative error for the second-order modal reduced system (modal_2o) of dimension 32.

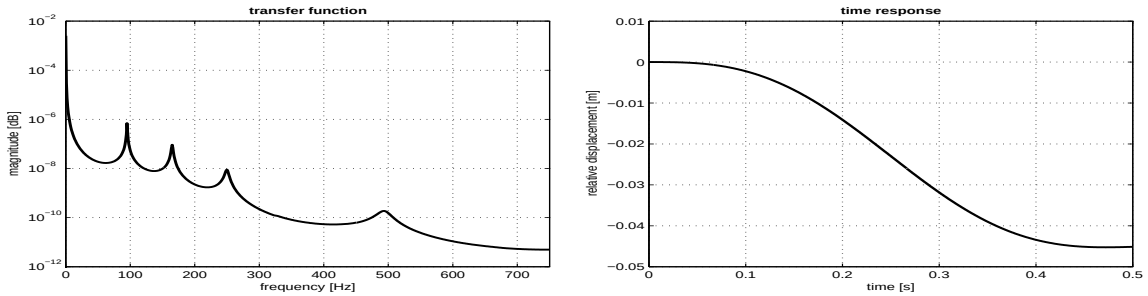

Figure 2: Transfer function and time response from the 1'st input to the 5'th output of the original system.

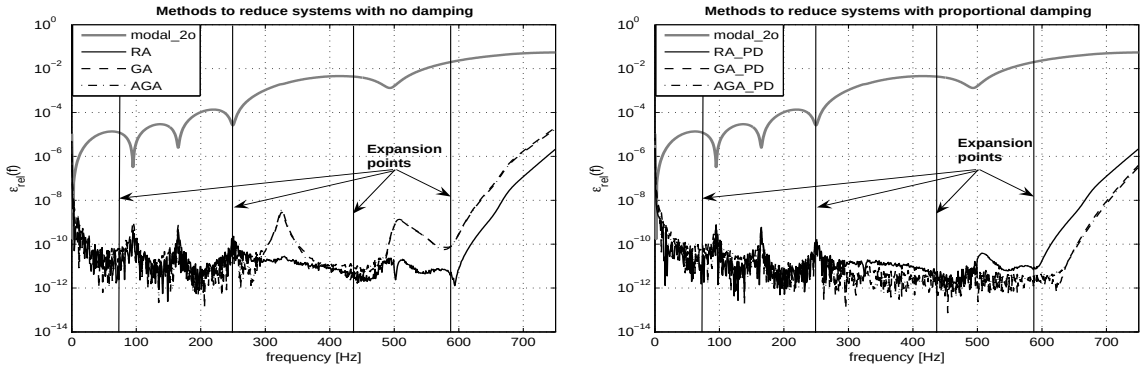

Figure 3: Relative error $\epsilon_{rel}(f)$.

Clearly, all Krylov subspace reduced systems approximate the original system more accurate than the modal reduced system in the frequency interval considered here. All methods achieve reduced systems with lower approximation errors than the modal reduced system of the same order. As expected all reduced systems approximate the original transfer function up to a frequency of 358 Hz and higher very accurately.

All Krylov subspace reduced systems obtained by reduction of a system without damping matrix have an error smaller than $5 \cdot 10^{-9}$ for frequencies up to 600 Hz. For higher frequencies the error increases. The maximum errors of the global Arnoldi methods GA and AGA (10^{-5}) are slightly higher than of the block Arnoldi method RA (10^{-6}).

All Krylov subspace reduced systems obtained by reduction of a system with proportional damping matrix have a maximal error of 10^{-9} for frequencies up to 600 Hz. Here the maximum errors of the block Arnoldi method RA_PD (10^{-6}) are slightly higher than that of the global Arnoldi methods GA and AGA ($5 \cdot 10^{-7}$). In the whole frequency range considered here the approximation errors of this systems are very similar or smaller than those of systems achieved by the methods without damping matrix.

4.2. Approximation of the time response

To analyze the approximation abilities of the response behavior in time domain an input signal $u(t) = 1000 \cdot sin(4\pi t)$ was used as torque on the motor shaft. In Figure 4 the absolute approximation errors $\epsilon_{abs}(t)$ from the 1'st input to the 5'th output of the reduced systems are displayed. The relevant time interval is from 0 to 0,5 sec. In the entire time

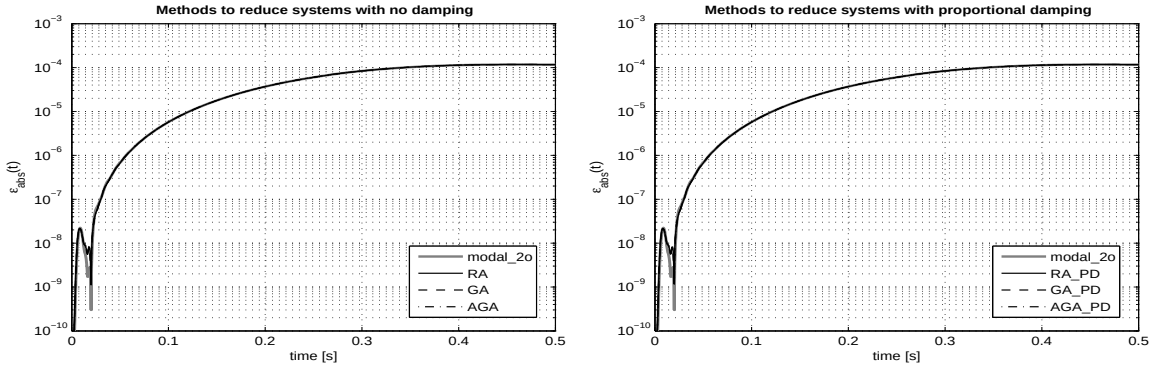

Figure 4: Absolute error $\epsilon_{abs}(t)$ of the reduced systems.

interval considered here all reduced systems approximate the time response of the original system very accurately with maximal absolute error smaller than $2 \cdot 10^{-4}$. The errors of all Krylov subspace reduced systems and the error of the modal reduced system are nearly the same.

5. Conclusions

In this paper we propose a novel multi point adaptive method combining the Adaptive Order Rational Global Arnoldi (AORGA) algorithm and a modified Iterative Rational Krylov Algorithm (IRKA) for application in model order reduction of second-order systems. Starting from a arbitrary initial set of expansion points the method determines the expansion points iteratively and the expansions per expansion point adaptively. Numerical experiments show good approximation results of the time response and the transfer function, especially for low frequencies. This frequency range is most important to simulate the behavior of mechanical structures. Further investigations by adoption this method to the global Lanczos algorithm and for Krylov subspaces of second kind are still in work.

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References

[1] S. Gugercin, C. Beattie, Krylov-based model reduction of second-order systems with proportional damping, in: Proceedings of the 44th IEEE Conference on Decision and Control, 2005, pp. 5905–5910.

[2] S. Salimbahrami, Structure Preserving Order Reduction of Large Scale Second Order Models, Ph.D. thesis, Technical University of Munich, Germany (2007).

[3] Z. Bai, Y. Su, SOAR: A Second-order Arnoldi Method for the Solution of the Quadratic Eigenvalue Problem, SIAM Journal on Matrix Analysis and Applications 26 (3) (2005) 640–659.

[4] N. Higham, D. Mackey, N. Mackey, F. Tisseur, Symmetric linearizations for matrix polynomials, SIAM J. Matrix Anal. Appl. 29 (1) (2006) 143–159.

[5] D. Mackey, N. Mackey, C. Mehl, V. Mehrmann, Vector spaces of linearizations for matrix polynomials, SIAM J. Matrix Anal. Appl. 28 (4) (2006) 971–1004.

[6] F. Tisseur, K. Merbergen, A survey of the quadratic eigenvalue problem, SIAM Review 43 (2001) 234–286.

[7] M. Bampton, R. Craig, Coupling of Substructures for Dynamic Analysis, AIAA Journal 6 (1968) 1313–1319.

[8] A. Antoulas, Approximation of Large-Scale Dynamical Systems, SIAM, Philadelphia, USA, 2005.

[9] P. Feldmann, R. W. Freund, Efficient linear circuit analysis by Padé approximation via the Lanczos process, IEEE Trans. Computer-Aided Design 14 (1995) 639–649.

[10] A. Odabasioglu, M. Celik, L. Pileggi, PRIMA: Passive Reduced-Order Interconnect Macromodeling Algorithm, IEEE Trans. Computer-Aided Design 17 (8) (1997) 58–65.

[11] K. Gallivan, E. Grimme, P. van Dooren, A rational Lanczos algorithm for model reduction, Numerical Algorithms 12 (1996) 33–63.

[12] E. Grimme, Krylov projection methods for model reduction, Ph.D. thesis, University of Illinois at Urbana-Champaign (1997).

[13] S. Gugercin, A. Antoulas, C. Beattie, $\mathcal{H}_2$-Model Reduction for Large-Scale Linear Systems, SIAM Journal on Matrix Analysis and Applications 30 (2) (2008) 609–638.

[14] E. Grimme, K. Gallivan, A rational lanczos algorithm for model reduction ii: Interpolation point selection, Numerical Algorithms 12 (1998) 33–63.

[15] S. Gugercin, A. Antoulas, An $\mathcal{H}_2$ error expression for the Lanczos procedure, in: Proceedings of the 2003 IEEE Conference on Decision and Control, 2003, pp. 1869–1872.

[16] M. Frangos, I. M. Jaimoukha, Adaptive rational interpolation: Arnoldi and Lanczos-like equations, European Journal of Control (2008) 342–354.

[17] H.-J. Lee, C.-C. Chu, W. Feng, An adaptive-order rational Arnoldi method for model-order reductions of linear time-invariant systems, Linear Algebra Appl. 415 (2006) 235–261.

[18] K. Jbilou, A. Messaoudi, H. Sadok, Global FOM and GMRES algorithms for matrix equations, Appl. Numer. Math. 31 (1999) 49–63.

[19] C.-C. Chu, M. Lai, W. Feng, MIMO Interconnects Order Reductions by Using the Multiple Point Adaptive-Order Rational Global Arnoldi Algorithm, IEICE Trans. Electron. E89-C(6) (2006) 792–802.

[20] C.-C. Chu, M. Lai, W. Feng, The multiple point global Lanczos method for multiple-inputs multiple-outputs interconnect order reduction, IEICE Trans. Fundam. E89-A(10) (2006) 2706–2716.

[21] C.-C. Chu, M. Lai, W. Feng, Lyapunov-based error estimations of MIMO interconnect reductions by using the global Arnoldi algorithm, IEICE Trans. Electron. E90-A(2) (2007) 415–418.

[22] C.-C. Chu, M. Lai, W. Feng, Model-order reductions for MIMO systems using global Krylov subspace methods, Mathematics and Computers in Simulation 79 (2008) 1153–1164.

[23] G.H.Golub, C. van Loan, Matrix Computations, John Hopkins University Press, 1996.

[24] G. Stewart, Matrix Algorithms, Volume II: Eigensystems, SIAM, Philadelphia, USA, 2001.

[25] M. Sadkane, Block Arnoldi and Davidson methods for unsymmetric large eigenvalue problems, Numer. Math. 64 (1993) 195–211.

[26] M. Heyouni, The global Hessenberg and CMRH methods for linear systems with multiple right-hand sides, Numer. Algorithms 26 (4) (2001) 317–332.

[27] K. Jbilou, A. Riquet, Projection methods for large Lyapunov matrix equations, Linear Algebra and its Applications 415 (2-3) (2006) 344–358.
[28] R. Horn, C. Johnson, Topics in Matrix Analysis, Cambridge University Press, 1991.
[29] M.-H. Lai, C.-C. Chu, W.-S. Feng, Applications of AOGL Model-Order Reduction Techniques in Interconnect Analysis, in: Proceedings of the 2007 IEEE International Symposium on Circuits and Systems, 2007, pp. 1133–1136.
[30] R. Eid, B. Salimbahrami, B. Lohmann, E. Rudnyi, J. Korvink, Parametric Order Reduction of Proportionally Damped Second Order Systems, Technical Report on Automatic Control TRAC-1, Technische Universität München, Lehrstuhl für Regelungstechnik.