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Global well-posedness for a class of 2D Boussinesq systems with fractional dissipation

Wanrong Yang [a,*], Quansen Jiu [b], Jiahong Wu [c,d]

[a] Department of Mathematics, Beifang University of Nationalities, Ningxia 750021, PR China
[b] School of Mathematics, Capital Normal University, Beijing 100037, PR China
[c] Department of Mathematics, Oklahoma State University, 401 Mathematical Sciences, Stillwater, OK 74078, USA
[d] Department of Mathematics, College of Natural Science, Chung-Ang University, Seoul 156-756, South Korea

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Abstract

The incompressible Boussinesq equations not only have many applications in modeling fluids and geophysical fluids but also are mathematically important. The well-posedness and related problem on the Boussinesq equations have recently attracted considerable interest. This paper examines the global regularity issue on the 2D Boussinesq equations with fractional Laplacian dissipation and thermal diffusion. Attention is focused on the case when the thermal diffusion dominates. We establish the global well-posedness for the 2D Boussinesq equations with a new range of fractional powers of the Laplacian.

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1. Introduction

This paper studies the following 2D incompressible Boussinesq system with fractional dissipation

* Corresponding author.
E-mail addresses: yangwanrong1618@163.com (W. Yang), jiuqs@mail.cnu.edu.cn (Q. Jiu), jiahong@math.okstate.edu (J. Wu).

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$$\begin{cases} \partial_t u + u \cdot \nabla u + \nu \Lambda^\alpha u = -\nabla p + \theta e_2, & x \in \mathbb{R}^2,\ t > 0, \\ \partial_t \theta + u \cdot \nabla \theta + \kappa \Lambda^\beta \theta = 0, & x \in \mathbb{R}^2,\ t > 0, \\ \nabla \cdot u = 0, & x \in \mathbb{R}^2,\ t > 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), & x \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ denotes the 2D velocity, $p = p(x,t)$ the pressure, $\theta = \theta(x,t)$ the temperature, $\nu > 0$, $\kappa > 0$, $\alpha \in (0,1)$ and $\beta \in (0,1)$ are real parameters, and e_2 denotes the unit vector in the vertical direction. $\Lambda = (-\Delta)^{\frac{1}{2}}$ is the Zygmund operator and Λ^α can be defined through the Fourier transform,

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \widehat{f}(\xi),$$

where

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x)\, dx.$$

When $\alpha = 2$ and $\beta = 2$, (1.1) becomes the 2D Boussinesq equations with Laplacian dissipation. The standard 2D Boussinesq equations and their fractional Laplacian generalizations have recently attracted considerable attention due to their physical applications and mathematical significance. The Boussinesq equations model geophysical flows such as atmospheric fronts and oceanic circulation, and play an important role in the study of Rayleigh–Bénard convection (see, e.g., [12,17,30,33]). Mathematically the 2D Boussinesq equations serve as a lower dimensional model of the 3D hydrodynamics equations. In fact, the Boussinesq equations retain some key features of the 3D Navier–Stokes and the Euler equations such as the vortex stretching mechanism. As pointed out in [31], the inviscid Boussinesq equations can be identified with the 3D Euler equations for axisymmetric flows.

The goal of this paper is to establish the global well-posedness of (1.1) for the parameters α and β in a new range. Our attention is focused on the situation when the dissipation in the θ equation dominates. More precisely, we assume

$$0 < \alpha < 1, \qquad 0 < \beta < 1, \qquad \alpha < \beta. \tag{1.2}$$

The research presented here complements the existing results on the 2D Boussinesq equations with only partial dissipation or fractional Laplacian dissipation (see, e.g., [1–3,6–9,11,13–16,18–21,23–27,29,32,36,37]). The global regularity problem of 2D Boussinesq equations with only partial dissipation is not easy when α and β are in the range (1.2). The key obstacle is how to obtain global *a priori* bounds for the Sobolev norms (or equivalent Besov norms) of the solutions. For example, to bound the derivative of the velocity u, or the vorticity $\omega = \nabla \times u$, one resorts to the vorticity equation

$$\partial_t \omega + u \cdot \nabla \omega + \nu \Lambda^\alpha \omega = \partial_1 \theta \tag{1.3}$$

and immediately realizes that $\alpha \geq 2$ is needed in order to obtain even a global bound for the L^2-norm of ω when no prior information on the derivative of θ is known. Indeed, even the partial dissipation cases $\alpha = 2$ and $\kappa = 0$ or $\beta = 2$ and $\nu = 0$ are not trivial and have been dealt with by Chae [8] and by Hou and Li [20]. When $\alpha \le 1$ and $\beta \le 1$, the situation becomes more difficult and special techniques have to be developed to overcome the difficulty.

As suggested by Jiu, Miao, Wu and Zhang in [21], we classify the parameters α and β into three categories:

(i) the subcritical case, $\alpha + \beta > 1$;
(ii) the critical case, $\alpha + \beta = 1$;
(iii) the supercritical case, $\alpha + \beta < 1$.

One rationale behind this division is that (1.1) in the critical case defined here can indeed be converted into the critical case for the surface quasi-geostrophic type equation, as demonstrated in [21]. Although it appears that the smaller the sum $\alpha + \beta$ is, the more difficult the global regularity problem is, we caution that even the subcritical case may be difficult to handle. In fact, the global regularity of (1.1) has been obtained for only two subcritical ranges of α and β. In [11], Constantin and Vicol verified the global regularity for the case

$$\nu > 0, \qquad \kappa > 0, \qquad \alpha \in (0, 2), \qquad \beta \in (0, 2), \qquad \beta > \frac{2}{2+\alpha}.$$

Miao and Xue in [27] proved the global existence and uniqueness for (1.1) with $\nu > 0$, $\kappa > 0$ and

$$\alpha \in \left(\frac{6-\sqrt{6}}{4}, 1\right), \qquad \beta \in \left(1-\alpha, \min\left\{\frac{7+2\sqrt{6}}{5}\alpha - 2, \frac{\alpha(1-\alpha)}{\sqrt{6}-2\alpha}, 2-2\alpha\right\}\right). \tag{1.4}$$

For α and β in the critical case, several results are available. The global well-posedness of (1.1) with either $\alpha = 1$ and $\kappa = 0$ or $\nu = 0$ and $\beta = 1$ was obtained in [18,19]. The more general critical case $\alpha + \beta = 1$ with $\alpha < 1$ and $\beta < 1$, namely when the dissipation is split between the velocity and the temperature equations, is extremely difficult and was recently examined by Jiu, Miao, Wu and Zhang [21]. They were able to obtain the global well-posedness for this general critical case when $\alpha \ge \alpha_0$, where $\alpha_0 = \frac{23-\sqrt{145}}{12} \approx 0.9132$. When α and β are in the supercritical range, the only result in the literature is a very recent work of Jiu, Wu and Yang [22], which established the eventual regularity of weak solutions of (1.1).

To complement the existing results described above, this paper focuses on the ranges of α and β specified in (1.2). The global well-posedness is not trivial and does not follow from any previous work. We first state our main result and then explain the approach.

Theorem 1.1. *Assume that $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ satisfy*

$$\frac{\alpha}{2} + \beta > 1, \qquad \beta \ge \frac{2}{3} + \frac{\alpha}{3} \quad \text{and} \quad \beta > \frac{10-5\alpha}{10-4\alpha}. \tag{1.5}$$

Consider (1.1) with (u_0, θ_0) satisfying $\nabla \cdot u_0 = 0$ and

$$u_0 \in H^1 \cap B^1_{\infty,1} \quad \text{and} \quad \theta_0 \in L^2 \cap B^1_{\infty,1}. \tag{1.6}$$

Then (1.1) has a unique global solution (u, θ) with

$$u \in L^\infty_{loc}\big([0,\infty); H^1 \cap B^1_{\infty,1}\big), \qquad \theta \in L^\infty_{loc}\big([0,\infty); L^2 \cap B^1_{\infty,1}\big). \tag{1.7}$$

Here $B^1_{\infty,1}$ denotes a Besov space. More details on Besov spaces can be found in Appendix A. The key component in the proof of Theorem 1.1 is to establish the global *a priori* bounds in the class defined in (1.7). This does not appear to be trivial and the energy methods are not sufficient for this purpose. Although the global bounds for $u \in L^\infty([0,T]; L^2)$ and $\theta \in L^\infty([0,T]; L^p)$ with $p \in [2,\infty]$ can be easily obtained, the global bounds for the derivatives are not evident. In fact, it appears to be difficult to obtain a global bound for the H^1-norm of u, or the L^2-norm of ω. When we perform a simple energy estimate on the vorticity equation (1.3),

$$\frac{d}{dt}\|\omega\|_{L^2}^2 + 2\left\|\Lambda^{\frac{\alpha}{2}}\omega\right\|_{L^2}^2 = 2\int \omega\partial_1\theta,$$

the right-hand side generated by the "vortex stretching" term $\partial_1\theta$ appears to prevent us from "closing" this inequality. The parameters ν and κ do not play any essential role and we set $\nu = \kappa = 1$ throughout the rest of this paper. A natural idea is to hide $\partial_1\theta$ by combining the vorticity equation and the temperature equation. Setting the operator

$$\mathcal{R}_\beta = \Lambda^{-\beta}\partial_1,$$

applying $\mathcal{R}_\beta$ to the temperature equation and then adding to the vorticity equation, we find that

$$G = \omega + \mathcal{R}_\beta\theta$$

satisfies

$$\partial_t G + u\cdot\nabla G + \Lambda^\alpha G = \Lambda^{\alpha-\beta}\partial_1\theta - [\mathcal{R}_\beta, u\cdot\nabla]\theta. \tag{1.8}$$

Here we have used the standard commutator notation

$$[\mathcal{R}_\beta, u\cdot\nabla]\theta = \mathcal{R}_\beta(u\cdot\nabla\theta) - u\cdot\nabla\mathcal{R}_\beta\theta.$$

Quantities similar to G have been introduced in [19] and [27] to deal with the cases when $\alpha = 0$ and $\beta = 1$ and when (α,β) satisfies (1.4). Although (1.8) appears to be more complicated than the vorticity equation, but the terms on the right of (1.8) are less singular than $\partial_1\theta$ in the vorticity equation. In fact, by obtaining a suitable bound for $[\mathcal{R}_\beta, u\cdot\nabla]\theta$, we are able to obtain a global bound for $\|G\|_{L^2}$ when α and β satisfy (1.5). The global L^2-bound allows us to obtain a global bound for $\|G\|_{L^p}$ with $2 < p < p_0 \equiv \frac{2(2-\alpha)}{3-\frac{3}{2}\alpha-\beta}$. To further the estimates, we exploit the smoothing effect of the dissipation in the temperature equation. By deriving the inequality from the temperature equation, for any $q \in (1,\infty)$ and $r \in [1,\infty)$,

$$2^{r\beta j}\int_0^t \|\Delta_j\theta\|_{L^p}^r\,d\tau \le C2^{(r-1)\beta j}\|\Delta_j\theta_0\|_{L^p}^r + C\|\theta_0\|_{L^\infty}^r\int_0^t \|\omega\|_{L^p}^r\,d\tau,$$

we can then bound $2^{r\beta j}\int_0^t \|\Delta_j\theta\|_{L^p}^r\,d\tau$ for $2 \le p < p_0$. Here Δ_j with $j \in \mathbb{N}$ denotes the Fourier localization operators (see Appendix A). This bound, in turn, can be used to globally bound $\|G\|_{L^p}$ for p in a bigger range. Through an iterative process, we establish a global bound for

$\|G\|_{L^p}$ with $2 \le p < \infty$. This global bound enables us to gain further regularity for $\|G\|_{B^0_{\infty,1}}$ and consequently a global bound for $\|u\|_{B^1_{\infty,1}}$, which yields Theorem 1.1.

The rest of this paper is divided into five regular sections and two appendices. Sections 2, 3 and 4 prove global bounds for $\|G\|_{L^2}$, $\|G\|_{L^p}$ with $2 < p < p_0$ and $\|G\|_{L^p}$ with any $2 < p < \infty$, respectively. Section 5 establishes global bounds for $\|G\|_{L^\infty}$, $\|G\|_{B^0_{\infty,1}}$ and $\|u\|_{B^1_{\infty,1}}$. Section 6 proves Theorem 1.1. Appendix A provides the definitions of some of the functional spaces and related facts and gives the proof of a commutator estimate in Besov space setting. Appendix B provides a statement of the Osgood inequality used in Section 6.

2. Global L^2-bound for G

This section establishes a global *a priori* bound for $\|G\|_{L^2}$. Recall that

$$G = \omega + \mathcal{R}_\beta \theta \quad \text{with } \mathcal{R}_\beta = \Lambda^{-\beta}\partial_1,$$

satisfies

$$\partial_t G + u \cdot \nabla G + \Lambda^\alpha G = \Lambda^{\alpha-\beta}\partial_1\theta - [\mathcal{R}_\beta, u\cdot\nabla]\theta. \tag{2.1}$$

This global bound is valid for any $\frac{\alpha}{2} + \beta > 1$ and $\beta \ge \frac{2}{3} + \frac{\alpha}{3}$.

Proposition 2.1. *Assume (u_0, θ_0) satisfies the assumptions stated in Theorem 1.1. Let (u, θ) be the corresponding solution of (1.1). If $\frac{\alpha}{2} + \beta > 1$ and $\beta \ge \frac{2}{3} + \frac{\alpha}{3}$, then G obeys a global L^2 bound, namely for any $T > 0$ and $t \le T$,*

$$\|G(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{\alpha}{2}} G(\tau)\|_{L^2}^2\, d\tau \le C(T, u_0, \theta_0),$$

where $C(T, u_0, \theta_0)$ is a constant depending on T and the initial data only.

To prove Proposition 2.1, the following elementary global *a priori* bounds will be used. Notice that θ_0 satisfying (1.6) especially implies $\theta_0 \in L^p$ for any $p \in [2, \infty]$.

Lemma 2.2. *Assume (u_0, θ_0) satisfies the assumptions stated in Theorem 1.1. Then the corresponding solution (u, θ) of (1.1) obeys the following global bounds, for any $t > 0$,*

$$\|\theta(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{\beta}{2}}\theta(\tau)\|_{L^2}^2 d\tau \le \|\theta_0\|_{L^2},$$

$$\|\theta(t)\|_{L^p} \le \|\theta_0\|_{L^p} \quad \textit{for any } 2 \le p \le \infty,$$

and

$$\|u(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{\alpha}{2}} u(\tau)\|_{L^2}^2 d\tau \le \left(\|u_0\|_{L^2} + t\|\theta_0\|_{L^2}\right)^2.$$

We will also need the following commutator estimate. Its proof is presented in Appendix A. We use extensively the Besov spaces $B^s_{p,r}$ and their definitions can also be found in Appendix A.

Proposition 2.3. *Let $\beta \in (0,1)$, $(p,r) \in [2,\infty) \times [1,\infty]$. Let $s \in (0,1)$ satisfy $s-\beta<0$. Then there exists a constant $C = C(p,r)$ such that*

$$\big\|[\mathcal{R}_\beta, f]g\big\|_{B^s_{p,r}} \le C\|\nabla f\|_{L^p}\|g\|_{B^{s-\beta}_{\infty,r}}.$$

We are now ready to prove Proposition 2.1.

Proof of Proposition 2.1. Taking the inner product of (2.1) with G, we obtain, after integration by parts,

$$\frac{1}{2}\frac{d}{dt}\|G\|^2_{L^2} + \big\|\Lambda^{\frac{\alpha}{2}}G\big\|^2_{L^2} = K_1 + K_2, \tag{2.2}$$

where

$$K_1 = \int G\Lambda^{\alpha-\beta}\partial_1\theta\,dx,$$

$$K_2 = -\int G[\mathcal{R}_\beta, u\cdot\nabla]\theta\,dx = \int G\nabla\cdot[\mathcal{R}_\beta, u]\theta\,dx.$$

By Hölder's inequality and the fact that $\beta \ge \frac{2}{3}+\frac{\alpha}{3}$,

$$\begin{aligned}|K_1| &\le \big\|\Lambda^{\frac{\alpha}{2}-\beta}\partial_1\theta\big\|_{L^2}\big\|\Lambda^{\frac{\alpha}{2}}G\big\|_{L^2}\\ &\le \|\theta\|_{H^{1+\frac{\alpha}{2}-\beta}}\big\|\Lambda^{\frac{\alpha}{2}}G\big\|_{L^2}\\ &\le \|\theta\|_{H^{\frac{\beta}{2}}}\big\|\Lambda^{\frac{\alpha}{2}}G\big\|_{L^2}.\end{aligned}$$

By Proposition 2.3,

$$\begin{aligned}|K_2| &\le \big\|\Lambda^{1-\frac{\alpha}{2}}[\mathcal{R}_\beta, u]\theta\big\|_{L^2}\big\|\Lambda^{\frac{\alpha}{2}}G\big\|_{L^2}\\ &\le \big\|[\mathcal{R}_\beta, u]\theta\big\|_{B^{1-\frac{\alpha}{2}}_{2,2}}\big\|\Lambda^{\frac{\alpha}{2}}G\big\|_{L^2}\\ &\le \|\nabla u\|_{L^2}\|\theta\|_{B^{1-\frac{\alpha}{2}-\beta}_{\infty,2}}\big\|\Lambda^{\frac{\alpha}{2}}G\big\|_{L^2}.\end{aligned}$$

Since $\beta \ge \frac{2}{3}+\frac{\alpha}{3}$,

$$\|\nabla u\|_{L^2} = \|\omega\|_{L^2} \le \|G\|_{L^2} + \big\|\Lambda^{1-\beta}\theta\big\|_{L^2} \le \|G\|_{L^2} + \|\theta\|_{H^{\frac{\beta}{2}}}.$$

Due to $1-\frac{\alpha}{2}-\beta<0$,

$$\|\theta\|_{B^{1-\frac{\alpha}{2}-\beta}_{\infty,2}} = \left[\sum_{j=-1}^{\infty} 2^{2j(1-\frac{\alpha}{2}-\beta)}\|\Delta_j\theta\|^2_{L^\infty}\right]^{1/2} \le C\|\theta_0\|_{L^\infty}.$$

Therefore,

$$|K_2| \le C\|G\|_{L^2}\big\|\Lambda^{\frac{\alpha}{2}}G\big\|_{L^2} + C\|\theta\|_{H^{\frac{\beta}{2}}}\big\|\Lambda^{\frac{\alpha}{2}}G\big\|_{L^2}.$$

Inserting the bounds for K_1 and K_2 in (2.2), applying Young's inequality and invoking the bounds for θ in Lemma 2.2 yield the desired bound. This completes the proof of Proposition 2.1. □

3. Global bound for $\|G\|_{L^p}$ with $2 < p < p_0$

By making use of the global L^2-bound, this section proves a global bound for $\|G\|_{L^p}$ with $2 < p < p_0$, where p_0 is specified in (3.1).

Proposition 3.1. *Assume that $\alpha \in (0,1)$ and $\beta \in (0,1)$ satisfy $\frac{\alpha}{2} + \beta > 1$ and $\beta \ge \frac{2}{3} + \frac{\alpha}{3}$. Assume (u_0, θ_0) satisfies (1.6) and let (u, θ) be the corresponding solution of (1.1). Assume*

$$2 < p < p_0 \quad \text{with } p_0 = \frac{4(2-\alpha)}{6 - 3\alpha - 2\beta}. \tag{3.1}$$

Then, for any $T > 0$ and $t \le T$,

$$\big\|G(t)\big\|_{L^p} \le C(T, u_0, \theta_0),$$

where $C(T, u_0, \theta_0)$ is a constant depending on T and the initial data only.

We will use the following lemma (see [19,21]).

Lemma 3.2. *Let $s \in (0,1)$, $\alpha \in (0,1)$ and $p \in [2,\infty)$. Then,*

$$\big\|\Lambda^s\big(G|G|^{p-2}\big)\big\|_{L^2} \le C\|G\|^{p-2}_{L^{\frac{2p}{2-\alpha}}}\|G\|_{\dot{B}^s_{\widetilde{p},2}},$$

where $\widetilde{p} = \frac{2p}{2(2-\alpha)-p(1-\alpha)}$ and $\dot{B}^s_{\widetilde{p},2}$ denotes a homogeneous Besov space (see Appendix A). Especially,

$$\big\|\Lambda^s\big(G|G|^{p-2}\big)\big\|_{L^2} \le C\|G\|^{p-2}_{L^{\frac{2p}{2-\alpha}}}\|G\|_{\dot{H}^{2+s-\alpha-\frac{2(2-\alpha)}{p}}}.$$

Proof of Proposition 3.1. Taking the inner product of (2.1) with $G|G|^{p-2}$, we have

$$\frac{1}{p}\frac{d}{dt}\|G\|^p_{L^p} + \int G|G|^{p-2}\Lambda^\alpha G\,dx = F_1 + F_2, \tag{3.2}$$

where

$$F_1 = -\int G|G|^{p-2}[\mathcal{R}_\beta, u\cdot\nabla]\theta\,dx,$$

$$F_2 = \int G|G|^{p-2}\Lambda^{\alpha-\beta}\partial_1\theta\,dx.$$

By a pointwise inequality for fractional Laplacians (see [10]) and a Sobolev embedding inequality,

$$\int G|G|^{p-2}\Lambda^\alpha G dx \geq C\int \left|\Lambda^{\frac{\alpha}{2}}|G|^{\frac{p}{2}}\right|^2 dx \geq C_0\|G\|^p_{L^{\frac{2p}{2-\alpha}}}.$$

Applying Proposition 2.3 and Lemma 3.2, we have for any $s\in(0,1)$,

$$\begin{aligned}
|F_1| &= \left|\int \Lambda^s\big(G|G|^{p-2}\big)\Lambda^{1-s}\big(\Lambda^{-1}\nabla\cdot[\mathcal{R}_\beta,u]\theta\big)dx\right| \\
&\leq \big\|\Lambda^s\big(G|G|^{p-2}\big)\big\|_{L^2}\big\|\Lambda^{1-s}[\mathcal{R}_\beta,u]\theta\big\|_{L^2} \\
&= \big\|\Lambda^s\big(G|G|^{p-2}\big)\big\|_{L^2}\big\|[\mathcal{R}_\beta,u]\theta\big\|_{\dot H^{1-s}} \\
&\leq C\|G\|^{p-2}_{L^{\frac{2p}{2-\alpha}}}\|G\|_{\dot H^{2+s-\alpha-\frac{2(2-\alpha)}{p}}}\|\nabla u\|_{L^2}\|\theta\|_{B^{1-s-\beta}_{\infty,2}}.
\end{aligned}$$

For p satisfying (3.1), we can choose $s\in(0,1)$ such that

$$1-s-\beta<0,\qquad 2+s-\alpha-\frac{2(2-\alpha)}{p}=\frac{\alpha}{2}.$$

Bounding $\|\nabla u\|_{L^2}$ by

$$\|\nabla u\|_{L^2}\leq\|G\|_{L^2}+\big\|\Lambda^{1-\beta}\theta\big\|_{L^2}\leq\|G\|_{L^2}+C\|\theta\|^{3-\frac{2}{\beta}}_{L^2}\big\|\Lambda^{\frac{\beta}{2}}\theta\big\|^{\frac{2}{\beta}-2}_{L^2},$$

we obtain, invoking the global bound for $\|G\|_{L^2}$,

$$\begin{aligned}
|F_1| &\leq C\|G\|^{p-2}_{L^{\frac{2p}{2-\alpha}}}\|G\|_{\dot H^{\frac{\alpha}{2}}}\big(\|G\|_{L^2}+\|\theta\|_{H^{\frac{\beta}{2}}}\big)\|\theta_0\|_{L^\infty} \\
&\leq \frac{C_0}{4}\|G\|^p_{L^{\frac{2p}{2-\alpha}}}+C\Big(\|G\|^{\frac{p}{2}}_{L^2}+\big\|\Lambda^{\frac{\beta}{2}}\theta\big\|^{(\frac{2}{\beta}-2)\frac{p}{2}}_{L^2}\Big)\|G\|^{\frac{p}{2}}_{\dot H^{\frac{\alpha}{2}}} \\
&\leq \frac{C_0}{4}\|G\|^p_{L^{\frac{2p}{2-\alpha}}}+C\|G\|^2_{\dot H^{\frac{\alpha}{2}}}+C\big\|\Lambda^{\frac{\beta}{2}}\theta\big\|^{(\frac{2}{\beta}-2)\frac{p}{2}\frac{4}{4-p}}_{L^2}.
\end{aligned}$$

It is easy to check that, for $\beta\geq\frac{2}{3}$ and p satisfying (3.1), we have

$$\left(\frac{2}{\beta}-2\right)\frac{p}{2}\frac{4}{4-p}\leq 2.$$

Again, by Proposition 2.3,

$$\begin{aligned}
|F_2| &\leq \big\|\Lambda^s\big(G|G|^{p-2}\big)\big\|_{L^2}\big\|\Lambda^{1+\alpha-\beta-s}\theta\big\|_{L^2} \\
&\leq C\|G\|^{p-2}_{L^{\frac{2p}{2-\alpha}}}\|G\|_{\dot H^{\frac{\alpha}{2}}}\big\|\Lambda^{\frac{\beta}{2}}\theta\big\|^a_{L^2}\|\theta\|^{1-a}_{L^2} \\
&\leq \frac{C_0}{4}\|G\|^p_{L^{\frac{2p}{2-\alpha}}}+C\|G\|^2_{\dot H^{\frac{\alpha}{2}}}+C\Big(\|\theta\|^{1-a}_{L^2}\big\|\Lambda^{\frac{\beta}{2}}\theta\big\|^a_{L^2}\Big)^{\frac{2p}{4-p}},
\end{aligned}$$

where $a = \frac{2(1+\alpha-\beta-s)}{\beta}$ and $a \cdot \frac{2p}{4-p} \le 2$. Inserting the bounds for F_1 and F_2 in (3.2) and then integrating with respect to t yield the desired bound. This completes the proof of Proposition 3.1. □

4. Iteration and global bound for $\|G\|_{L^p}$ with any $2 < p < \infty$

The goal of this section is to show that, for any $2 < p < \infty$, $\|G\|_{L^p}$ admits a global bound. More precisely, we prove the following proposition.

Proposition 4.1. *Assume that α and β satisfy (1.5). Assume (u_0, θ_0) satisfies (1.6) and let (u, θ) be the corresponding solution of (1.1). Then, for $2 \le p < \infty$ and any $T > 0$, $t \le T$,*

$$\|G(t)\|_{L^p} \le C(T, u_0, \theta_0),$$

where $C(T, u_0, \theta_0)$ is a constant depending on p, T and the initial data only.

In order to achieve this bound, we exploit the dissipation in the temperature equation and derive the inequality bounding $2^{r\beta j}\|\Delta_j\theta\|_{L_t^r L^q}$ in terms of $\|\omega\|_{L_t^r L_x^q}$ (see Lemma 4.2). To be more precise, we consider the transport–diffusion equation

$$\begin{cases} \partial_t\theta + u \cdot \nabla\theta + \Lambda^\beta\theta = 0, \\ \theta(x, 0) = \theta_0(x). \end{cases} \tag{4.1}$$

Lemma 4.2. *Let $p \in (1, \infty)$. Assume that $\theta_0 \in L^p \cap L^\infty$. Assume that (u, θ) solves (4.1). Let ω be the corresponding vorticity. Then, for any $r \in [1, \infty)$ and any integer $j \ge 0$,*

$$2^{r\beta j}\int_0^t \|\Delta_j\theta\|_{L^p}^r\, d\tau \le C2^{(r-1)\beta j}\|\Delta_j\theta_0\|_{L^p}^r + C\|\theta_0\|_{L^\infty}^r \int_0^t \|\omega\|_{L^p}^r\, d\tau, \tag{4.2}$$

where C is a constant independent of j.

Proof. Applying Δ_j to the equation in (4.1) yields

$$\partial_t\Delta_j\theta + u \cdot \nabla\Delta_j\theta + \Lambda^\beta\Delta_j\theta = -[\Delta_j, u \cdot \nabla]\theta.$$

Multiplying the above equation by $\Delta_j\theta|\Delta_j\theta|^{p-2}$, integrating by parts and using Hölder's inequality, we get

$$\frac{1}{p}\frac{d}{dt}\|\Delta_j\theta\|_{L^p}^p + \int \Delta_j\theta|\Delta_j\theta|^{p-2}\left(\Lambda^\beta\Delta_j\theta\right)dx \le \|\Delta_j\theta\|_{L^p}^{p-1}\left\|[\Delta_j, u \cdot \nabla]\theta\right\|_{L^p}.$$

Recalling the following generalized Bernstein inequality (see [28]),

$$c_p 2^{\beta j}\|\Delta_j\theta\|_{L^p}^p \le \int \Delta_j\theta|\Delta_j\theta|^{p-2}\left(\Lambda^\beta\Delta_j\theta\right)dx$$

and applying Lemma A.7,

$$\big\|[\Delta_j, u\cdot\nabla]\theta\big\|_{L^p} \le C\|\nabla u\|_{L^p}\|\theta\|_{L^\infty},$$

we obtain

$$\frac{d}{dt}\|\Delta_j\theta\|_{L^p} + c2^{\beta j}\|\Delta_j\theta\|_{L^p} \le C\|\nabla u\|_{L^p}\|\theta\|_{L^\infty}.$$

Integrating in time and using the fact that

$$\|\nabla u\|_{L^p} \le C\|\omega\|_{L^p} \quad \text{for any } p\in(1,\infty),$$

we have

$$\big\|\Delta_j\theta(t)\big\|_{L^p} \le C\|\Delta_j\theta_0\|_{L^p}e^{-ct2^{\beta j}} + C\|\theta_0\|_{L^\infty}\int_0^t e^{-c(t-\tau)2^{\beta j}}\|\omega\|_{L^p}\,d\tau.$$

Taking the L^r-norm in time each side and using Young's inequality for convolution, we obtain (4.2). This complete the proof of Lemma 4.2. □

Proof of Proposition 4.1. The proof relies on an iterative process. By Proposition 3.1, for p_0 given by (3.1),

$$\|G\|_{L^p} \le C \quad \text{for any } 2<p<p_0.$$

Consequently, for $2<p<p_0$ and $1\le r\le 2$,

$$\begin{aligned}\int_0^t \|\omega\|_{L^p}^r\,d\tau &\le \int_0^t \|G\|_{L^p}^r\,d\tau + \int_0^t \big\|\Lambda^{1-\beta}\theta\big\|_{L^p}^r\,d\tau \\ &\le \int_0^t \|G\|_{L^p}^r\,d\tau + C\int_0^t \|\theta\|_{H^{\frac{\beta}{2}}}^r\,d\tau \le C(t),\end{aligned}$$

where $C(t)$ is a constant depending on t and the initial data. Therefore, by Lemma 4.2, for $2<p<p_0$ and $1\le r\le 2$,

$$\begin{aligned}2^{r\beta j}\int_0^t \|\Delta_j\theta\|_{L^p}^r\,d\tau &\le C2^{(r-1)\beta j}\|\Delta_j\theta_0\|_{L^p}^r + C\|\theta_0\|_{L^\infty}^r\int_0^t\|\omega\|_{L^p}^r\,d\tau \\ &\le C(t).\end{aligned} \tag{4.3}$$

Let p_* be close to p_0, say $p_*=p_0-\delta$ for a small $\delta>0$. Since $1+\alpha-2\beta<0$, we can choose p_1 satisfying

$$p_1>p_0, \qquad 1+\alpha-2\beta+\frac{2}{p_*}-\frac{2}{p_1}<0. \tag{4.4}$$

We first show the global bound of $\|G\|_{L^p} \le C$ for any $p_0 \le p \le p_1$. Taking the inner product of (2.1) with $G|G|^{p-2}$, we have

$$\frac{1}{p}\frac{d}{dt}\|G\|_{L^p}^p + \int G|G|^{p-2}\Lambda^\alpha G\,dx = F_1 + F_2, \tag{4.5}$$

where

$$F_1 = -\int G|G|^{p-2}[\mathcal{R}_\beta, u\cdot\nabla]\theta\,dx,$$
$$F_2 = \int G|G|^{p-2}\Lambda^{\alpha-\beta}\partial_1\theta\,dx.$$

We start with the estimate of F_2, which is simpler. By Hölder's inequality,

$$|F_2| \le \|G\|_{L^p}^{p-1}\left\|\Lambda^{\alpha-\beta}\partial_1\theta\right\|_{L^p}.$$

By Bernstein's inequality,

$$\begin{aligned}\left\|\Lambda^{\alpha-\beta}\partial_1\theta\right\|_{L^p} &\le \sum_{j\ge -1}\left\|\Delta_j\Lambda^{\alpha-\beta}\partial_1\theta\right\|_{L^p}\\ &\le C\|\theta_0\|_{L^2} + \sum_{j\ge 0}2^{(1+\alpha-\beta)j}2^{2j(\frac{1}{p_*}-\frac{1}{p})}\|\Delta_j\theta\|_{L^{p_*}}\\ &\le C\|\theta_0\|_{L^2} + \sum_{j\ge 0}2^{(1+\alpha-2\beta+\frac{2}{p_*}-\frac{2}{p})j}2^{\beta j}\|\Delta_j\theta\|_{L^{p_*}}.\end{aligned} \tag{4.6}$$

By Hölder's inequality,

$$|F_1| \le \|G\|_{L^p}^{p-1}\left\|[\mathcal{R}_\beta, u\cdot\nabla]\theta\right\|_{L^p}.$$

The commutator will be estimated as follows. We first apply the trivial inequality

$$\left\|[\mathcal{R}_\beta, u\cdot\nabla]\theta\right\|_{L^p} \le \sum_{k\ge -1}\left\|\Delta_k[\mathcal{R}_\beta, u\cdot\nabla]\theta\right\|_{L^p}$$

and then bound the right-hand side as in the proof of Proposition A.6. Writing

$$\Delta_k[\mathcal{R}_\beta, u\cdot\nabla]\theta = J_1 + J_2 + J_3,$$

where

$$J_1 = \sum_{|j-k|\le 2}\Delta_k\big(\mathcal{R}_\beta(S_{j-1}u\Delta_j\nabla\theta) - S_{j-1}u\mathcal{R}_\beta\Delta_j\nabla\theta\big),$$
$$J_2 = \sum_{|j-k|\le 2}\Delta_k\big(\mathcal{R}_\beta(\Delta_j u S_{j-1}\nabla\theta) - \Delta_j u\mathcal{R}_\beta S_{j-1}\nabla\theta\big),$$

$$J_3 = \sum_{j\ge k-1} \Delta_k\big(\mathcal{R}_\beta(\Delta_j u\tilde{\Delta}_j\nabla\theta) - \Delta_j u\mathcal{R}_\beta\tilde{\Delta}_j\nabla\theta\big),$$

we have

$$\sum_{k\ge -1}\big\|\Delta_k[\mathcal{R}_\beta, u\cdot\nabla]\theta\big\|_{L^p} \le \sum_{k\ge -1}\big(\|J_1\|_{L^p} + \|J_2\|_{L^p} + \|J_3\|_{L^p}\big).$$

Applying Lemma A.7 and by Bernstein's inequality, we have

$$\begin{aligned}\sum_{k\ge -1}\|J_1\|_{L^p} &\le C\sum_{k\ge -1} 2^{(1-\beta)k}\|\nabla S_{k-1}u\|_{L^p}\|\Delta_k\theta\|_{L^\infty}\\ &\le C\|\nabla u\|_{L^p}\sum_{k\ge -1} 2^{(1-\beta)k}\|\Delta_k\theta\|_{L^\infty}\\ &\le C\|\nabla u\|_{L^p}\left(\|\theta_0\|_{L^2} + \sum_{k\ge 0} 2^{(1-2\beta+\frac{2}{p_*})k}2^{\beta k}\|\Delta_k\theta\|_{L^{p_*}}\right).\end{aligned}$$

By Hölder's and Bernstein's inequalities and the fact that $S_{k-1}\nabla\theta = 0$ for $k < 1$,

$$\begin{aligned}\sum_{k\ge -1}\|J_2\|_{L^p} &\le C\sum_{k\ge 1} 2^{-\beta k}\|\nabla\Delta_k u\|_{L^p}\|S_{k-1}\nabla\theta\|_{L^\infty}\\ &\le C\|\nabla u\|_{L^p}\sum_{k\ge 1} 2^{-\beta k}\sum_{m\le k-2}\|\Delta_m\nabla\theta\|_{L^\infty}\\ &\le C\|\nabla u\|_{L^p}\sum_{k\ge 1} 2^{-\beta k}\sum_{m\le k-2} 2^{(1+\frac{2}{p_*}-\beta)m}2^{\beta m}\|\Delta_m\theta\|_{L^{p_*}}\\ &\le C\|\nabla u\|_{L^p}\sum_{k\ge 1} 2^{(1+\frac{2}{p_*}-2\beta)k}\sum_{m\le k-2} 2^{(1+\frac{2}{p_*}-\beta)(m-k)}2^{\beta m}\|\Delta_m\theta\|_{L^{p_*}}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\sum_{k\ge -1}\|J_3\|_{L^p} &\le \sum_{k\ge -1}\sum_{j\ge k-1} 2^{k-j}2^{(1-\beta)j}\|\nabla\Delta_j u\|_{L^p}\|\Delta_j\theta\|_{L^\infty}\\ &\le \|\nabla u\|_{L^p}\sum_{k\ge -1}\sum_{j\ge k-1} 2^{k-j}2^{(1+\frac{2}{p_*}-2\beta)j}2^{\beta j}\|\Delta_j\theta\|_{L^{p_*}}.\end{aligned}$$

$\|\nabla u\|_{L^p}$ can be bounded by

$$\|\nabla u\|_{L^p} \le C\|\omega\|_{L^p} \le C\|G\|_{L^p} + C\big\|\Lambda^{-\beta}\partial_1\theta\big\|_{L^p}$$

while $\|\Lambda^{-\beta}\partial_1\theta\|_{L^p}$ can be bounded as in (4.6) by

$$\big\|\Lambda^{-\beta}\partial_1\theta\big\|_{L^p} \le C\|\theta_0\|_{L^2} + \sum_{j\ge 0} 2^{(1-2\beta+\frac{2}{p_*}-\frac{2}{p})j}2^{\beta j}\|\Delta_j\theta\|_{L^{p_*}}.$$

Therefore,

$$\big\|[\mathcal{R}_\beta, u\cdot\nabla]\theta\big\|_{L^p} \le Cg(t)\bigg(\|G\|_{L^p} + \|\theta_0\|_{L^2} + \sum_{j\ge 0} 2^{(1-2\beta+\frac{2}{p_*}-\frac{2}{p})j}2^{\beta j}\|\Delta_j\theta\|_{L^{p_*}}\bigg),$$

where $g(t)$ is given by

$$\begin{aligned} g(t) \equiv{}& \|\theta_0\|_{L^2} + \sum_{k\ge 0} 2^{(1-2\beta+\frac{2}{p_*})k}2^{\beta k}\|\Delta_k\theta\|_{L^{p_*}} \\ &+ \sum_{k\ge 1} 2^{(1+\frac{2}{p_*}-2\beta)k}\sum_{m\le k-2} 2^{(1+\frac{2}{p_*}-\beta)(m-k)}2^{\beta m}\|\Delta_m\theta\|_{L^{p_*}} \\ &+ \sum_{k\ge -1} 2^{k-j}\sum_{j\ge k-1} 2^{(1+\frac{2}{p_*}-2\beta)j}2^{\beta j}\|\Delta_j\theta\|_{L^{p_*}}. \end{aligned}$$

Inserting the bounds for F_1 and F_2 in (4.5), we obtain

$$\begin{aligned} \frac{d}{dt}\|G\|_{L^p} \le{}& C\|\theta_0\|_{L^2} + C\sum_{j\ge 0} 2^{(1+\alpha-2\beta+\frac{2}{p_*}-\frac{2}{p})j}2^{\beta j}\|\Delta_j\theta\|_{L^{p_*}} + Cg(t)\|G\|_{L^p} \\ &+ Cg(t)\bigg(\|\theta_0\|_{L^2} + \sum_{j\ge 0} 2^{(1-2\beta+\frac{2}{p_*}-\frac{2}{p})j}2^{\beta j}\|\Delta_j\theta\|_{L^{p_*}}\bigg). \end{aligned} \tag{4.7}$$

We then integrate in time. Due to (4.4) and (4.3) with $r = 1$,

$$\begin{aligned} \int_0^t \sum_{j\ge 0} 2^{(1+\alpha-2\beta+\frac{2}{p_*}-\frac{2}{p})j}2^{\beta j}\|\Delta_j\theta\|_{L^{p_*}}\,d\tau &= \sum_{j\ge 0} 2^{(1+\alpha-2\beta+\frac{2}{p_*}-\frac{2}{p})j}2^{\beta j}\int_0^t \|\Delta_j\theta\|_{L^{p_*}}\,d\tau \\ &\le \sup_{j\ge 0} 2^{\beta j}\int_0^t \|\Delta_j\theta\|_{L^{p_*}}\,d\tau \\ &\le C(t). \end{aligned}$$

Since α and β satisfy (1.5), especially $\beta > \frac{10-5\alpha}{10-4\alpha}$, we have $1-2\beta+\frac{2}{p_*} < 0$. Thus,

$$\int_0^t g(\tau)\,d\tau \le C\sup_{j\ge 0} 2^{\beta j}\int_0^t \|\Delta_j\theta\|_{L^{p_*}}\,d\tau \le C(t)$$

and

$$\int_0^t g(\tau)\sum_{j\ge 0} 2^{(1-2\beta+\frac{2}{p_*}-\frac{2}{p})j}2^{\beta j}\|\Delta_j\theta\|_{L^{p_*}}\,d\tau \le C\sup_{j\ge 0} 2^{2\beta j}\int_0^t \|\Delta_j\theta\|^2_{L^{p_*}}\,d\tau \le C(t)$$

due to (4.3) with $r=2$. Integrating (4.7) and using the bounds above, we obtain

$$\big\|G(t)\big\|_{L^p} \le C(t)$$

for any $p_0 \le p \le p_1$. The process above can be iterated for $p_1 < p \le p_2$ with the gap $p_2 - p_1$ as large as $p_1 - p_0$. Therefore, an iterative process would allow us to extend the global bound to any $2 < p < \infty$. This completes the proof of Proposition 4.1. □

5. Global bounds for $\|G\|_{B^0_{\infty,1}}$ and for $\|u\|_{B^1_{\infty,1}}$

This section establishes a global bound for $\|G\|_{B^0_{\infty,1}}$ and consequently a global bound for $\|u\|_{L^1_t B^1_{\infty,1}}$ and then $\|\omega\|_{L^\infty_t B^0_{\infty,1}}$. We state our results in two propositions. The first proposition is a global L^∞ for G for the completeness. The second proposition proves a global bound for $\|G\|_{B^0_{\infty,1}}$, which consequently yields $\|u\|_{L^1_t B^1_{\infty,1}}$. Once we have this bound for the velocity, then all other *a priori* bounds follow.

Proposition 5.1. *Assume that α and β satisfy (1.5). Assume (u_0, θ_0) satisfies (1.6) and let (u,θ) be the corresponding solution of (1.1). Then, for any $T > 0$ and $t \le T$,*

$$\big\|G(t)\big\|_{L^\infty} \le C(T, u_0, \theta_0),$$

where $C(T, u_0, \theta_0)$ is a constant depending on T and the initial data only.

Proposition 5.2. *Assume that α and β satisfy (1.5). Assume (u_0, θ_0) satisfies (1.6) and let (u,θ) be the corresponding solution of (1.1). Then, for any $T > 0$ and $t \le T$,*

$$\big\|G(t)\big\|_{B^0_{\infty,1}} \le C(T, u_0, \theta_0), \tag{5.1}$$

where $C(T, u_0, \theta_0)$ is a constant depending on T and the initial data only. A special consequence is the global bound

$$\|u\|_{B^1_{\infty,1}} \le C(T, u_0, \theta_0).$$

Furthermore,

$$\|\omega\|_{L^2 \cap B^0_{\infty,1}} \le C(T, u_0, \theta_0), \qquad \|\theta\|_{B^1_{\infty,1}} \le C(T, u_0, \theta_0).$$

To prove Proposition 5.2, we need the following lemma (see, e.g., [28]).

Lemma 5.3. *Let $(p,r) \in [1,\infty]$. If v is a Lipschitz divergence-free vector field, u is a solution of the following equation*

$$\begin{cases} \partial_t u + v \cdot \nabla u + \nu \Lambda^\alpha u = f, \quad 0 \le \alpha \le 2 \\ u(x,0) = u_0 \in B^0_{p,r} \end{cases}$$

Then for any $t > 0$, there exists a constant C such that

$$\|u\|_{\mathcal{L}_t^\infty B^0_{p,r}} \le C\left(\|u_0\|_{B^0_{p,r}} + \|f\|_{\mathcal{L}_t^1 B^0_{p,r}}\right)\left(1 + \int_0^t \|\nabla v\|_{L^\infty} dt\right),$$

where the space–time Besov space $\mathcal{L}_t^\infty B^0_{p,r}$ is defined Appendix A.

Proof of Proposition 5.1. Recall that G satisfies

$$\partial_t G + u \cdot \nabla G + \Lambda^\alpha G = \Lambda^{\alpha-\beta}\partial_1\theta - [\mathcal{R}_\beta, u \cdot \nabla]\theta. \tag{5.2}$$

By the maximum principle,

$$\|G\|_{L^\infty} \le \|G_0\|_{L^\infty} + \int_0^t \left(\left\|\Lambda^{\alpha-\beta}\partial_1\theta\right\|_{L^\infty} + \left\|[\mathcal{R}_\beta, u \cdot \nabla]\theta\right\|_{L^\infty}\right) d\tau. \tag{5.3}$$

Since $1 + \alpha - 2\beta < 0$, we take p sufficiently large such that $1 + \alpha - 2\beta + \frac{4}{p} < 0$. Then,

$$\begin{aligned}\left\|\Lambda^{\alpha-\beta}\partial_1\theta\right\|_{L^\infty} &\le \left\|\Lambda^{\alpha-\beta}\partial_1\theta\right\|_{B^0_{\infty,1}} = \sum_{j\ge -1}\left\|\Delta_j\Lambda^{\alpha-\beta}\partial_1\theta\right\|_{L^\infty}\\ &\le C\|\theta_0\|_{L^2} + \sum_{j\ge 0} 2^{(1+\alpha-2\beta+\frac{2}{p})j} 2^{\beta j}\|\Delta_j\theta\|_{L^p}.\end{aligned} \tag{5.4}$$

By Bernstein's inequality and by Proposition A.6,

$$\begin{aligned}\left\|[\mathcal{R}_\beta, u \cdot \nabla]\theta\right\|_{L^\infty} &\le \left\|[\mathcal{R}_\beta, u \cdot \nabla]\theta\right\|_{B^0_{\infty,1}} = \sum_{j\ge -1}\left\|\Delta_j[\mathcal{R}_\beta, u \cdot \nabla]\theta\right\|_{L^\infty}\\ &\le \sum_{j\ge -1} 2^{\frac{2}{p}j}\left\|\Delta_j[\mathcal{R}_\beta, u \cdot \nabla]\theta\right\|_{L^p}\\ &= \left\|[\mathcal{R}_\beta, u \cdot \nabla]\theta\right\|_{B^{\frac{2}{p}}_{p,1}}\\ &\le \|\nabla u\|_{L^p}\|\theta\|_{B^{\frac{2}{p}+1-\beta}_{\infty,1}}.\end{aligned} \tag{5.5}$$

In addition,

$$\begin{aligned}\|\nabla u\|_{L^p} &\le \|G\|_{L^p} + C\left\|\Lambda^{-\beta}\partial_1\theta\right\|_{L^p}\\ &\le \|G\|_{L^p} + C\|\theta_0\|_{L^2} + C\sum_{j\ge 0} 2^{(1-2\beta)j} 2^{\beta j}\|\Delta_j\theta\|_{L^p}\end{aligned}$$

and

$$\begin{aligned}\|\theta\|_{B^{\frac{2}{p}+1-\beta}_{\infty,1}} &\leq C\sum_{j\geq -1} 2^{(1-\beta+\frac{2}{p})j}\|\Delta_j\theta\|_{L^\infty}\\ &\leq C\|\theta_0\|_{L^2}+C\sum_{j\geq 0}2^{(1-2\beta+\frac{4}{p})j}2^{\beta j}\|\Delta_j\theta\|_{L^p}.\end{aligned}$$

Inserting the bounds above in (5.3) and noticing (4.3), we obtain

$$\|G\|_{L^\infty}\leq \|G_0\|_{L^\infty}+C(t).$$

This completes the proof of Proposition 5.1. □

Proof of Proposition 5.2. Applying Lemma 5.3 with $p=\infty$ and $r=1$ to (5.2) yields

$$\begin{aligned}\|G\|_{B^0_{\infty,1}} \leq C\Big(\|G_0\|_{B^0_{\infty,1}} &+ \big\|\Lambda^{\alpha-\beta}\partial_1\theta\big\|_{L^1_tB^0_{\infty,1}}\\ &+\big\|[\mathcal{R}_\beta, u\cdot\nabla]\theta\big\|_{L^1_tB^0_{\infty,1}}\Big)\left(1+\int_0^t\|\nabla u\|_{L^\infty}dt\right).\end{aligned} \tag{5.6}$$

Making use of the bounds in (5.4) and (5.5), we have the estimate

$$\|G_0\|_{B^0_{\infty,1}}+\big\|\Lambda^{\alpha-\beta}\partial_1\theta\big\|_{L^1_tB^0_{\infty,1}}+\big\|[\mathcal{R}_\beta, u\cdot\nabla]\theta\big\|_{L^1_tB^0_{\infty,1}}\leq C(t).$$

Using Littlewood–Paley decomposition and the fact that $\|\Delta_j u\|_{L^\infty}\approx 2^{-j}\|\Delta_j\omega\|_{L^\infty}$ for any $j\geq 0$, we get

$$\begin{aligned}\|\nabla u\|_{L^\infty} &\leq \|\Delta_{-1}\nabla u\|_{L^\infty}+\sum_{j\geq 0}\|\Delta_j\nabla u\|_{L^\infty}\\ &\leq C\|u\|_{L^2}+\sum_{j\geq 0}\|\Delta_j\omega\|_{L^\infty}\\ &\leq C(t)+\|\omega\|_{B^0_{\infty,1}}.\end{aligned}$$

In addition, by $G=\omega+\mathcal{R}_\beta\theta$

$$\int_0^t\|\omega\|_{B^0_{\infty,1}}d\tau\leq\int_0^t\|G\|_{B^0_{\infty,1}}d\tau+\int_0^t\|\mathcal{R}_\beta\theta\|_{B^0_{\infty,1}}d\tau\leq C(t)+\int_0^t\|G\|_{B^0_{\infty,1}}d\tau.$$

It then follows from (5.6) that

$$\|G\|_{B^0_{\infty,1}}\leq C(t)\left(C(t)+\int_0^t\|G\|_{B^0_{\infty,1}}d\tau\right).$$

Gronwall's inequality then implies (5.1). Consequently,

$$\int_0^t \|\nabla u\|_{L^\infty}\, d\tau \le \int_0^t \|u\|_{B^1_{\infty,1}}\, d\tau \le C(t) + \int_0^t \|\omega\|_{B^0_{\infty,1}}\, d\tau \le C(t).$$

Therefore, according to the equation of θ,

$$\|\nabla\theta\|_{L^2} \le \|\nabla\theta_0\|_{L^2} e^{\int_0^t \|\nabla u\|_{L^\infty}\, d\tau} \le C(t),$$
$$\|\theta\|_{B^1_{\infty,1}} \le \|\theta_0\|_{B^1_{\infty,1}} e^{\int_0^t \|u\|_{B^1_{\infty,1}}\, d\tau} \le C(t).$$

It then follows from the vorticity equation that

$$\|\omega\|_{L^2 \cap B^0_{\infty,1}} \le C(t).$$

This completes the proof of Proposition 5.2. $\square$

6. Proof of Theorem 1.1

This section proves Theorem 1.1. The proof of Theorem 1.1 is divided into two main parts: the uniqueness and existence. To prove the uniqueness, we need the following simple fact.

Lemma 6.1. *Let $s \in (-1, 1)$, $\varrho \in [1, \infty]$, and v be a Lipschitz divergence-free vector field. Assume that u solves*

$$\begin{cases} \partial_t u + v \cdot \nabla u + \Lambda^\alpha u + \nabla p = f, \quad 0 \le \alpha \le 2, \\ u(x, 0) = u_0 \in B^0_{p,r}, \end{cases}$$

where $p, r \in [1, \infty]$. Then, for any $t > 0$, there exists a constant C such that

$$\|u\|_{L^\infty_t B^s_{2,\infty}} \le C e^{CV(t)} \Big(\|u_0\|_{B^s_{2,\infty}} + \big(1 + t^{1-\frac{1}{\varrho}}\big) \|f\|_{\mathcal{L}^\varrho_t B^{s+\frac{\alpha}{\varrho}-\alpha}_{2,\infty}} \Big),$$

where $V(t) := \int_0^t \|\nabla v\|_{L^\infty} dt$.

Proof of Theorem 1.1. We first prove the uniqueness. We show that any two solutions satisfying (1.7) must be the same. We draw ideas from [19], and [27]. Let $(u^{(i)}, \theta^{(i)})$ with $i = 1, 2$ be two solutions of (1.1) satisfying (1.7). We set $u = u^{(1)} - u^{(2)}$ and $\theta = \theta^{(1)} - \theta^{(2)}$. Then

$$\partial_t u + u^{(1)} \cdot \nabla u + \Lambda^\alpha u + \nabla p = -u \cdot \nabla u^{(2)} + \theta e_2,$$
$$\partial_t \theta + u^{(1)} \cdot \nabla \theta + \Lambda^\beta \theta = -u \cdot \nabla \theta^{(2)},$$
$$(u, \theta)|_{t=0} = (u_0, \theta_0).$$

To estimate u, we apply Lemma 6.1. The two terms on the right of the equation for u are estimated differently. For this purpose, we write $u = U_1 + U_2$ where U_i solves

$$\partial_t U_i + u^{(1)} \cdot \nabla U_i + \Lambda^\alpha U_i + \nabla p_i = F_i, \quad i = 1, 2,$$

with $F_1 = -u \cdot \nabla u^{(2)}$ and $F_2 = \theta e_2$. To estimate U_1, we use Lemma 6.1 with $\varrho = 1$ and $s = 0$ while, to estimate U_2, we use Lemma 6.1 with $\varrho = +\infty$ and $s = 0$. This yields, for every $t \in [0, T]$,

$$\|u\|_{L_t^\infty B^0_{2,\infty}} \le C e^{C\|u^{(1)}\|_{L_t^1 B^1_{\infty,1}}} \left(\|u_0\|_{B^0_{2,\infty}} + \left\|u \cdot \nabla u^{(2)}\right\|_{L_t^1 B^0_{2,\infty}} + (1+t)\|\theta\|_{L_t^\infty B^{-\alpha}_{2,\infty}} \right). \tag{6.1}$$

It is easy to check by the paraproduct decomposition that

$$\left\|u \cdot \nabla u^{(2)}\right\|_{B^0_{2,\infty}} \le \|u\|_{L^2} \left\|u^{(2)}\right\|_{B^1_{\infty,1}}.$$

Using the logarithmic interpolation inequality

$$\|u\|_{L^2} \le \|u\|_{B^0_{2,\infty}} \log\left(e + \frac{1}{\|u\|_{B^0_{2,\infty}}}\right) \log\left(e + \|u\|_{H^1}\right),$$

we obtain

$$\left\|u \cdot \nabla u^{(2)}\right\|_{B^0_{2,\infty}} \le \left\|u^{(2)}\right\|_{B^1_{\infty,1}} \mu\left(\|u\|_{B^0_{2,\infty}}\right) \log\left(e + \|u\|_{H^1}\right), \tag{6.2}$$

where $\mu(x) = x \log(e + \frac{1}{x})$. On the other hand, applying Lemma 6.1 with $\varrho = 1$ and $s = 0$ to the equation for θ yields

$$\|\theta\|_{L_t^\infty B^{-\alpha}_{2,\infty}} \le C e^{C\|u^{(1)}\|_{L_t^1 B^1_{\infty,1}}} \left(\|\theta_0\|_{B^{-\alpha}_{2,\infty}} + \left\|u \cdot \nabla \theta^{(2)}\right\|_{L_t^1 B^{-\alpha}_{2,\infty}} \right). \tag{6.3}$$

To estimate the right-hand side, we have the following product estimate

$$\begin{aligned} \left\|u \cdot \nabla \theta^{(2)}\right\|_{B^{-\alpha}_{2,\infty}} &\le \|u\|_{L^2} \left\|\theta^{(2)}\right\|_{B^{1-\alpha}_{\infty,1}} \\ &\le \left\|\theta^{(2)}\right\|_{B^{1-\alpha}_{\infty,1}} \mu\left(\|u\|_{B^0_{2,\infty}}\right) \log\left(e + \|u\|_{H^1}\right). \end{aligned} \tag{6.4}$$

Inserting (6.2) and (6.3) with (6.4) in (6.1) leads to an inequality of the form for $\mathcal{Z}(t) := \|u\|_{L_t^\infty B^0_{2,\infty}} + \|\theta\|_{L_t^\infty B^{-\alpha}_{2,\infty}}$,

$$\mathcal{Z}(t) \le f(t) \left[\mathcal{Z}(0) + \int_0^t \left(\left\|u^2(\tau)\right\|_{B^1_{\infty,1}} + \left\|\theta^2(\tau)\right\|_{B^{1-\alpha}_{\infty,1}} \right) \mu\left(\mathcal{Z}(\tau)\right) d\tau \right],$$

where $f(t)$ is an explicit function depending continuously on t and $\|u^{(i)}\|_{L_t^\infty H^1 \cap L_t^1 B^1_{\infty,1}}$ with $i = 1, 2$. The uniqueness then follows from the Osgood Lemma B.1 and the fact that $\mathcal{Z}(0) = 0$. In addition, for the purpose of later applications, we have, by Remark B.2,

$$\mathcal{Z}(0) \le \alpha(T) \quad \Longrightarrow \quad \mathcal{Z}(t) \le \beta(T)\left(\mathcal{Z}(0)\right)^{\gamma(T)}, \tag{6.5}$$

where α, β and γ are explicit functions depending continuously on T and on the norms $\|u^{(i)}\|_{L_t^\infty H^1 \cap L_t^1 B_{\infty,1}^1}$ with $i = 1, 2$.

We now prove the existence. First we smooth the data to get the following approximate system

$$\begin{cases} \partial_t u^{(n)} + u^{(n)} \cdot \nabla u^{(n)} + \Lambda^\alpha u^{(n)} = -\nabla p^{(n)} + \theta^{(n)} e_2, \quad x \in \mathbb{R}^2, \ t > 0, \\ \partial_t \theta^{(n)} + u^{(n)} \cdot \nabla \theta^{(n)} + \Lambda^\beta \theta^{(n)} = 0, \quad x \in \mathbb{R}^2, \ t > 0, \\ \nabla \cdot u^{(n)} = 0, \quad x \in \mathbb{R}^2, \ t > 0, \\ u^{(n)}(x,0) = S_n u_0, \quad \theta^{(n)}(x,0) = S_n \theta_0(x), \quad x \in \mathbb{R}^2. \end{cases} \tag{6.6}$$

For $(u_0, \theta_0) \in L^2$, $S_n u_0$ and $S_n \theta_0$ are H^s for any $s \in \mathbb{R}$. An application of the Picard type theorem would yield the local well-posedness of (6.6). As shown in the previous sections, $(u^{(n)}, \theta^{(n)})$ obeys the global *a priori* estimates (uniform with respect to n), for any $T > 0$,

$$\|u^{(n)}\|_{L_T^\infty (H^1 \cap B_{\infty,1}^1)} \le C(T), \qquad \|\theta^{(n)}\|_{L_T^\infty (L^2 \cap B_{\infty,1}^1)} \le C(T).$$

In particular, $u^{(n)}$ is Lipschitz for all time, which implies that $(u^{(n)}, \theta^{(n)})$ is global in time. In addition, up to the extraction of a subsequence of $(u^{(n)}, \theta^{(n)})$, $(u^{(n)}, \theta^{(n)})$ converges weakly to (u, θ), which satisfies the same estimate as above. Furthermore, as shown in the uniqueness part, we have by (6.5)

$$\|u^{(n)} - u^{(m)}\|_{L_T^\infty B_{2,\infty}^0} + \|\theta^{(n)} - \theta^{(m)}\|_{L_T^\infty B_{2,\infty}^{-\alpha}} \le \beta(T) (a_{n,m})^{\gamma(T)},$$

where

$$a_{n,m} = \|(S_n - S_m) u_0\|_{B_{2,\infty}^0} + \|(S_n - S_m)\theta_0\|_{B_{2,\infty}^{-\alpha}}.$$

This proves that $u^{(n)}$ is a Cauchy sequence and hence that it converges strongly to u in the space $L_T^\infty B_{2,\infty}^0$. By interpolation we can easily get the strong convergence of $u^{(n)}$ to u in $L^2([0,T] \times \mathbb{R}^2)$. This implies that $u^{(n)} \otimes u^{(n)}$ converges in $L^1([0,T] \times \mathbb{R}^2)$. But since $\theta^{(n)}$ converges to θ weakly in $L^2([0,T] \times \mathbb{R}^2)$, then, by weak strong convergence, we have also that $u^{(n)}\theta^{(n)}$ converges weakly to $u\theta$. This allows us to pass to the limit in the system (6.6) and to get that (u, θ) is a solution of our original problem, namely (1.1). This completes the proof of Theorem 1.1. □

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Appendix A. Functional spaces and commutator estimates

This appendix serves two purposes. First, it provides the definitions of some of the functional spaces and related facts used in the previous sections. Materials presented here can be found in several books and many papers (see, e.g., [4,5,28,34,35]). Second, we give the proof of a commutator estimate used in the previous sections.

We start with several notation. $\mathcal{S}$ denotes the usual Schwartz class and $\mathcal{S}'$ its dual, the space of tempered distributions. $\mathcal{S}_0$ denotes a subspace of $\mathcal{S}$ defined by

$$\mathcal{S}_0 = \left\{\phi \in \mathcal{S}\colon \int\limits_{\mathbb{R}^d} \phi(x)x^\gamma\,dx = 0,\ |\gamma| = 0, 1, 2, \cdots\right\}$$

and $\mathcal{S}_0'$ denotes its dual. $\mathcal{S}_0'$ can be identified as

$$\mathcal{S}_0' = \mathcal{S}'/\mathcal{S}_0^\perp = \mathcal{S}'/\mathcal{P}$$

where $\mathcal{P}$ denotes the space of multinomials. To introduce the Littlewood–Paley decomposition, we write for each $j \in \mathbb{Z}$

$$A_j = \left\{\xi \in \mathbb{R}^d\colon 2^{j-1} \le |\xi| < 2^{j+1}\right\}.$$

The Littlewood–Paley decomposition asserts the existence of a sequence of functions $\{\Phi_j\}_{j\in\mathbb{Z}} \in \mathcal{S}$ such that

$$\operatorname{supp}\widehat{\Phi}_j \subset A_j, \qquad \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0\big(2^{-j}\xi\big) \quad \text{or} \quad \Phi_j(x) = 2^{jd}\Phi_0\big(2^j x\big),$$

and

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi) = \begin{cases} 1, & \text{if } \xi \in \mathbb{R}^d \setminus \{0\}, \\ 0, & \text{if } \xi = 0. \end{cases}$$

Therefore, for a general function $\psi \in \mathcal{S}$, we have

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}.$$

In addition, if $\psi \in \mathcal{S}_0$, then

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for any } \xi \in \mathbb{R}^d.$$

That is, for $\psi \in \mathcal{S}_0$,

$$\sum_{j=-\infty}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\sum_{j=-\infty}^{\infty} \Phi_j * f = f, \quad f \in \mathcal{S}_0'$$

in the sense of weak-$*$ topology of $\mathcal{S}_0'$. For notational convenience, we define

$$\dot{\Delta}_j f = \Phi_j * f, \quad j \in \mathbb{Z}. \tag{A.1}$$

Definition A.1. For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the homogeneous Besov space $\dot{B}^s_{p,q}$ consists of $f \in \mathcal{S}_0'$ satisfying

$$\|f\|_{\dot{B}^s_{p,q}} \equiv \big\| 2^{js} \|\dot{\Delta}_j f\|_{L^p} \big\|_{l^q} < \infty.$$

We now choose $\Psi \in \mathcal{S}$ such that

$$\widehat{\Psi}(\xi) = 1 - \sum_{j=0}^{\infty} \widehat{\Phi}_j(\xi), \quad \xi \in \mathbb{R}^d.$$

Then, for any $\psi \in \mathcal{S}$,

$$\Psi * \psi + \sum_{j=0}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f$$

in $\mathcal{S}'$ for any $f \in \mathcal{S}'$. To define the inhomogeneous Besov space, we set

$$\Delta_j f = \begin{cases} 0, & \text{if } j \le -2, \\ \Psi * f, & \text{if } j = -1, \\ \Phi_j * f, & \text{if } j = 0, 1, 2, \cdots. \end{cases} \tag{A.2}$$

Definition A.2. The inhomogeneous Besov space $B^s_{p,q}$ with $1 \le p, q \le \infty$ and $s \in \mathbb{R}$ consists of functions $f \in \mathcal{S}'$ satisfying

$$\|f\|_{B^s_{p,q}} \equiv \big\| 2^{js} \|\Delta_j f\|_{L^p} \big\|_{l^q} < \infty.$$

The Besov spaces $\dot{B}^s_{p,q}$ and $B^s_{p,q}$ with $s \in (0, 1)$ and $1 \le p, q \le \infty$ can be equivalently defined by the norms

$$\|f\|_{\dot{B}^s_{p,q}} = \left(\int_{\mathbb{R}^d} \frac{(\|f(x+t) - f(x)\|_{L^q})^q}{|t|^{d+sq}} dt \right)^{1/q},$$

$$\|f\|_{B^s_{p,q}} = \|f\|_{L^q} + \left(\int_{\mathbb{R}^d} \frac{(\|f(x+t) - f(x)\|_{L^p})^q}{|t|^{d+sq}} dt \right)^{1/q}.$$

When $q = \infty$, the expressions are interpreted in the normal way.

Definition A.3. For $t > 0$, $s \in \mathbb{R}$ and $1 \le p, q, r \le \infty$, the space–time spaces $\mathcal{L}^r_t \dot{B}^s_{p,q}$ and $\mathcal{L}^r_t B^s_{p,q}$ are defined through the norms

$$\|f\|_{\mathcal{L}^r_t \dot{B}^s_{p,q}} \equiv \left\| 2^{js} \|\dot{\Delta}_j f\|_{L^r_t L^p} \right\|_{l^q},$$

$$\|f\|_{\mathcal{L}^r_t B^s_{p,q}} \equiv \left\| 2^{js} \|\Delta_j f\|_{L^r_t L^p} \right\|_{l^q}.$$

These spaces are related to the classical space–time spaces $L^r_t \dot{B}^s_{p,q}$ and $L^r_t B^s_{p,q}$ via the Minkowski inequality. Many frequently used function spaces are special cases of Besov spaces. The following proposition lists some useful equivalence and embedding relations.

Proposition A.4. *For any $s \in \mathbb{R}$,*

$$\dot{H}^s \sim \dot{B}^s_{2,2}, \qquad H^s \sim B^s_{2,2}.$$

For any $s \in \mathbb{R}$ and $1 < q < \infty$,

$$\dot{B}^s_{q,\min\{q,2\}} \hookrightarrow \dot{W}^s_q \hookrightarrow \dot{B}^s_{q,\max\{q,2\}}.$$

In particular, $\dot{B}^0_{q,\min\{q,2\}} \hookrightarrow L^q \hookrightarrow \dot{B}^0_{q,\max\{q,2\}}$.

For notational convenience, we write Δ_j for $\dot{\Delta}_j$. There will be no confusion if we keep in mind that Δ_j's associated with the homogeneous Besov spaces are defined in (A.1) while those associated with the inhomogeneous Besov spaces are defined in (A.2). Besides the Fourier localization operators Δ_j, the partial sum S_j is also a useful notation. For an integer j,

$$S_j \equiv \sum_{k=-1}^{j-1} \Delta_k,$$

where Δ_k is given by (A.2). For any $f \in \mathcal{S}'$, the Fourier transform of $S_j f$ is supported on the ball of radius 2^j.

Bernstein's inequalities are useful tools in dealing with Fourier localized functions and these inequalities trade integrability for derivatives. The following proposition provides Bernstein type inequalities for fractional derivatives.

Proposition A.5. *Let $\alpha \geq 0$. Let $1 \leq p \leq q \leq \infty$.*

1) *If f satisfies*

$$\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^d\colon\ |\xi| \leq K 2^j\},$$

for some integer j and a constant $K > 0$, then

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_1 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}.$$

2) *If f satisfies*

$$\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^d\colon\ K_1 2^j \leq |\xi| \leq K_2 2^j\}$$

for some integer j and constants $0 < K_1 \leq K_2$, then

$$C_1 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^d)} \leq \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_2 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)},$$

where C_1 and C_2 are constants depending on α, p and q only.

The rest of this appendix provides a proof for the following commutator estimate.

Proposition A.6. *Let $\beta \in (0,1)$, $(p,r) \in [2,\infty) \times [1,\infty]$. Let $s \in (0,1)$ satisfy $s - \beta < 0$, then there exists a constant $C = C(p,r)$ such that*

$$\|[\mathcal{R}_\beta, f]g\|_{B^s_{p,r}} \leq \|\nabla f\|_{L^p} \|g\|_{B^{s-\beta}_{\infty,r}}. \tag{A.3}$$

To prove this estimate, we need a simple inequality (see, e.g. [18]).

Lemma A.7. *Given $(p,m) \in [1,\infty]^2$ such that $p > m_1$ with m_1 the conjugate exponent of m, let f, g and h be three functions such that $\nabla f \in L^p$, $g \in L^m$ and $xh \in L^{m_1}$. Then*

$$\|h * (fg) - f(h * g)\|_{L^p} \leq \|xh\|_{L^{m_1}} \|\nabla f\|_{L^p} \|g\|_{L^m}.$$

Proof of Proposition A.6. Let $k \geq -1$ be an integer. By the notion of paraproducts, we write

$$\Delta_k[\mathcal{R}_\beta, f]g = J_1 + J_2 + J_3,$$

where

$$J_1 = \sum_{|j-k|\leq 2} \Delta_k\big(\mathcal{R}_\beta(S_{j-1} f \Delta_j g) - S_{j-1} f \mathcal{R}_\beta \Delta_j g\big),$$

$$J_2 = \sum_{|j-k|\leq 2} \Delta_k\big(\mathcal{R}_\beta(\Delta_j f S_{j-1} g) - \Delta_j f \mathcal{R}_\beta S_{j-1} g\big),$$

$$J_3 = \sum_{j\geq k-1} \Delta_k\big(\mathcal{R}_\beta(\Delta_j f \tilde{\Delta}_j g) - \Delta_j f \mathcal{R}_\beta \tilde{\Delta}_j g\big),$$

with $\tilde{\Delta}_j = \Delta_{j-1} + \Delta_j + \Delta_{j+1}$. We first note that, if the Fourier transform of F is supported in the annulus around radius 2^j, then $\mathcal{R}_\beta F$ can be represented as a convolution,

$$\mathcal{R}_\beta F = h_j * F, \quad h_j(x) = 2^{d+1-\beta} h_0\big(2^j x\big)$$

for a function h_0 in the Schwartz class $\mathcal{S}$ whose spectrum does not meet the origin. This can be obtained by simply examining the Fourier transform of $\mathcal{R}_\beta F$. By the definition of $B^s_{p,r}$,

$$\begin{aligned}\big\|[\mathcal{R}_\beta, f]g\big\|_{B^s_{p,r}} &= \big\|2^{sk}\big\|\Delta_k[\mathcal{R}_\beta, f]g\big\|_{L^p}\big\|_{l^r} \\ &\le \big\|2^{sk}\|J_1\|_{L^p}\big\|_{l^r} + \big\|2^{sk}\|J_2\|_{L^p}\big\|_{l^r} + \big\|2^{sk}\|J_3\|_{L^p}\big\|_{l^r}.\end{aligned} \tag{A.4}$$

Applying Lemma A.7, we have

$$\begin{aligned}\|J_1\|_{L^p} &\le C2^{(1-\beta)k}\big\||x|2^{dk}h_0\big(2^k x\big)\big\|_{L^1}\|\nabla S_{k-1} f\|_{L^p}\|\Delta_k g\|_{L^\infty} \\ &\le C2^{-\beta k}\|\nabla f\|_{L^p}\|\Delta_k g\|_{L^\infty}.\end{aligned}$$

Thus,

$$\big\|2^{sk}\|J_1\|_{L^p}\big\|_{l^r} \le C\|\nabla f\|_{L^p}\|g\|_{B^{s-\beta}_{\infty,r}}. \tag{A.5}$$

By Bernstein's inequality, we have

$$\begin{aligned}\|J_2\|_{L^p} &\le C2^{-\beta k}\|\nabla f\|_{L^p}\|S_{k-1} g\|_{L^\infty} \\ &\le C2^{-\beta k}\|\nabla f\|_{L^p}\sum_{m\le k-2}\|\Delta_m g\|_{L^\infty} \\ &\le C\|\nabla f\|_{L^p}\sum_{m\le k-2} 2^{-\beta(k-m)}2^{-\beta m}\|\Delta_m g\|_{L^\infty}.\end{aligned}$$

Since $s-\beta<0$, we obtain, by applying Young's inequality for series,

$$\big\|2^{sk}\|J_2\|_{L^p}\big\|_{l^r} \le C\|\nabla f\|_{L^p}\|g\|_{B^{s-\beta}_{\infty,r}}. \tag{A.6}$$

Similarly, we have

$$\|J_3\|_{L^p} \le C\sum_{j\ge k-1} 2^{-\beta j}\|\nabla f\|_{L^p}\|\Delta_j g\|_{L^\infty}.$$

Therefore, for $s>0$, by Young's inequality for series,

$$\begin{aligned}\big\|2^{sk}\|J_3\|_{L^p}\big\|_{l^r} &\le C\left\|2^{sk}\left(\sum_{j\ge k-1} 2^{-\beta j}\|\nabla f\|_{L^p}\|\Delta_j g\|_{L^\infty}\right)\right\|_{l^r} \\ &\le C\|\nabla f\|_{L^p}\|g\|_{B^{s-\beta}_{\infty,r}}.\end{aligned} \tag{A.7}$$

Combining (A.4), (A.5), (A.6) and (A.7), we obtain the desired bound in (A.3). This completes the proof of Proposition A.6. □

Appendix B. Some basic inequalities

For the convenience of readers, we provide a statement of the Osgood lemma used in Section 6. In addition, a simple estimate used in Section 6 is also stated.

Lemma B.1 *(Osgood lemma). Let $\gamma \in L^1_{loc}(\mathbb{R}_+;\mathbb{R}_+)$, μ be a continuous nondecreasing function, $a \in \mathbb{R}_+$ and α be a measurable function satisfying*

$$0 \le \alpha(t) \le a + \int_0^t \gamma(\tau)\mu\big(\alpha(\tau)\big)d\tau, \quad \forall t \in \mathbb{R}_+.$$

If we assume that $a > 0$, then

$$-\mathcal{M}\big(\alpha(t)\big) + \mathcal{M}(a) \le \int_0^t \gamma(\tau)d\tau \quad \text{with } \mathcal{M}(x) := \int_x^1 \frac{1}{\mu(r)}dr.$$

If we assume $a = 0$ and $\lim_{x\to 0^+} \mathcal{M}(x) = +\infty$, then $\alpha(t) = 0$, $\forall t \in \mathbb{R}_+$.

Remark B.2. In the particular case $\mu(r) = r(1 - \log r)$, one can show, for any $t > 0$,

$$\alpha(0) \le e^{1-\exp(\int_0^t \gamma(\tau)d\tau)} \quad \Longrightarrow \quad \alpha(t) \le \alpha(0)^{\exp(-\int_0^t \gamma(\tau)d\tau)} e^{1-\exp(-\int_0^t \gamma(\tau)d\tau)}.$$

References

[1] D. Adhikari, C. Cao, J. Wu, The 2D Boussinesq equations with vertical viscosity and vertical diffusivity, J. Differential Equations 249 (2010) 1078–1088.

[2] D. Adhikari, C. Cao, J. Wu, Global regularity results for the 2D Boussinesq equations with vertical dissipation, J. Differential Equations 251 (2011) 1637–1655.

[3] D. Adhikari, C. Cao, J. Wu, X. Xu, Small global solutions to the damped two-dimensional Boussinesq equations, J. Differential Equations 256 (2014) 3594–3613.

[4] H. Bahouri, J.-Y. Chemin, R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, Springer, 2011.

[5] J. Bergh, J. Löfström, Interpolation Spaces, an Introduction, Springer-Verlag, Berlin, Heidelberg, New York, 1976.

[6] L. Brandolese, M. Schonbek, Large time decay and growth for solutions of a viscous Boussinesq system, Trans. Amer. Math. Soc. 364 (2012) 5057–5090.

[7] C. Cao, J. Wu, Global regularity for the 2D anisotropic Boussinesq equations with vertical dissipation, Arch. Ration. Mech. Anal. 208 (2013) 985–1004.

[8] D. Chae, Global regularity for the 2D Boussinesq equations with partial viscosity terms, Adv. Math. 203 (2006) 497–513.

[9] D. Chae, J. Wu, The 2D Boussinesq equations with logarithmically supercritical velocities, Adv. Math. 230 (2012) 1618–1645.

[10] A. Córdoba, D. Córdoba, A maximum principle applied to quasi-geostrophic equations, Comm. Math. Phys. 249 (2004) 511–528.

[11] P. Constantin, V. Vicol, Nonlinear maximum principles for dissipative linear nonlocal operators and applications, Geom. Funct. Anal. 22 (2012) 1289–1321.

[12] P. Constantin, C.R. Doering, Infinite Prandtl number convection, J. Stat. Phys. 94 (1999) 159–172.
[13] X. Cui, C. Dou, Q. Jiu, Local well-posedness and blow up criterion for the inviscid Boussinesq system in Hölder spaces, J. Partial Differ. Equ. 25 (2012) 220–238.
[14] R. Danchin, M. Paicu, Global well-posedness issues for the inviscid Boussinesq system with Yudovich's type data, Comm. Math. Phys. 290 (2009) 1–14.
[15] R. Danchin, M. Paicu, Global existence results for the anisotropic Boussinesq system in dimension two, Math. Models Methods Appl. Sci. 21 (2011) 421–457.
[16] W. E, C. Shu, Small-scale structures in Boussinesq convection, Phys. Fluids 6 (1994) 49–58.
[17] A.E. Gill, Atmosphere-Ocean Dynamics, Academic Press, London, 1982.
[18] T. Hmidi, S. Keraani, S. Rousset, Global well-posedness for Euler–Boussinesq system with critical dissipation, J. Differential Equations 249 (2010) 2147–2174.
[19] T. Hmidi, S. Keraani, S. Rousset, Global well-posedness for Boussinesq–Navier–Stokes system with critical dissipation, Comm. Partial Differential Equations 36 (2011) 420–445.
[20] T. Hou, C. Li, Global well-posedness of the viscous Boussinesq equations, Discrete Contin. Dyn. Syst. 12 (2005) 1–12.
[21] Q. Jiu, C. Miao, J. Wu, Z. Zhang, The 2D incompressible Boussinesq equations with general critical dissipation, arXiv:1212.3227v1 [math.AP], 2012.
[22] Q. Jiu, J. Wu, W. Yang, Eventual regularity of the two-dimensional Boussinesq equations with supercritical dissipation, J. Nonlinear Sci., accepted for publication.
[23] D. KC, D. Regmi, L. Tao, J. Wu, Generalized 2D Euler–Boussinesq equations with a singular velocity, J. Differential Equations 257 (1) (2014) 82–108.
[24] D. KC, D. Regmi, L. Tao, J. Wu, The 2D Boussinesq–Navier–Stokes equations with logarithmically supercritical dissipation, submitted for publication.
[25] M. Lai, R. Pan, K. Zhao, Initial boundary value problem for two-dimensional viscous Boussinesq equations, Arch. Ration. Mech. Anal. 199 (2011) 739–760.
[26] A. Larios, E. Lunasin, E.S. Titi, Global well-posedness for the 2D Boussinesq system with anisotropic viscosity and without heat diffusion, J. Differential Equations 255 (2013) 2636–2654.
[27] C.X. Miao, L.T. Xue, On the global well-posedness of a class of Boussinesq–Navier–Stokes systems, NoDEA Nonlinear Differential Equations Appl. 18 (2011) 707–735.
[28] C. Miao, J. Wu, Z. Zhang, Littlewood–Paley Theory and Its Applications in Partial Differential Equations of Fluid Dynamics, Science Press, Beijing, China, 2012 (in Chinese).
[29] H.K. Moffatt, Some remarks on topological fluid mechanics, in: R.L. Ricca (Ed.), An Introduction to the Geometry and Topology of Fluid Flows, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001, pp. 3–10.
[30] A.J. Majda, Introduction to PDEs and Waves for the Atmosphere and Ocean, Courant Lect. Notes Math., vol. 9, AMS/CIMS, 2003.
[31] A.J. Majda, A.L. Bertozzi, Vorticity and Incompressible Flow, Cambridge University Press, 2001.
[32] K. Ohkitani, Comparison between the Boussinesq and coupled Euler equations in two dimensions, in: Tosio Kato's Method and Principle for Evolution Equations in Mathematical Physics, Sapporo, 2001, in: Sūrikaisekikenkyūsho Kōkyūroku, vol. 1234, 2001, pp. 127–145.
[33] J. Pedlosky, Geophysical Fluid Dynamics, Springer-Verlag, New York, 1987.
[34] T. Runst, W. Sickel, Sobolev Spaces of Fractional Order, Nemytskij Operators and Nonlinear Partial Differential Equations, Walter de Gruyter, Berlin, New York, 1996.
[35] H. Triebel, Theory of Function Spaces II, Birkhäuser Verlag, 1992.
[36] X. Xu, Global regularity of solutions of 2D Boussinesq equations with fractional diffusion, Nonlinear Anal. 72 (2010) 677–681.
[37] K. Zhao, 2D inviscid heat conductive Boussinesq equations on a bounded domain, Michigan Math. J. 59 (2010) 329–352.