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Research Article

Series Solutions of Time-Fractional PDEs by Homotopy Analysis Method

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The homotopy analysis method (HAM) is applied to solve linear and nonlinear fractional partial differential equations (fPDEs). The fractional derivatives are described by Caputo's sense. Series solutions of the fPDEs are obtained. A convergence theorem for the series solution is also given. The test examples, which include a variable coefficient, inhomogeneous and hyperbolic-type equations, demonstrate the capability of HAM for nonlinear fPDEs.

1. Introduction

Fractional calculus has been given considerable popularity and importance during the past three decades, due mainly to its applications in numerous fields of science and engineering. For example, phenomena in the areas of fluid flow, rheology, electrical networks, probability and statistics, control theory of dynamical systems, electrochemistry of corrosion, chemical physics, optics and signal processing, and so on can be successfully modelled by linear or nonlinear fractional differential equations (fDEs) [1–4].

Finding accurate methods for solving nonlinear differential equations has become important. Some of the analytical methods for nonlinear differential equations are the Adomian decomposition method (ADM) [5–14], the homotopy-perturbation method (HPM) [15–19], variational iteration method (VIM) [12, 20–24], and the EXP-function method [25]. Another analytical approach that can be applied to solve nonlinear differential equations is to employ the homotopy analysis method (HAM) [26–29]. Some of the recent applications of HAM can be found in [30–41]. An account of the recent developments of HAM was given by Liao [42]. HAM has been successfully applied into engineering fields. The method has been applied to give an explicit solution for the Riemann problem of the nonlinear shallow-water equations [43]. The obtained Riemann solver has been implemented into a numerical model to simulate long waves, such as storm surge or tsunami, propagation and run-up.

Very recently, Song and Zhang [44] applied HAM to solve fractional KdV-Burgers-Kuramoto equation. Cang et al. [45] solved nonlinear Riccati differential equations of fractional order using HAM. Hashim et al. [46] employed HAM to solve fractional initial value problems (fIVPs) for ordinary differential equations. In [47], the applicability of the HAM was extended to construct numerical solution for the fractional BBM-Burgers equation. The HAM solutions for systems of nonlinear fractional differential equations were presented by Bataineh et al. [48].

A specific linear, nonhomogeneous time fractional partial differential equation (fPDE) with variable coefficients was first transformed to two fractional ordinary differential equations which were then solved by HAM in [49]. Recently, Xu et al. [50] applied the HAM to linear, homogeneous one- and two-dimensional fractional heat-like PDEs subject to the Neumann boundary conditions. Jafari and Seifi [51] applied HAM to linear and nonlinear homogeneous fractional diffusion-wave equations. Very recently, the HAM was shown to be capable of solving linear and nonlinear systems of fPDEs [52].

In this paper, we shall consider linear and nonlinear fPDEs of the form

$$D_t^{\alpha} u(x,t) = f(u, u_x, u_{xx}), \quad n-1 < \alpha \le n, \ t > 0, \tag{1.1}$$

subject to the initial conditions

$$u^{(k)}(x,0) = g_k(x), \quad k = 0,1,2,\ldots,n-1, \tag{1.2}$$

where n is an integer, f is a linear/nonlinear function, and $D_t^{\alpha}(\cdot) = \partial^{\alpha}(\cdot)/\partial t^{\alpha}$ is a fractional differential operator. We shall demonstrate the applicability of HAM to fPDEs through several linear and nonlinear test examples.

2. Preliminaries

The fractional derivative is defined in the Caputo sense as in [53],

$$D^{\alpha} w(t) = J^{n-\alpha} D^{n} w(t). \tag{2.1}$$

Here D^n is the usual integer differential operator of order m and J^{β} is the Riemann-Liouville fractional integral operator of order $\beta > 0$, defined by

$$\begin{gathered} J^{\beta} w(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} w(\tau) d\tau, \quad (t > 0), \\ J^{0} w(t) = w(t). \end{gathered} \tag{2.2}$$

Some of the properties of the operator J^{β}, which we will need in our work, are as follows [2, 3]:

(1) $J^{\beta} J^{\mu} w(t) = J^{\beta+\mu} w(t)$,

(2) $J^{\beta} J^{\mu} w(t) = J^{\mu} J^{\beta} w(t)$,

(3) $J^{\beta} t^{\gamma} = (\Gamma(\gamma+1)/\Gamma(\beta+\gamma+1)) t^{\beta+\gamma}$.

Caputo's fractional derivative has a useful property [54]

$$\left(J^{\mu}D^{\mu}\right)w(t) = w(t) - \sum_{k=0}^{n-1} w^{(k)}\left(0^{+}\right)\frac{t^{k}}{k!}, \quad (n-1 < \mu \leq n). \tag{2.3}$$

The operator form of the nonlinear fPDEs (1.1) can be written as follows:

$$D_{t}^{\alpha}\left(u(x,t)\right) = A\left(u,u_{x},u_{xx}\right) + B\left(u,u_{x},u_{xx}\right) + C(x,t), \quad n-1 < \alpha \leq n,\ t > 0, \tag{2.4}$$

subject to the initial conditions

$$u^{(k)}(x,0) = g_{k}(x), \quad k = 0,1,2,\ldots,n-1, \tag{2.5}$$

where A is a linear operator which might include other fractional derivatives of order less that α, B is a nonlinear operator which also might include other fractional derivatives of order less that α and C is a known analytic function.

Applying the operator J^{α}, the inverse operator of D^{α}, to both sides of (2.4) with considering the initial conditions (2.5) according to (2.3), we obtain

$$u(x,t) = \sum_{k=0}^{n-1} g_{k}(x)\frac{t^{k}}{k!} + J^{\alpha}C(x,t) + J^{\alpha}A\left(u,u_{x},u_{xx}\right) + J^{\alpha}B\left(u,u_{x},u_{xx}\right), \quad n-1 < \alpha \leq n,\ t > 0. \tag{2.6}$$

3. Homotopy analysis method (HAM)

3.1. The zeroth-order deformation equation

Let $\mathcal{L}$ denote an auxiliary linear operator, $u_0(x,t)$ is an initial approximation of $u(x,t)$ which satisfies the initial conditions (2.5). Note that, in this paper, the auxiliary linear operator $\mathcal{L}$ is not the same linear operator A of (2.4).

Note that the original equation (1.1) contains the linear operator D_{t}^{α}. So, it is straightforward for us to choose the auxiliary linear operator

$$\mathcal{L}(\phi) = D_{t}^{\alpha}(\phi). \tag{3.1}$$

According to (2.6), we can choose the initial approximation to be

$$u_{0}(x,t) = \sum_{k=0}^{m-1} g_{k}(x)\frac{t^{k}}{k!} + J^{\alpha}C(x,t). \tag{3.2}$$

For simplicity, let us define, according to (2.4), the nonlinear operator

$$\mathcal{N}(\phi) = D_{t}^{\alpha}(\phi) - A\left(\phi,\phi_{x},\phi_{xx}\right) - B\left(\phi,\phi_{x},\phi_{xx}\right) - C(x,t). \tag{3.3}$$

Hence, in the frame of HAM [29], we can construct the so-called zeroth-order deformation

$$(1-q)\mathcal{L}\big(U(x,t;q)-u_0(x,t)\big)=q\hbar\mathcal{N}\big(U(x,t;q)\big), \tag{3.4}$$

subject to the following initial conditions:

$$U^{(k)}(x,0;q)=g_k(x), \quad k=0,1,2,\dots,m-1, \tag{3.5}$$

where $q\in[0,1]$ is the embedding parameter, $\hbar\neq 0$ is an auxiliary parameter, and $U(x,t;q)$ is an unknown function on the independent variables x, t, and q.

When $q=0$, since $u_0(x,t)$ satisfies all the initial conditions (2.5), and $\phi=0$ is a solution of $\mathcal{L}\phi=0$, we have obviously

$$U(x,t;0)=u_0(x,t), \tag{3.6}$$

and when $q=1$, the zeroth-order deformation equations (3.4) and (3.5) are equivalent to the original equations (2.4) and (2.5), provided

$$U(x,t;1)=u(x,t). \tag{3.7}$$

Using the parameter q, we expand $U(x,t;q)$ in Taylor series as follows:

$$U(x,t;q)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)q^m, \tag{3.8}$$

where

$$u_m(x,t)=\frac{1}{m!}\frac{\partial^m U(x,t;q)}{\partial^m q}\bigg|_{q=0}. \tag{3.9}$$

Assume that the auxiliary linear operator $\mathcal{L}$, the initial guess u_0 and the auxiliary parameter $\hbar$ are properly chosen such that the series (3.8) is convergent at $q=1$. Thus, due to (3.7) we have

$$u(x,t)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t). \tag{3.10}$$

3.2. The mth-order deformation equation

Let us define the vector

$$\vec{u}_n=\big\{u_0(x,t),u_1(x,t),\dots,u_n(x,t)\big\}. \tag{3.11}$$

Following Liao [26–29], differentiating (3.4) m times with respect to the embedding parameter q, then setting $q = 0$, and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$\mathcal{L}\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = \hbar \mathcal{R}_m\left(\vec{u}_{m-1}\right), \tag{3.12}$$

subject to the initial conditions

$$u_m^{(k)}(x,0) = 0, \quad k = 0,1,2,\ldots,m-1, \tag{3.13}$$

where

$$\mathcal{R}_m\left(\vec{u}_{m-1}\right) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}\left(U(x,t;q)\right)}{\partial^{m-1}q}\bigg|_{q=0}, \tag{3.14}$$

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{3.15}$$

Substituting (3.3) into (3.14), and since A is a linear operator, $\mathcal{R}_m(\vec{u}_{m-1})$ can be given by

$$\begin{aligned} \mathcal{R}_m\left(\vec{u}_{m-1}\right) &= D_t^{\alpha} u_{m-1} - A\left(u_{(m-1)}, u_{(m-1)x}, u_{(m-1)xx}\right) \\ &\quad - \frac{1}{(m-1)!}\frac{\partial^{m-1}B\left(U, U_x, U_{xx}\right)}{\partial^{m-1}q}\bigg|_{q=0} - \left(1-\chi_m\right)C(x,t). \end{aligned} \tag{3.16}$$

According to (3.1), we can apply the operator J^{α} to both sides of (3.12) to obtain

$$J^{\alpha} D^{\alpha}\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = \hbar J^{\alpha}\left[\mathcal{R}_m\left(\vec{u}_{m-1}\right)\right]. \tag{3.17}$$

Using the property (2.3) and the initial conditions (1.1), we have

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar J^{\alpha}\left[\mathcal{R}_m\left(\vec{u}_{m-1}\right)\right]. \tag{3.18}$$

Finally, for the purpose of computation, we will approximate the HAM solution (3.10) by the following truncated series:

$$\phi_m(x,t) = \sum_{k=0}^{m-1} u_k(x,t). \tag{3.19}$$

3.3. Convergence theorem

Theorem 3.1. *As long as the series* $u(x,t) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t)$ *converges, where* $u_m(x,t)$ *is governed by* (3.12) *under the definitions* (3.14) *and* (3.15)*, it must be a solution of* (2.4).

Proof. If the series $\sum_{m=0}^{+\infty} u_m(x,t)$ is convergent, we can write

$$S(x,t) = \sum_{m=0}^{+\infty} u_m(x,t), \tag{3.20}$$

and it holds

$$\lim_{n\to+\infty} u_n(x,t) = 0. \tag{3.21}$$

From (3.12) and by using (3.15), it yields

$$\begin{aligned}
\hbar \sum_{m=1}^{+\infty} \mathcal{R}_m(\vec{u}_{m-1}) &= \sum_{m=1}^{+\infty} \mathcal{L}[u_m(x,t) - \chi_m u_{m-1}(x,t)] \\
&= \lim_{n\to+\infty} \sum_{m=1}^{n} \mathcal{L}[u_m(x,t) - \chi_m u_{m-1}(x,t)] \\
&= \mathcal{L}\left[\lim_{n\to+\infty} \sum_{m=1}^{n} (u_m(x,t) - \chi_m u_{m-1}(x,t))\right] \\
&= \mathcal{L}\left[\lim_{n\to+\infty} u_n(x,t)\right] = 0.
\end{aligned} \tag{3.22}$$

Since $\hbar \neq 0$, then

$$\sum_{m=1}^{+\infty} \mathcal{R}_m(\vec{u}_{m-1}) = 0. \tag{3.23}$$

Substituting (3.16) into the above equation and simplifying it, due to the convergence of the series $u(x,t) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t)$ and since A is a linear operator, yield

$$\begin{aligned}
\sum_{m=1}^{+\infty} \mathcal{R}_m(\vec{u}_{m-1}) &= \sum_{m=1}^{+\infty} \left[D_t^{\alpha} u_{m-1} - A\left(u_{(m-1)}, u_{(m-1)x}, u_{(m-1)xx}\right)\right] \\
&\quad - \sum_{m=1}^{+\infty} \left[\frac{1}{(m-1)!} \frac{\partial^{m-1} B(U, U_x, U_{xx})}{\partial^{m-1} q}\bigg|_{q=0} - (1-\chi_m) C(x,t)\right] \\
&= D_t^{\alpha}\left(\sum_{m=0}^{+\infty} u_m\right) - A\left(\sum_{m=0}^{+\infty} u_m, \sum_{m=0}^{+\infty} u_{mx}, \sum_{m=0}^{+\infty} u_{mxx}\right) \\
&\quad - \sum_{m=0}^{+\infty} \left[\frac{1}{m!} \frac{\partial^{m} B(U, U_x, U_{xx})}{\partial^{m} q}\bigg|_{q=0}\right] - C(x,t).
\end{aligned} \tag{3.24}$$

Now, expanding the nonlinear term $B(U(x,t;q),U_x(x,t;q),U_{xx}(x,t;q))$ by using the general Taylor theorem at $q=0$ yields

$$B\big(U(x,t;q),U_x(x,t;q),U_{xx}(x,t;q)\big)=\sum_{m=0}^{+\infty}\left[\frac{1}{m!}\frac{\partial^m B(U,U_x,U_{xx})}{\partial^m q}\bigg|_{q=0}q^m\right]. \tag{3.25}$$

Setting $q=1$ in the above equation and using (3.8), we obtain

$$B\left(\sum_{m=0}^{+\infty}u_m,\sum_{m=0}^{+\infty}u_{mx},\sum_{m=0}^{+\infty}u_{mxx}\right)=\sum_{m=0}^{+\infty}\left[\frac{1}{m!}\frac{\partial^m B(U,U_x,U_{xx})}{\partial^m q}\bigg|_{q=0}\right]. \tag{3.26}$$

Then

$$\begin{aligned}
&\sum_{m=1}^{+\infty}\mathcal{R}_m(\vec{u}_{m-1})\\
&\quad=D_t^\alpha\left(\sum_{m=0}^{+\infty}u_m\right)-A\left(\sum_{m=0}^{+\infty}u_m,\sum_{m=0}^{+\infty}u_{mx},\sum_{m=0}^{+\infty}u_{mxx}\right)-B\left(\sum_{m=0}^{+\infty}u_m,\sum_{m=0}^{+\infty}u_{mx},\sum_{m=0}^{+\infty}u_{mxx}\right)-C(x,t)\\
&\quad=D_t^\alpha(S(x,t))-A\big(S(x,t),S_x(x,t),S_{xx}(x,t)\big)-B\big(S(x,t),S_x(x,t),S_{xx}(x,t)\big)-C(x,t).
\end{aligned} \tag{3.27}$$

From the initial conditions (3.5) and (3.13), it holds that

$$S^{(k)}(x,0)=\sum_{m=0}^{+\infty}u_m^{(k)}(x,0)=u_0^{(k)}(x,0)+\sum_{m=1}^{+\infty}u_m^{(k)}(x,0)=g_k(x). \tag{3.28}$$

Thus, $S(x,t)$is satisfied and also must be the exact solution for (2.4). □

4. Test examples

In this section, we shall illustrate the applicability of HAM to several linear and nonlinear fPDEs.

4.1. Problem 1

Let us consider the following linear time-fractional wave-like equations:

$$D_t^\alpha u(x,t)=\frac{1}{2}x^2u_{xx}(x,t),\quad t>0,\ x\in R,\ 1<\alpha\le 2, \tag{4.1}$$

$$u(x,0)=x,\qquad u_t(x,0)=x^2. \tag{4.2}$$

We note that the heat-like counterpart of (4.1) was solved by HAM in [50] without direct comparison with the result by the ADM. According to (3.2), we can choose the initial guess to be

$$u_0(x,t) = x + x^2 t. \tag{4.3}$$

From (3.18), we have

$$u_m = (\chi_m + \hbar) u_{m-1} - \frac{\hbar}{2} x^2 J^\alpha \left[(u_{m-1})_{xx} \right]. \tag{4.4}$$

Consequently, the first few terms of HAM series solutions are as follows:

$$\begin{aligned} u_1(x,t) &= -\hbar \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} x^2, \\ u_2(x,t) &= -\hbar(\hbar+1) \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} x^2 + \hbar^2 \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} x^2, \\ u_3(x,t) &= -\hbar(\hbar+1)^2 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} x^2 + 2\hbar^2(\hbar+1) \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} x^2 - \hbar^3 \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} x^2, \end{aligned} \tag{4.5}$$

and so on. Hence, the HAM series solution is

$$\begin{aligned} u(x,t) &= u_0 + u_1 + u_2 + u_3 + \cdots \\ &= x + x^2 t - \hbar \left[1 + (\hbar+1) + (\hbar+1)^2 + \cdots \right] \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} x^2 \\ &\quad + \hbar^2 \left[1 + 2(\hbar+1) + \cdots \right] \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} x^2 \\ &\quad - \hbar^3 \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} x^2 + \cdots. \end{aligned} \tag{4.6}$$

Since we choose the initial guess $u_0(x,t)$ to be the same initial guess used by ADM [12], we can notice that when $\hbar = -1$, the above expression gives the same solution given by ADM. Table 1 shows the HAM approximation solutions for (4.1)-(4.2) when $\alpha = 1.5, 1.75$, and 2 with $\hbar = -1$ and -1.0453. It is to be noted that the first four terms of the HAM series were used to evaluate the approximate solutions in Table 1.

4.2. Problem 2

In this example, we consider the following one-dimensional linear inhomogeneous time-fractional equation:

$$D^\alpha u(x,t) + x u_x(x,t) + u_{xx}(x,t) = 2t + 2x^2 + 2, \quad t > 0,\ x \in R,\ 0 < \alpha \le 1, \tag{4.7}$$

Table 1: Approximate solution of (4.1) for some values of α using the 4-term HAM approximation, ϕ_4, with $\hbar = -1$ and $\hbar = -1.0453$.

t	x	$\alpha = 1.5$		$\alpha = 1.75$		$\alpha = 2.0$		
		$\hbar = -1$	$\hbar = -1.045$	$\hbar = -1$	$\hbar = -1.045$	$\hbar = -1$	$\hbar = -1.045$	Exact
0.2	0.25	0.26284062	0.26284062	0.26266989	0.26266991	0.26258350	0.26258350	0.26258350
	0.50	0.55136246	0.55136250	0.55067959	0.55067964	0.55033400	0.55033402	0.55033400
	0.75	0.86556554	0.86556562	0.86402909	0.86402919	0.86325150	0.86325156	0.86325150
	1.0	1.20544985	1.20544999	1.20271839	1.20271856	1.20133600	1.20133610	1.20133600
0.4	0.25	0.27697114	0.27697109	0.27615668	0.27615669	0.27567202	0.27567205	0.27567202
	0.50	0.60788455	0.60788437	0.60462675	0.60462679	0.60268808	0.60268820	0.60268808
	0.75	0.99274025	0.99273983	0.98541019	0.98541027	0.98104818	0.98104846	0.98104818
	1.0	1.43153821	1.43153748	1.41850702	1.41850715	1.41075232	1.41075282	1.41075232
0.6	0.25	0.29309481	0.29309501	0.29109009	0.29108997	0.28979084	0.28979084	0.28979084
	0.50	0.67237924	0.67238006	0.66436039	0.66435990	0.65916338	0.65916339	0.65916339
	0.75	1.13785328	1.13785513	1.11981088	1.11980978	1.10811762	1.10811764	1.10811764
	1.0	1.68951694	1.68952024	1.65744156	1.65743961	1.63665355	1.63665358	1.63665358

subject to the initial condition

$$u(x,0) = x^2. \tag{4.8}$$

In Section 3, we chose the initial guess to contain the initial conditions and the source term $C(x,t)$. In this example, due to the appearance of noise terms and also to get the exact solution, we will modify the way we choose the initial guess. The initial guess is set to contain only the initial condition (4.8), and the source term, $C(x,t) = 2t + 2x^2 + 2$, will be added to $u_1(x,t)$. The other terms are obtained the same as described in Section 3.

Hence, the initial guess is given by

$$u_0(x,t) = x^2, \tag{4.9}$$

and according to (3.18), we have

$$\begin{aligned} u_1 &= J^\alpha\left[2t + 2x^2 + 2\right] + \hbar J^\alpha\left[x(u_0)_x + (u_0)_{xx}\right], \\ u_m &= (\chi_m + \hbar)u_{m-1} + \hbar J^\alpha\left[x(u_{m-1})_{xx} + (u_{m-1})_{xx}\right], \quad m \geq 2. \end{aligned} \tag{4.10}$$

The terms of the HAM solution series can be given by

$$\begin{aligned} u_1(x,t) &= \frac{2t^{\alpha+1}}{\Gamma(\alpha+2)} + 2(\hbar+1)(x^2+1)\frac{t^\alpha}{\Gamma(\alpha+1)}, \\ u_2(x,t) &= 2(\hbar+1)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 2(\hbar+1)(x^2+1)\left[\frac{(\hbar+1)t^\alpha}{\Gamma(\alpha+1)} + \frac{2\hbar t^{2\alpha}}{\Gamma(2\alpha+1)}\right], \end{aligned} \tag{4.11}$$

and so on. Hence, the HAM series solution is

$$\begin{aligned} u(x,t) &= u_0 + u_1 + u_2 + \cdots \\ &= x^2 + 2\left[1 + (\hbar+1) + \cdots\right]\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \\ &\quad + 2(\hbar+1)\left(x^2+1\right)\left[1 + (\hbar+1) + \cdots\right]\frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ &\quad + 4\hbar(\hbar+1)\left(x^2+1\right)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+1)}x^2 + \cdots . \end{aligned} \tag{4.12}$$

Taking $\hbar = -1$ in (4.12), we obtain the exact solution,

$$u(x,t) = x^2 + \frac{2t^{\alpha}}{\Gamma(\alpha+2)}. \tag{4.13}$$

4.3. Problem 3

Consider the following nonlinear time-fractional hyperbolic equation:

$$D_t^{\alpha}u(x,t) = \frac{\partial}{\partial x}\left(u(x,t)\frac{\partial u(x,t)}{\partial x}\right), \quad t > 0,\ x \in R,\ 1 < \alpha \leq 2, \tag{4.14}$$

subject to the initial conditions

$$u(x,0) = x^2, \qquad u_t(x,0) = -2x^2. \tag{4.15}$$

Equation (4.14) can be rewritten as follows:

$$D_t^{\alpha}u(x,t) = \left(\frac{\partial u(x,t)}{\partial x}\right)^2 + u(x,t)\frac{\partial^2 u(x,t)}{\partial x^2}. \tag{4.16}$$

From (3.4), construct the following zeroth-order deformation:

$$(1-q)\mathcal{L}\left(U(x,t;q) - u_0(x,t)\right) = q\hbar\mathcal{N}\left(U(x,t;q)\right), \tag{4.17}$$

subject to the following initial conditions:

$$U(x,0) = x^2, \qquad U_t(x,0) = -2x^2, \tag{4.18}$$

where

$$\mathcal{N}(\theta) = D_t^{\alpha}\theta - \theta_x^2 - \theta\theta_{xx}. \tag{4.19}$$

The auxiliary linear operator can be chosen as follows:

$$\mathcal{L}(\theta) = D_t^{\alpha}(\theta), \tag{4.20}$$

with the property

$$\mathcal{L}(\theta) = 0 \quad \text{when } \theta = 0, \tag{4.21}$$

while, the initial guess is

$$u_0(x,t) = x^2 - 2x^2 t. \tag{4.22}$$

Again from (3.12), the high-order deformation equation can be given by

$$\mathcal{L}\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = \hbar \mathcal{R}_m\left(\vec{u}_{m-1}\right), \tag{4.23}$$

subject to the initial conditions

$$u_m^{(k)}(x,0) = 0, \quad k = 0,1, \tag{4.24}$$

where

$$\mathcal{R}_m\left(\vec{u}_{m-1}\right) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathcal{N}(U(x,t;q))}{\partial^{m-1} q}\bigg|_{q=0}. \tag{4.25}$$

Then, $\mathcal{R}_m(\vec{u}_{m-1})$ can be given by

$$\mathcal{R}_m\left(\vec{u}_{m-1}\right) = D_t^{\alpha} u_{m-1} - \sum_{i=0}^{m-1} u_{ix} u_{(m-1-i)x} - \sum_{i=0}^{m-1} u_i u_{(m-1-i)xx}. \tag{4.26}$$

Accordingly, the governing equation is as follows:

$$u_m = \chi_m u_{m-1} + \hbar J^{\alpha}\left[D_t^{\alpha} u_{m-1} - \sum_{i=0}^{m-1} u_{ix} u_{(m-1-i)x}\right] - \hbar J^{\alpha}\left[\sum_{i=0}^{m-1} u_i u_{(m-1-i)xx}\right], \quad m \geq 1. \tag{4.27}$$

Consequently, the first few terms of HAM series solutions are given by

$$\begin{aligned} u_1(x,t) &= -6\hbar x^2\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)} - 4\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 8\frac{t^{\alpha+2}}{\Gamma(\alpha+3)}\right], \\ u_2(x,t) &= -6\hbar x^2\left[(\hbar+1)\frac{t^{\alpha}}{\Gamma(\alpha+1)} - 4(\hbar+1)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 8(\hbar+1)\frac{t^{\alpha+2}}{\Gamma(\alpha+3)}\right] \\ &\quad + 72\hbar^2 x^2\left[\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - 2\left(\frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)}+2\right)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right] \\ &\quad + 576\hbar^2 x^2\left[\left(\frac{\Gamma(\alpha+3)}{\Gamma(\alpha+2)}+1\right)\frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)} + 2\Gamma(\alpha+4)\frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)}\right], \end{aligned} \tag{4.28}$$

and so on. Hence, the HAM series solution is

$$\begin{aligned} u(x,t) &= u_0 + u_1 + u_2 + u_3 + \cdots \\ &= x^2 - 2x^2 t - 6\hbar x^2\left[1 + (\hbar+1) + \cdots\right]\frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ &\quad + 24\hbar x^2\left[1 + (\hbar+1) + \cdots\right]\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \\ &\quad - 48\hbar x^2\left[1 + (\hbar+1) + \cdots\right]\frac{t^{\alpha+2}}{\Gamma(\alpha+3)} \\ &\quad + 72h\hbar^2 x^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots. \end{aligned} \tag{4.29}$$

The four-term HAM approximate solutions for (4.14)-(4.15), when $\alpha = 1.5$, 1.75, and 2 with $\hbar = -1$ and -1.0966, are shown in Table 2. Notice that the HAM approximate solution when $\alpha = 2$ with $\hbar = -1.0966$ is in good agreement with the exact solution, $u(x,t) = (x/(t+1))^2$.

4.4. Problem 4

Consider the following nonlinear time-fractional Fisher's equation:

$$D^{\alpha}u(x,t) = u_{xx}(x,t) + 6u(x,t)\big(1 - u(x,t)\big), \tag{4.30}$$

for $t > 0,\ x \in R,\ 0 < \alpha \le 1$, subject to the initial condition

$$u(x,0) = \frac{1}{(1+e^{x})^2}. \tag{4.31}$$

According to (3.2), we can choose the initial guess to be

$$u_0(x,t) = \frac{1}{(1+e^{x})^2}, \tag{4.32}$$

Table 2: Approximate solution of (4.14) for some values of α using the 4-term HAM approximation, ϕ_4, with $\hbar = -1$ and $\hbar = -1.0966$.

		$\alpha = 1.5$		$\alpha = 1.75$		$\alpha = 2.0$		
t	x	$\hbar = -1$	$\hbar = -1.0966$	$\hbar = -1$	$\hbar = -1.0966$	$\hbar = -1$	$\hbar = -1.0966$	Exact
0.2	0.25	0.060049	0.060199	0.048780	0.048788	0.043403	0.043404	0.043403
	0.50	0.240195	0.240795	0.195119	0.195152	0.173610	0.173617	0.173611
	0.75	0.540439	0.541789	0.439018	0.439092	0.390623	0.390638	0.390625
	1.0	0.960781	0.963180	0.780477	0.780607	0.694441	0.694468	0.694444
0.4	0.25	0.074536	0.076976	0.045434	0.045821	0.031853	0.031904	0.031888
	0.50	0.298142	0.307903	0.181736	0.183285	0.127412	0.127616	0.127551
	0.75	0.670821	0.692783	0.408907	0.412392	0.286677	0.287137	0.286990
	1.0	1.192570	1.231614	0.726946	0.733141	0.509649	0.510465	0.510204
0.6	0.25	0.091469	0.099717	0.045907	0.048048	0.023920	0.024414	0.024414
	0.50	0.365878	0.398867	0.183626	0.192191	0.095680	0.097656	0.097656
	0.75	0.823225	0.897450	0.413159	0.432430	0.215280	0.219727	0.219727
	1.0	1.463511	1.595467	0.734504	0.768765	0.382720	0.390626	0.390625

Table 3: Approximate solution of (4.30) for some values of α using the 3-term HAM approximation, ϕ_3, with $\hbar = -1$ and -1.05133.

		$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1.0$		
t	x	$\hbar = -1$	$\hbar = -1.0513$	$\hbar = -1$	$\hbar = -1.0513$	$\hbar = -1$	$\hbar = -1.0513$	Exact
0.1	0.25	0.946129	0.984066	0.488195	0.496911	0.317948	0.319612	0.316042
	0.50	0.843908	0.883590	0.405740	0.414926	0.250500	0.252292	0.250000
	0.75	0.715013	0.752540	0.324453	0.333180	0.190964	0.192689	0.191689
	1.0	0.576466	0.609185	0.249683	0.257315	0.140979	0.142501	0.142537
0.2	0.25	1.475318	1.551785	0.791250	0.816533	0.481199	0.488421	0.461284
	0.50	1.359827	1.439679	0.690146	0.716646	0.396941	0.404576	0.387456
	0.75	1.180981	1.256421	0.574405	0.599498	0.315266	0.322535	0.316042
	1.0	0.970078	1.035807	0.456647	0.478543	0.241175	0.247540	0.250000
0.3	0.25	1.967448	2.082543	1.124230	1.171123	0.681440	0.698116	0.604195
	0.50	1.845234	1.965336	1.009482	1.058531	0.581860	0.599391	0.534447
	0.75	1.622908	1.736324	0.859509	0.905896	0.475833	0.492464	0.461284
	1.0	1.345505	1.444292	0.695479	0.735922	0.372917	0.387446	0.387456

and according to (3.18), we have

$$u_m = \left(\chi_m + \hbar\right)u_{m-1} - \hbar J^{\alpha}\left[\left(u_{m-1}\right)_{xx} + 6u_{m-1} - 6\sum_{i=0}^{m-1} u_i u_{m-1-i}\right]. \tag{4.33}$$

Consequently, the first few terms of HAM series solutions are as follows:

$$\begin{aligned} u_1(x,t) &= -\frac{10\hbar e^x}{\left(1+e^x\right)^3}\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \\ u_2(x,t) &= -\frac{10\hbar e^x}{\left(1+e^x\right)^3}\left[(\hbar+1)\frac{t^{\alpha}}{\Gamma(\alpha+1)} - 5\hbar\frac{\left(2e^x-1\right)}{\left(1+e^x\right)}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right], \end{aligned} \tag{4.34}$$

and so on. Hence, the HAM series solution is

$$
\begin{aligned}
u(x,t) &= u_0 + u_1 + u_2 + \cdots \\
&= \frac{1}{(1+e^x)^2} - \frac{10\hbar e^x}{(1+e^x)^3}\left(1 + (\hbar+1) + \cdots\right)\frac{t^\alpha}{\Gamma(\alpha+1)} \\
&\quad + 50\hbar^2 \frac{e^x(2e^x-1)}{(1+e^x)^4}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots .
\end{aligned}
\tag{4.35}
$$

Table 3 shows the 3-term HAM approximate solutions for (4.30)-(4.31), ϕ_3, when $\alpha = 0.5, 0.75$, and 1 with $\hbar = -1$ and -1.05133. We notice that the HAM approximate solution when $\alpha = 2$ with $\hbar = -1.05133$ is in good agreement with the exact solution, $u(x,t) = 1/(1+\exp(x-5t))^2$.

5. Conclusions

In this work, the homotopy analysis method (HAM) was implemented to derive exact and approximate analytical solutions for both linear and nonlinear partial differential equations of fractional order. The convergence region of the series solution obtained by HAM can be controlled and adjusted by the auxiliary parameter $\hbar$. We give some examples to show the efficiency and accuracy of the suggested method. It was also demonstrated that the Adomian decomposition method (ADM) is a special case of HAM for the first and second test examples.

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