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Attractor and self-similar group of generalized fuzzy contraction mapping in fuzzy metric space

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Abstract: In this paper, we construct a deterministic fractal in fuzzy metric space using generalized fuzzy contraction mapping and its fixed-point theorem in hyperspace of non-empty compact sets. Moreover, we present the self-similar group of $\mathcal{H}$-contraction in fuzzy metric space and prove some familiar results of self-similar group for fuzzy metric space.

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1. Introduction

At the origin, fractal was defined by rough or fragmented geometric shape that can be split into parts where each smaller part is reduced size of the whole. That is, fractal can be defined through self-similar property. According to self-similar property, fractal can be characterized into two types, they are, an object having approximate or statistical self-similarity called random fractal and another one is an object having regular or exact self-similarity called deterministic or regular fractal. Mathematically, sets with non-integral Hausdorff dimension which exceed their topological dimension are called fractals by Mandelbort (1983).

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PUBLIC INTEREST STATEMENT

The Euclidean geometry handles regular objects with integer dimension, while the fractal geometry directs irregular objects with noninteger dimension. According to self-similar property, fractal can be characterized into two types, they are, an object having approximate self-similarity is called random fractal and another one is an object having exact self-similarity is called deterministic fractal. Natural objects that approximate fractals to a degree include clouds, mountain ranges, lightning bolts, coastlines, and snowflakes. However, not all self-similar objects are fractals for example, the real line is formally self-similar but fails to have other fractal characteristics. Fractals and Fuzzy spaces play a significant role in the Nonlinear Analysis. Hence, the above studies motivate our direction to investigate the fractal concepts in particular self-similar property in fuzzy setting.

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Hutchinson (1981) introduced the formal definition of iterated function systems (IFS) and Barnsley (1993) developed the theory of IFS called the Hutchinson–Barnsley theory (HB Theory) in order to define and construct the fractal as a compact invariant subset of a complete metric space generated by the IFS of Banach contractions. That is, Hutchinson introduced an operator on hyperspace of non-empty compact sets called as Hutchinson–Barnsley operator (HB operator) to define a fractal set as a unique fixed point using the Banach contraction theorem in the complete metric space, in order to generate fractal as a unique fixed point using Banach fixed-point theorem having the aforesaid exact self-similar property. Moreover, these fractal sets have Hausdorff dimension greater than its topological dimension, in such a way that self-similarity is the most fundamental property of the fractals. In order to analyze self-similar sets in depth, we must realize their group structure. In this study, we present the self-similar group in fuzzy setting. Self-similar group is defined through Banach contraction and topological group in the classical metric space, while the fuzzy self-similar group is defined by fuzzy H-contraction and fuzzy topological group in the fuzzy metric space.

Fuzzy set theory was introduced by Zadeh (1965). Kramosil and Michalek (1975) introduced the notion of fuzzy metric space. Many authors have introduced and discussed several notions of fuzzy metric space in different ways and also proved fixed-point theorems with interesting consequent results in the fuzzy metric spaces (Farnoosh, Aghajani, & Azhdari, 2009; George & Veeramani, 1997; Grabiec, 1988; Gregori & Sapena, 2002; Mihet, 2007; Rodriguez-Lopez & Romaguera, 2004; Uthayakumar & Gowrisankar, 2014; Wardowski, 2013). George and Veeramani (1994) imposed some stronger conditions on the fuzzy metric space in order to obtain a Hausdorff topology. Rodriguez-Lopez and Romaguera defined the Hausdorff metric on fuzzy hyperspace and constructed the Hausdorff fuzzy metric space. Besides that, the necessary results of the Hausdorff fuzzy metric on fuzzy hyperspaces are proved in Rodriguez-Lopez and Romaguera (2004). Uthayakumar and Easwaramoorthy (2011), Easwaramoorthy and Uthayakumar (2011), investigated the fuzzy IFS fractals in the fuzzy metric space. On the basis of self-similar group of Banach contraction in classical metric space given by Saltan and Demir (2013), in this paper, we introduce the definition and property of self-similar group and strong self-similar group of $\mathcal{H}$-contraction. If G is a self-similar group (strong self-similar group) of $\mathcal{H}$-contraction, then G is also described as the attractor of a $\mathcal{H}$-IFS and one of the $\mathcal{H}$-contractions of $\mathcal{H}$-IFS is a group homomorphism (isomorphism). The image of G under this $\mathcal{H}$-contraction map is its proper subgroup H being homomorphic (isomorphic) to G. Fractal set can be defined as a self-similar and strong self-similar group in the sense of $\mathcal{H}$-IFS of compact topological space.

The paper is organized into two directions, first one is to construct the fractals in fuzzy metric space using generalized fuzzy contraction mapping. Second direction is that we investigate a fuzzy metric group on self-similar property of fractal set in order to define the topological group with generalized fuzzy contraction. In this paper, we will start with short introduction of deterministic fractals in fuzzy metric space in Section 2 and some of its properties which will be used frequently in the sequel. In Section 3, we present generalization of the fuzzy contraction mappings together with their fixed-point properties. Further, in Section 4, we define the self-similar groups in fuzzy metric space and investigate the properties of these groups. At the end of the paper, two substantial examples are given, which shows the existence of fuzzy self-similar groups.

2. Fuzzy iterated function system

In this section, we recall some pertinent concepts on fuzzy metric spaces in the sense of George and Veeramani. Hausdorff fuzzy metric for a given fuzzy metric space on the set of its non-empty compact subsets as well as Fuzzy IFS Fractals in the fuzzy metric space.

Definition 2.1   (George & Veeramani, 1997, 1994) A binary operation $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a continuous t-norm, if $([0, 1], *)$ is a topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c, b \leq d$, and $a, b, c, d \in [0, 1]$.

George and Veeramani modified the Kramosil and Michalek (1975) fuzzy metric space as follows.

Definition 2.2 (George & Veeramani, 1997, 1994) The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions:

(1) $M(x, y, t) > 0$,
(2) $M(x, y, t) = 1$ if and only if $x = y$,
(3) $M(x, y, t) = M(y, x, t)$,
(4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(5) $M(x, y, \cdot) : (0, \infty) \longrightarrow [0, 1]$ is continuous,

$x, y, z \in X$ and $t, s > 0$.

Definition 2.3 (Gregori & Sapena, 2002; Rodriguez-Lopez & Romaguera, 2004) Let $(X, M, *)$ be a fuzzy metric space. The mapping $f : X \longrightarrow X$ is fuzzy contractive if there exists $k \in (0, 1)$ such that

$$\frac{1}{M(f(x), f(y), t)} - 1 \leq k \left(\frac{1}{M(x, y, t)} - 1 \right)$$

for all $x, y \in X$ and $t > 0$. Here, k is called the fuzzy contractivity ratio of f.

THEOREM 2.1 *(Gregori & Sapena 2002. Fuzzy Banach Contraction Theorem) Let $(X, M, *)$ be a complete fuzzy metric space in which fuzzy contractive sequence are Cauchy. Let $f : X \longrightarrow X$ be a fuzzy contractive mapping with contractivity ratio k. Then f has a unique fixed point.*

Definition 2.4 (Rodriguez-Lopez & Romaguera, 2004) Let $(X, M, *)$ be a fuzzy metric space. Let $\mathscr{K}(X)$ be set of all non-empty compact subsets of X. Define, $M(x, B, t) = \sup_{y \in B} M(x, y, t)$ and $M(A, B, t) = \inf_{x \in A} M(x, B, t)$ for all $x \in X$ and $A, B \in \mathscr{K}(X)$. Then Hausdorff fuzzy metric (H_M) is a function $H_M : \mathscr{K}(X) \times \mathscr{K}(X) \times (0, \infty) \longrightarrow [0, 1]$ defined by

$$H_M(A, B, t) = \min \left\{ M(A, B, t), M(B, A, t) \right\}$$

then H_M is a fuzzy metric on $\mathscr{K}(X)$, and hence $(\mathscr{K}(X), H_M, *)$ is called a Hausdorff fuzzy metric space.

Definition 2.5 (Uthayakumar & Easwaramoorthy, 2011; Easwaramoorthy & Uthayakumar, 2011) Let $(X, M, *)$ be a fuzzy metric space and $f_n : X \longrightarrow X, n = 1, 2, 3, \ldots, N$ $(N \in \mathbb{N})$ be N - fuzzy contractive mappings with the corresponding contractivity ratios $k_n, n = 1, 2, 3, \ldots, N$. Then the system $\{X; f_n, n = 1, 2, 3, \ldots, N\}$ is called a Fuzzy Iterated Function System (FIFS) of fuzzy contractions in the fuzzy metric space $(X, M, *)$.

Definition 2.6 (Uthayakumar & Easwaramoorthy, 2011; Easwaramoorthy & Uthayakumar, 2011) Let $(X, M, *)$ be a fuzzy metric space. Let $\{X; f_n, n = 1, 2, 3, \ldots, N; N \in \mathbb{N}\}$ be a FIFS of fuzzy contractions. Then the Fuzzy Hutchinson–Barnsley operator (FHB operator) of the FIFS of fuzzy contractions is a function $F : \mathscr{K}(X) \longrightarrow \mathscr{K}(X)$ defined by

$$F(B) = \bigcup_{n=1}^{N} f_n(B), \quad \text{for all } B \in \mathscr{K}(X).$$

Definition 2.7 (Uthayakumar & Easwaramoorthy, 2011; Easwaramoorthy & Uthayakumar, 2011) Let $(X, M, *)$ be a complete fuzzy metric space. Let $\{X; f_n, n = 1, 2, 3, \ldots, N; N \in \mathbb{N}\}$ be a FIFS of fuzzy contractions and F be the FHB operator of the FIFS of fuzzy contractions. We say that the set $A_\infty \in \mathscr{K}(X)$ is Fuzzy Attractor (Fuzzy Fractal) of the given FIFS of fuzzy contractions, if A_∞ is a unique

fixed point of the FHB operator F of fuzzy contractions. Usually, such $A_\infty \in \mathscr{K}(X)$ is also called as Fractal generated by the FIFS of fuzzy contractions.

3. Attractor of generalized fuzzy contraction

In this section, we generate a fractal in fuzzy metric space, which is a generalization of a fractal initiated in the article (Easwaramoorthy & Uthayakumar, 2011). Moreover, we develop the $\mathcal{H}$-IFS theory in order to define and construct the fractal as a compact invariant subset of M-complete fuzzy metric space generated by the fixed-point theorem.

$\mathcal{H}$ denotes a collection of mappings $\eta : (0, 1] \longrightarrow [0, \infty)$ such that η maps $(0, 1]$ onto $[0, \infty)$ and $s > t$ implies $\eta(s) < \eta(t)$ for all $s, t \in (0, 1]$.

Definition 3.1 (Wardowski, 2013) Let $(X, M, *)$ be a fuzzy metric space. A mapping $f : X \longrightarrow X$ is said to be $\mathcal{H}$-contractive with respect to $\eta \in \mathcal{H}$ if there exists $k \in (0, 1)$ such that

$$\eta(M(f(x), f(y), t)) \leq k\eta(M(x, y, t)) \tag{1}$$

for all $x, y \in X$ and $t > 0$.

Remark 3.1 (Wardowski, 2013) Consider a mapping $\eta \in \mathcal{H}$ of the form $\eta(t) = \frac{1}{t} - 1, t \in (0, 1]$. Then the condition (1) reduces to

$$\frac{1}{M(f(x), f(y), t)} - 1 \leq k\left(\frac{1}{M(x, y, t)} - 1\right)$$

for all $x, y \in X$ and $t > 0$.

PROPOSITION 3.1 *(Wardowski, 2013) Let $(X, M, *)$ be a fuzzy metric space and $\eta \in \mathcal{H}$. A sequence $< x_n >_{n \in \mathbb{N}} \subset X$ is M-Cauchy if and only if for given $\epsilon > 0, t > 0$ there exits $n_0 \in \mathbb{N}$ scuh that*

$$\eta(M(x_m, x_n, t)) < \epsilon,$$

for all $m, n \geq n_0$.

PROPOSITION 3.2 *(Wardowski, 2013) Let $(X, M, *)$ be a fuzzy metric space and $\eta \in \mathcal{H}$. A sequence $< x_n >_{n \in \mathbb{N}} \subset X$ is convergent to $x \in X$ if and only if $\lim_{n \longrightarrow \infty} \eta(M(x_n, x, t)) = 0$ for all $t > 0$.*

THEOREM 3.1 *Let $(X, M, *)$ be a fuzzy metric space. Let $(\mathscr{K}(X), H_M, *)$ be a corresponding Hausdorff fuzzy metric space. If $f : X \longrightarrow X$ is a fuzzy $\mathcal{H}$-contraction with respect to $\eta \in \mathcal{H}$ on $(X, M, *)$, then f is a fuzzy $\mathcal{H}$-contraction with respect to $\eta \in \mathcal{H}$ on $(\mathscr{K}(X), H_M, *)$.*

Proof Fix $t > 0$. Let $A, B \in \mathscr{K}(X)$. f is fuzzy $\mathcal{H}$-contraction with respect to $\eta \in \mathcal{H}$ on $(X, M, *)$. Hence, there exists $k \in (0, 1)$ such that

$$\eta(M(f(x), f(y), t)) \leq k\eta(M(x, y, t)), \forall x, y \in X$$
$$\eta(\sup_{y \in B} M(f(x), f(y), t)) \leq k\eta(\sup_{y \in B} M(x, y, t)), \forall x \in A, y \in B$$
$$\eta(M(f(x), f(B), t)) \leq k\eta(M(x, B, t)), \forall x \in A, B \in \mathscr{K}(X)$$
$$\eta(\inf_{x \in A} M(f(x), f(B), t)) \leq k\eta(\inf_{x \in A} M(x, B, t)), \forall x \in A, B \in \mathscr{K}(X)$$
$$\eta(M(f(A), f(B), t)) \leq k\eta(M(A, B, t)), \forall A, B \in \mathscr{K}(X)$$
Similarly, $\eta(M(f(B), f(A), t)) \leq k\eta(M(B, A, t)), \forall A, B \in \mathscr{K}(X)$
Hence, $\eta(H_M(f(B), f(A), t)) \leq k\eta(H_M(B, A, t)), \forall A, B \in \mathscr{K}(X)$

THEOREM 3.2 *Let $(X, M, *)$ be a M-complete fuzzy metric space. Let $(\mathscr{K}(X), H_M, *)$ be a corresponding Hausdorff fuzzy metric space and let $f : \mathscr{K}(X) \longrightarrow \mathscr{K}(X)$ be a fuzzy $\mathcal{H}$-contractive with respect to $\eta \in \mathcal{H}$ such that*

(i) $\prod_{i=1}^{k} H_M(A, f(A), t_i) \neq 0$, for all $A \in \mathcal{K}(X), k \in \mathbb{N}$ and any sequence $< t_i >_{i \in \mathbb{N}} \subset (0, \infty), t_i \to 0$,

(ii) $r * s > 0$ implies $\eta(r * s) \leq \eta(r) + \eta(s)$, for all $r, s \in \{H_M(A, f(A), t) : A \in \mathcal{K}(X), t > 0\}$,

(iii) $\{H_M(A, f(A), t_i) : i \in \mathbb{N}\}$ is bounded for all $A \in \mathcal{K}(X)$ and any sequence $< t_i >_{i \in N} \subset (0, \infty), t_i \to 0$.

Then f has a unique fixed point A^* and for each A_0, the sequence $< f^n(A_0) >_{n \in \mathbb{N}}$ converges to A^*.

Proof Fix $A_0 \in \mathcal{K}(X)$. Define $A_1 = f(A_0)$ and $A_n = f(A_{n-1})$ for $n \geq 2$, we have a sequence $< A_n >_{n \in N}$. For $t > 0$,

$$\eta(H_M(A_1, A_2, t)) = \eta(H_M(f(A_0), f(A_1), t)) \leq k\eta(H_M(A_0, A_1, t)),$$

$$\eta(H_M(A_2, A_3, t)) = \eta(H_M(f(A_1), f(A_2), t)) \leq k\eta(H_M(A_1, A_2, t)) \leq k^2 \eta(H_M(A_0, A_1, t)),$$

$$\eta(H_M(A_n, A_{n+1}, t)) = \eta(H_M(f(A_{n-1}), f(A_n), t)) \leq k\eta(H_M(A_{n-1}, A_n, t)) \leq \ldots \leq k^n \eta(H_M(A_0, A_1, t)).$$

Hence, $\eta(H_M(A_n, A_{n+1}, t)) \leq k^n \eta(H_M(A_0, A_1, t))$ for all $n \geq 1$.

Clearly, $H_M(A_n, A_{n+1}, t) \geq \eta(H_M(A_0, A_1, t)$ for all $n \geq 1$ and $t > 0$. For $m, n \in \mathbb{N}, m < n, t < 0$ and let $< a_i >_{i \in \mathbb{N}}$ be a strictly decreasing sequence of positive number such that $\sum_{i=1}^{\infty} a_i = 1$.

$$H_M(A_m, A_n, t) = H_M\left(A_m, A_n, t + \sum_{i=m}^{n-1} a_i t - \sum_{i=m}^{n-1} a_i t\right)$$

$$\geq H_M\left(A_m, A_m, t - \sum_{i=m}^{n-1} a_i t\right) * H_M\left(A_m, A_n, \sum_{i=m}^{n-1} a_i t\right)$$

$$= 1 * H_M\left(A_m, A_n, \sum_{i=m}^{n-1} a_i t\right)$$

$$\geq \prod_{i=m}^{n-1} H_M(A_i, A_{i+1}, a_i t)$$

$$\geq \prod_{i=m}^{n-1} H_M(A_0, A_1, a_i t) < 0$$

$H_M(A_m, A_n, t) \geq \prod_{i=m}^{n-1} H_M(A_i, A_i, a_i t) \implies \eta(H_M(A_m, A_n, t)) \leq \eta(\prod_{i=m}^{n-1} H_M(A_i, A_i, a_i t)) \leq \sum_{i=m}^{n-1} k^i \eta(H_M(A_i, A, a_i t)). \eta(H_M(A_m, A_n, t)) \leq \sum_{i=m}^{n-1} k^i \eta(H_M(A_i, A_i, a_i t)).$

Easy to verify that a sequence $< \eta(H_M(A_0, A_1, a_i t)) <_{i \in \mathbb{N}}$ is non-decreasing and, by (c), bounded, hence we have a convergence of the series $\sum_{i=1}^{\infty} \eta(H_M(A_i, A_i, a_i t))$. Consquently, for any $\epsilon > 0$, there exist $k \in \mathbb{N}$ such that $\sum_{i=m}^{n-1} \eta(H_M(A_i, A_i, a_i t)) < \epsilon$ for all $m, n \geq N, m < n$. Thus, by Proposition 3.1, $< A_n <_{i \in \mathbb{N}}$ is an M-Cauchy sequence. By the M-completeness of X, there exists $A^* \in X$ such that $\lim_{n \to \infty} A_n = A^*$. Due to Proposition 3.2, $\lim_{n \to \infty} \eta(H_M(A_n, A^*, t)) = 0$ for each $t > 0$. Hence for all $t > 0$, we obtain $\eta(H_M(f(A^*), A_{n+1}, t)) \leq k\eta(H_M(A^*, A_n, t)) \longrightarrow 0$ as $n \longrightarrow \infty$. Finally, from the Proposition 3.2, we have $A^* = \lim_{n \to \infty} A_{n+1} = f(A^*)$.

Suppose that there exists $A' \in \mathcal{K}(X), A' \neq A^*$ such that $f(A') = A'$. Then, any $t > 0$,

$$\eta(H_M(A', A^*, t)) = \eta(H_M(f(A'), f(A^*), t)) \leq k\eta(H_M(A', A^*, t)).$$

Since $H_M(A', A^*, t) \neq 1$, it is contradiction to the definition of η. Hence, $A' = A^*, A^*$ is a unique fixed point of f.

Definition 3.2   Let $(X, M, *)$ be a fuzzy metric space and $f_n : X \longrightarrow X$, $n = 1, 2, 3, \dots, N(N \in \mathbb{N})$ be N-$\mathcal{H}$ contractive mappings. Then the system $\{X; f_n, \ n = 1, 2, 3, \dots, N\}$ is called a $\mathcal{H}$-Iterated Function System ($\mathcal{H}$-IFS) of $\mathcal{H}$-contractions in $(X, M, *)$. The Hutchinson–Barnsley operator (HB operator) of the $\mathcal{H}$-IFS is a function $F : \mathscr{K}(X) \longrightarrow \mathscr{K}(X)$ defined by

$$F(B) = \bigcup_{n=1}^{N} f_n(B), \quad \text{for all} \quad B \in \mathscr{K}(X)$$

Definition 3.3   Let $(X, M, *)$ be a complete fuzzy metric space. Let $\{X; f_n, \ n = 1, 2, 3, \dots, N; N \in \mathbb{N}\}$ be a $\mathcal{H}$-IFS and F be the HB operator of the $\mathcal{H}$-IFS. If F has a unique fixed point A^* in $(X, M, *)$, then the set $A^* \in \mathscr{K}(X)$ is called the Attractor (or Fractal) generated by the $\mathcal{H}$-IFS of $\mathcal{H}$-contractions.

THEOREM 3.3   Let $\{X; f_0, f_1, \dots, f_n\}$ be a $\mathcal{H}$-IFS with attractor A. If the $\mathcal{H}$-contraction mappings $f_0, f_1, \dots, f_n$ are one-to-one on A and

$$f_i(A) \cap f_j(A) = \phi \text{ for all } i, j \in \{0, 1, 2, \dots, n\} \text{ with} \pi \neq j$$

then A is totally disconnected set.

Proof   Suppose that there exist a connected subset B of A contains more than two points. A is an attractor of given $\mathcal{H}$-IFS, therefore $f_0(A) \cup f_2(A) \cup \cdots \cup f_n(A) = A$. $\mathcal{H}$-contraction mappings are t-uniform continuous, hence $f_i(B)$ is connected and $f_i(B) \cap f_j(B) = B$ for all $i, j \in \{0, 1, 2, \dots, n\}$. Clearly, it gives the contradiction to $f_0, f_2, \dots, f_n$ are one-to-one on A. Therefore, only connected subset of A is single point set, there are no other connected subsets in A. Hence, A is totally disconnected.

4. Fuzzy self-similar group

Romaguera and Sanchis (2001) extended the classical theorems on metric groups to the fuzzy setting. According to the definition of self-similar group of Banach contraction in classical metric space given by Saltan and Demir (2013), in this section, we introduce the definition and property of fuzzy self-similar group and strong fuzzy self-similar group of fuzzy contraction in fuzzy metric space. Then, we investigate some properties of strong fuzzy self-similar and fuzzy profinite groups.

Topological groups can be defined concisely as group objects in the category of topological spaces, in the same way that ordinary groups are group objects in the category of sets. Now we recall the definition of self-similar group in compact topological space and profinite group.

Definition 4.1   (Demir & Saltan, 2012; Saltan & Demir, 2013) Let (G, d) be a compact topological group with a translation-invariant metric d. G is called a self-similar group, if there exists a proper subgroup H of finite index and a surjective homomorphism $\phi : G \longrightarrow H$, which is a contraction with respect to d.

Definition 4.2   (Saltan & Demir, 2013) Let (G, d) be a compact topological group with a translation-invariant metric d. G is called a strong self-similar group, if there exists a proper subgroup H of finite index and a group isomorphism $\phi : G \longrightarrow H$, which is a contraction with respect to d.

Definition 4.3   (Dixon, Du Sautoy, Mann, & Segal, 1999; Saltan & Demir, 2013) A topological group G is profinite, if it is topologically isomorphic to an inverse limit of finite discrete topological groups. Equivalently, a profinite group is a compact, Hausdorff, and totally disconnected topological group.

A fuzzy metric group is a 4-tuple $(G, ., M, *)$ such that $(G, M, *)$ is a fuzzy metric space and $(G, ., \tau_M)$ is a topological group, where τ_M is a topology induced by the fuzzy metric $(M, *)$.

Definition 4.4   Let $(G, ., M, *)$ be a compact topological fuzzy metric group (simply fuzzy group) with a translation-invariant fuzzy metric $(M, *)$. Then G is called a self-similar group of $\mathcal{H}$-contraction, if

there exists a proper subgroup H of finite index and a surjective homomorphism $\phi{:}G \longrightarrow H$, which is a $\mathcal{H}$-contraction with respect to $\eta \in \mathcal{H}$ on fuzzy metric $(M, *)$.

Definition 4.5 Let $(G, ., M, *)$ be a compact fuzzy topological group with a translation-invariant fuzzy metric $(M, *)$. Then G is called a strong self-similar group of $\mathcal{H}$-contraction, if there exists a proper subgroup H of finite index and a group isomorphism $\phi{:}G \longrightarrow H$, which is a $\mathcal{H}$-contraction with respect to $\eta \in \mathcal{H}$ on fuzzy metric $(M, *)$.

Definition 4.6 A fuzzy topological group G is fuzzy profinite, if it is topologically isomorphic to an inverse limit of finite discrete topological fuzzy groups.

THEOREM 4.1 *If $(G, ., M, *)$ is a fuzzy profinite topological group, then G is profinite if and only if it is Hausdorff compact and totally disconnected.*

Proof Assume that G is profinite group, then G is Hausdorff, compact, and totally disconnected. Since every topological group is Hausdorff, and finite discrete groups are compact and totally disconnected.

Conversely, Let G be Hausdorff, compact, and totally disconnected. Since all components, i.e. all points of G, are closed and $e = comp\{e\}$ is the intersection of all open–closed neighborhoods of e. It is easy to show that every open–closed neighborhood of e contains an open normal subgroup, which implies G has a topological isomorphic to an inverse limit of finite discrete topological group.

PROPOSITION 4.1 *A strong self-similar group of $\mathcal{H}$-contraction is the attractor of $\mathcal{H}$-IFS.*

Proof Let $(X, ., M, *)$ be a strong self-similar group of $\mathcal{H}$-contraction. Hence, there is a proper subgroup H of X with $[X{:}H] = n$ such that the mapping $\phi_o{:}X \longrightarrow H$ is a group isomorphism and is a $\mathcal{H}$-contraction with respect to $\eta \in \mathcal{H}$. Let $x_o = e$ be the identity element of X. For all $i, j \in \{0, 1, 2 \ldots, n-1\}$ and $i \neq j$, there are cosets of H in X such that $(H.x_i) \cap (H.x_j) = \phi$ and $X = H \cup (H.x_1) \cup (H.x_2) \cup \cdots \cup (H.x_n - 1)$. Define $\phi_i{:}X \longrightarrow X$ by $\phi_i(g) = \phi_o(g).x_i$, $i = 1, 2, 3, \ldots, n-1$. Clearly,

$$\phi_i(X) = H.x_i$$

because of ϕ_o is surjective. Since ϕ_o is a $\mathcal{H}$-contraction mapping with respect to η and $(M, *)$ is a translation invariant fuzzy metric, we obtain that

$$
\begin{aligned}
M((\phi_i(g), \phi_i(h)), t) &= M(\phi_o(g).x_i, \phi_o(h).x_i, t) \\
&= M(\phi_o(g), \phi_o(h), t) \\
\eta(M((\phi_i(g), \phi_i(h)), t)) &= \eta(M(\phi_o(g), \phi_o(h), t)) \\
&\leq k\eta(M(g, h, t)),
\end{aligned}
$$

for all $g, h \in X$. Therefore, ϕ_i is a $\mathcal{H}$-contraction mapping with respect to η for $i = 1, 2, 3, \ldots, n-1$ and

$$
\begin{aligned}
X &= H \cup (H.x_1) \cup (H.x_2) \cup \cdots \cup (H.x_{n-1}) \\
&= \phi_o(X) \cup \phi_1(X) \cup \phi_2(X) \cup \cdots \cup \phi_{n-1}(X) \\
X &= \bigcup_{n=0}^{n-1} \phi_i(X)
\end{aligned}
$$

Thus, X is the attractor of the $\mathcal{H}$-IFS $\{X; \phi_o, \phi_1, \ldots, \phi_{n-1}\}$.

THEOREM 4.2 *Let $(G, ., M, *)$ and $(G', .', M', *')$ be compact fuzzy topological groups. If G is a strong self-similar group of $\mathcal{H}$-contraction and $f{:}G \longrightarrow G'$ is both an isometry map and a group isomorphism, so is G'.*

Proof   First, we show that M' is a translation-invariant fuzzy metric. There exists $x, y, z \in G$ such that $f(x) = x'$, $f(y) = y'$, and $f(z) = z'$ for all $x', y', z' \in G'$, since f is surjective and isometric, M is a translation-invariant fuzzy metric, hence we get

$$M'(x'.'z', y'.'z', t) = M'(f(x).'f(z), f(y).'f(z), t)$$
$$= M'(f(x.z), f(y.z), t)$$
$$= M(x.z, y.z, t)$$
$$= M(x, y, t)$$
$$= M'(f(x), f(y), t)$$
$$= M'(x', y', t).$$

G is a strong self-similar group of $\mathcal{H}$-contraction, there exists a subgroup H with $[G:H] = n$ and a group isomorphism $\phi: G \longrightarrow H$. Let $f(H) = H'$. f is a group isomorphism on G, hence f maps subgroup H of G into subgroup H' of G' with $[G':H'] = n$.

Define $\phi': G' \longrightarrow H'$ by $\phi'(g) = f_{|H} \circ \phi \circ f^{-1}(g)$, where $f_{|H}$ is a function from H to G' defined by $f_{|H}(g) = f(g), \forall x \in H$.

$f, f_{|H}$ and ϕ are group isomorphisms, it is clear that ϕ' is also a group isomorphism. Further, ϕ is a $\mathcal{H}$-contraction mapping with respect to $\eta \in \mathcal{H}$ and $f, f_{|H}$ are isometries, hence

$$M'(\phi'(g'), \phi'(h'), t) = M'(f_{|H} \circ \phi \circ f^{-1}(g'), f_{|H} \circ \phi \circ f^{-1}(h'), t)$$
$$= M'(f_{|H}(\phi \circ f^{-1})(g'), f_{|H}(\phi \circ f^{-1})(h'), t)$$
$$= M(\phi \circ f^{-1}(g'), \phi \circ f^{-1}(h'), t)$$
$$\eta(M'(\phi'(g'), \phi'(h'), t)) = \eta(M(\phi \circ f^{-1}(g'), \phi \circ f^{-1}(h'), t))$$
$$\leq k\, \eta((M(f^{-1}(g'), f^{-1}(h'), t))$$
$$= k\, \eta(M'(g', h', t))$$

for all $g', h' \in G'$. It gives that ϕ' is $\mathcal{H}$-contraction mapping on G'. Therefore, there exists a $\phi': G' \longrightarrow H'$ such that ϕ' is a group isomorphism and $\mathcal{H}$-contraction on G', hence G' is a strong self-similar group.

THEOREM 4.3   *If $G_1, G_2, \ldots, G_n$ are strong fuzzy self-similar groups of $\mathcal{H}$-contraction, so is $G_1 \times G_2 \times \cdots \times G_n$.*

Proof   *Consider the product fuzzy metric $M_p(X, Y, t) = M_1(x_1, y_1, t) * M_2(x_2, y_2, t) * \cdots * M_n(x_n, y_n, t)$ for all $t > 0$, $X, Y \in G_1 \times G_2 \times \cdots \times G_n$, $(G_1, M_1, *, .), (G_2, M_2, *, .), \ldots, (G_n, M_n, *, .)$ are compact fuzzy topological groups, hence $G_1 \times G_2 \times \cdots \times G_n$ is a compact fuzzy topological group. Moreover, there are subgroups $H_1, H_2, \ldots, H_n$ of $G_1, G_2, \ldots, G_n$, respectively, such that $[G_i:H_i] = m_i$ and the mappings*

$$\varphi_i: G_i \longrightarrow H_i$$

are $\mathcal{H}$-contraction with respect to η and group isomorphisms for $i = 1, 2, \ldots, n$, since these groups are strong self-similar groups of $\mathcal{H}$-contraction. Define the mapping $\phi: G_1 \times G_2 \times \cdots \times G_n \longrightarrow H_1 \times H_2 \times \cdots \times H_n$ by $\phi(g_1, g_2, \ldots, g_n) = (\varphi_1(g_1), \varphi_2(g_2), \ldots, \varphi_n(g_n))$. Clearly, $H_1 \times H_2 \times \cdots \times H_n$ is a subgroup of $G_1 \times G_2 \times \cdots \times G_n$ and $[G_1 \times G_2 \times \cdots \times G_n : H_1 \times H_2 \times \cdots \times H_n] = m_1 m_2 \ldots m_n$. $\varphi_1, \varphi_2, \ldots, \varphi_n$ are group homomorphisms, hence

$$\phi(g.h) = \phi((g_1, g_2, \ldots, g_n).(h_1, h_2, \ldots, h_n))$$
$$= \phi((g_1 \cdot_1 h_1, g_2 \cdot_2 h_2, \ldots, g_n \cdot_n h_n))$$
$$= (\varphi_1(g_1 \cdot_1 h_1), \varphi_2(g_2 \cdot_2 h_2), \ldots, \varphi_n(g_n \cdot_n h_n))$$
$$= (\varphi_1(g_1) \cdot_1 \varphi_1(h_1), \varphi_2(g_2) \cdot_2 \varphi_2(h_2), \ldots, \varphi_n(g_n) \cdot_n \varphi_n(h_n))$$
$$= (\varphi_1(g_1), \ldots, \varphi_n(g_n)).(\varphi_1(h_1), \ldots, \varphi_n(h_n))$$
$$= \phi(g_1, g_2, \ldots, g_n).\phi(h_1, h_2, \ldots, h_n)$$
$$= \phi(g).\phi(h).$$

It gives that ϕ is group homomorphism. $\varphi_1, \varphi_2, \ldots, \varphi_n$ are bijective implies ϕ is bijective. Take $\mathcal{H}$-contraction ratio b_i of $\mathcal{H}$-contraction mappings φ_i for $i = 1, 2, \ldots, n$ and choose $b = max\{b_1, b_2, \ldots, b_n\}$. Then,

$$\eta(M_p(\phi(g), \phi(h), t)) = \eta(M_p(\phi(g_1, g_2, \ldots, g_n), \phi(h_1, h_2, \ldots, h_n), t))$$
$$= \eta(M_p((\varphi_1(g_1), \ldots, \varphi_n(g_n)), (\varphi_1(h_1), \ldots, \varphi_n(h_n)), t))$$
$$= \eta(M_1(\varphi_1(g_1), \varphi_1(h_1), t)).\eta(M_2(\varphi_2(g_2), \varphi_2(h_2), t)). \cdots .\eta(M_n(\varphi_n(g_n), \varphi_n(h_n), t))$$
$$\leq b_1\eta(M_1(g_1, h_1, t)).b_2\eta(M_2(g_2, h_2, t)). \cdots .b_n\eta(M_n(g_n, h_n, t))$$
$$\leq b\eta(M_1(g_1, h_1, t)).b\eta(M_2(g_2, h_2, t)). \cdots .b\eta(M_n(g_n, h_n, t))$$
$$= b\eta(M_p((g_1, \ldots, g_n), (h_1, \ldots, h_n), t))$$
$$= b\eta(M_p(g, h, t))$$

Hence, ϕ is $\mathcal{H}$-contraction mapping with respect to $\eta \in \mathcal{H}$. It gives $G_1 \times G_2 \times \cdots \times G_n$ is a strong self-similar group of $\mathcal{H}$-contraction.

The above Theorem 4.3 shows that, finite product of strong self-similar groups of $\mathcal{H}$-contraction is also a strong self-similar group of $\mathcal{H}$-contraction.

PROPOSITION 4.2   *A self-similar group of $\mathcal{H}$-contraction is a disconnected set.*

Proof   Let G be a self-similar group of $\mathcal{H}$-contraction. Then G is a topological fuzzy group. Proposition 4.1 shows that G is the attractor of the $\mathcal{H}$-IFS $\{\phi_0, \ldots, \phi_{n-1}\}$. Hence,

$$G = \phi_0(G) \cup \phi_1(G) \cup \cdots \cup \phi_{n-1}(G)$$
$$\Phi = \phi_i(G) \cap \phi_j(G)$$

for all $i, j \in \{0, 1, 2, \ldots, n-1\}$ and $i \neq j$. For every $i = 1, 2, \ldots, n-1$, the mappings $\phi_i: G \longrightarrow \phi_i(G)$ are $\mathcal{H}$-contraction with respect to $\eta \in \mathcal{H}$. $\mathcal{H}$-contraction mapping is t-uniform continuous. Further, t-uniformly continuous function maps compact set into compact set. Hence, $\phi_i(G)$ is compact subspace in $(G, M, *)$. $\phi_i(G)$ is a closed set in $(G, M, *)$ for all $i \in \{0, 1, 2, \ldots, n-1\}$, since $(G, M, *)$ is Hausdroff space. Due to the fact that

$$G = \phi_0(G) \cup [\phi_1(G) \cup \cdots \cup \phi_{n-1}(G)]$$
$$\Phi = \phi_0(G) \cap [\phi_1(G) \cup \cdots \cup \phi_{n-1}(G)])$$

hence G can be written as disjoint union of non-empty closed sets $\phi_0(G), [\phi_1(G) \cup \cdots \cup \phi_{n-1}(G)]$. That is, G is a disconnected set.

PROPOSITION 4.3   *A strong self-similar group of $\mathcal{H}$-contraction is a totally disconnected set.*

Proof   Let G be a strong self-similar group of $\mathcal{H}$-contraction. Proposition 4.1 shows that G is the attractor of a $\mathcal{H}$-IFS $\{\phi_0, \ldots, \phi_{n-1}\}$. Since $\phi_0: G \longrightarrow H$ is one-to-one, we get

$$\phi_i(g) = \phi_i(h)$$
$$\phi_0(g) * x_i = \phi_0(h) * x_i$$
$$\phi_0(g) = \phi_0(h)$$
$$g = h$$

for all $g, h \in G$. This shows that ϕ_i is one-to-one for $i = 1, 2, \ldots, n - 1$. In addition that, $\phi_i(G) \cap \phi_j(G) = \Phi$, for all $i, j \in \{0, 1, 2, \ldots, n - 1\}$ and $i \neq j$. As per the Theorem 3.3, G is a totally disconnected set.

The following theorem illuminates connections between fuzzy profinite group and strong self-similar group of $\mathcal{H}$-contraction.

THEOREM 4.4   *A strong self-similar group of $\mathcal{H}$-contraction is a fuzzy profinite group.*

Proof   Let A be a strong self-similar group of $\mathcal{H}$-contraction. By the Definition 4.5, A is a compact topological fuzzy group and also A is Hausdorff since every fuzzy metric space is Hausdorff. Proposition 4.3 shows that A is a totally disconnected set. Finally, we get A is compact, Hausdorff, and totally disconnected. Thus, we have the properties which characterize profinite groups. This shows that a strong self-similar group of $\mathcal{H}$-contraction is a profinite group.

Remark 4.1   As per the Remark 3.1, $\mathcal{H}$-contraction is the generalization of fuzzy contraction. Hence, the above theorems and propositions are all true in the sense of the self-similar group of fuzzy contraction, in this case, $\eta \in \mathcal{H}$ of the form $\eta(t) = \frac{1}{t} - 1, t \in (0, 1]$. We illustrate our result by the following examples.

Example 4.1   Consider the continuous t-norm $a * b = ab$. Given a finite group G with fuzzy metric induced by discrete metric. Since G is a discrete topological space, it is a compact topological group with respect to the fuzzy metric. Let $|G| = m$ and $H = \{e\}$, where e is the identity element of G. So, it is clear that $[G:H] = m$. Moreover, the mapping

$$\phi: G \to H$$
$$g \mapsto e$$

is a surjective group homomorphism. If $\eta(t) = \frac{1}{t} - 1, t \in (0, 1]$, then ϕ is fuzzy contraction. As a result, G is a self-similar group but not a strong self-similar group. Since, no finite group is isomorphic to its proper subgroup.

4.1. Construction of fuzzy self-similar group

Example 4.2   Self-similar group of fuzzy contraction on Cantor set
Consider the direct product group

$$G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots$$

Define the fuzzy metric on G by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

where $d(x, y) = 2 \left| \sum_{i=1}^{\infty} \frac{x_i - y_i}{3^i} \right|$ for all $x, y \in G$ and $t > 0$. G is compact topological group with respect to the fuzzy metric defined above. Consider $\eta(t) = \frac{1}{t} - 1$, define the $\mathcal{H}$-contraction mappings with respect to η as follows:

$$f_0 : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots$$
$$(x_1, x_2, x_3, \ldots) \mapsto (0, 0, x_2, x_3, \ldots)$$
$$f_1 : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots$$
$$(x_1, x_2, x_3, \ldots) \mapsto (0, 1, x_2, x_3, \ldots)$$
$$f_2 : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots$$
$$(x_1, x_2, x_3, \ldots) \mapsto (1, 0, x_2, x_3, \ldots)$$
$$f_3 : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots$$
$$(x_1, x_2, x_3, \ldots) \mapsto (1, 1, x_2, x_3, \ldots)$$

Then, G is the attractor of the FIFS $\{G; f_0, f_1, f_2, f_3\}$.

$$f_0(G) = \{0\} \times \{0\} \times \mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z} \times \cdots$$

f_0 is a surjective group homomorphism but it is not one-to-one. Therefore, G is a fuzzy self-similar group of $\mathcal{H}$-contraction. The $\mathcal{H}$-contraction mappings f_0, f_1, f_2, f_3 are defined on the Cantor set C by

$$f_0 : C \to C$$
$$x \mapsto f_0(x) = \begin{cases} \frac{x}{3} & \text{for } 0 \leq x \leq \frac{1}{3} \\ \frac{1}{3}\left(x - \frac{2}{3}\right) & \text{for } \frac{2}{3} \leq x \leq 1 \end{cases}$$

$$f_1 : C \to C$$
$$x \mapsto f_1(x) = \begin{cases} \frac{x}{3} + \frac{2}{9} & \text{for } 0 \leq x \leq \frac{1}{3} \\ \frac{1}{3}\left(x - \frac{2}{3}\right) + \frac{2}{9} & \text{for } \frac{2}{3} \leq x \leq 1 \end{cases}$$

$$f_2 : C \to C$$
$$x \mapsto f_2(x) = \begin{cases} \frac{x}{3} + \frac{2}{3} & \text{for } 0 \leq x \leq \frac{1}{3} \\ \frac{1}{3}\left(x - \frac{2}{3}\right) + \frac{2}{3} & \text{for } \frac{2}{3} \leq x \leq 1 \end{cases}$$

$$f_3 : C \to C$$
$$x \mapsto f_3(x) = \begin{cases} \frac{x}{3} + \frac{8}{9} & \text{for } 0 \leq x \leq \frac{1}{3} \\ \frac{1}{3}\left(x - \frac{2}{3}\right) + \frac{8}{9} & \text{for } \frac{2}{3} \leq x \leq 1 \end{cases}$$

That is, the mappings f_0, f_1, f_2, and f_3 are fuzzy $\mathcal{H}$-contractions with contraction constant $\frac{1}{3}$ and translations $0, \frac{2}{9}, \frac{2}{3}, \frac{8}{9}$, respectively.

Example 4.3   Strong self-similar group of fuzzy contraction on Cantor set

Consider the compact topological group $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots$ with the fuzzy metric defined in the previous example.

Define $f_0 : G \to G$ by $(x_1, x_2, x_3, \ldots) \mapsto (0, x_1, x_2, \ldots)$

$f_1 : G \to G$ by $(x_1, x_2, x_3, \ldots) \mapsto (1, x_1, x_2, \ldots)$. f_0, f_1 are fuzzy $\mathcal{H}$-contractions with respect to $\eta(t) = \frac{1}{t} - 1$ and G is the attractor of the IFS $\{G; f_0, f_1\}$. Moreover,

$$f_0(G) = \{0\} \times \mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z} \times \cdots$$

is a subgroup of G and f_0 is an isomorphism. This shows that G is a strong fuzzy self-similar group.

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