Proceedings of the ASME 2007 International Design Engineering Technical Conferences & Computers and Information in Engineering Conference
IDETC/CIE 2007
September 4-7, 2007, Las Vegas, Nevada, USA

DETC2007-34171

DETERMINATION OF PARAMETRIC RESONANCES IN DISTRIBUTED PARAMETER SYSTEMS USING FREQUENCY ANALYSIS

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ABSTRACT

Theoretical analysis of parametric instability for the control systems with distributed parameters shall be given. The approach to the solution of such systems can be composed of two parts, i.e. modeling and estimation of the distributed parameters and instability estimation of the periodical time-variant elements using parametric circumference. A control system with mechanical distributed parameters such as robot manipulators is introduced as an example. Theoretical analysis shows that the parametric instabilities occur by digital controllers or time-varying elements which excite the resonance regions of distributed parameters. An electro-mechanical transformer which consists of constant current motor and synchronous generator is applied as another example. Inductance between stator windings and rotor of the synchronous generator serves as a periodical time-varying parameter and long electrical line plays a role of an element with distributed parameters. Instability condition of the transformer rotation owing to the parametric resonance excitement was obtained.

1. INTRODUCTION

There are numerous oscillatory systems whose interaction with the external world amounts only to a periodic time dependence of their parameters. The corresponding resonance is called *parametric* [1,2]. A textbook example is a simple pendulum with a vertically oscillating point of suspension [1]. The main resonance occurs when the excitation frequency ω is nearly twice the natural frequency of the oscillator ω_o [1,2]. Applications of this basic phenomenon in physics and technology are ubiquitous.

Periodical time variant dynamic objects with distributed parameters are widely used in control engineering, electro-mechanics, mechanics, thermo-, hydro-, gas dynamics and the like [2-7]. Examples of Partial Differential Equations (PDEs) in which some of the coefficients are spatially periodic functions and parametric resonance occurs are given in the literature [3]. Coordinate control system of robot-manipulator can be considered as a similar example. In this case time-periodic element is a MDM (modulator-demodulator) amplifier and a resilient shaft with a driving gripper serves as an element with distributed parameters. Another example of periodical time variant system with distributed parameters is an electro-mechanical transformer which consists of a constant current motor and a synchronous generator. In this system mutual inductance between stator windings and a rotor of the synchronous generator serves as a time-periodic parameter, and long electrical line plays a role of element with distributed parameters. Methodology and procedures for solving time variant systems were presented in many literatures [1~2, 8-11].

This paper slightly modifies Chechurin's idea [8] about the stability of the general lumped system and extends it to the stability analysis of distributed parameter systems. The main idea of the modification is to centralize the parametric circle presented in the literature [8]. Such small modification allows us to find regions of parametric resonance in the frequency domain more easily than the previous one and use Frequency Response Functions (FRF) of the distributed parameter sub-system from the total such as shells and plates in mechanical system and electrical lines in the electronic system. Two examples in the works given in the present paper will explain the existence of parametric resonance of systems with distributed parameters and usefulness of the modification of the parametric resonance circle.

2. PARAMETRIC RESONANCE IN PERIODICAL TIME VARIANT SYSTEMS WITH DISTRIBUTED PARAMETERS

As the systems with distributed parameters have theoretically infinite number of natural frequencies, it is intuitively clear that the parametric resonances in such systems may occur in many regions of frequencies as well. However, it is not easy to find whole frequencies or frequency regions of parametric resonances by a simple approach. A simple approximate approach presented in the literature [8] which uses parametric circle for a time-periodic element in frequency domain was well approved by many examples and comparisons with other approaches [8, 11]. The proposed parametric circle shall be applied in order to find regions of parametric resonance for the systems with distributed parameters as well.

Firstly, we review the technique of approximate frequency domain parametric resonance analysis presented in

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the literature [8]. Time variant periodical element with τ period $a(t\text{-}\tau)$ can be written as Fourier series

$$a(t-\tau)=\sum_{r=-\infty}^{+\infty}a_r e^{+jr\Omega(t-\tau)},\quad \Omega=\frac{2\pi}{T} \tag{1}$$

where Fourier coefficients

$$a_r=\frac{1}{T}\int_0^T a(t-\tau)e^{-jr\Omega(t-\tau)}dt\,, \tag{2}$$

Ω is the alternating excitation frequency

Harmonic linearization by sinusoidal input $x(t)=\sin(r\Omega t/2)$ and output $y(t)=a(t\text{-}\tau)\sin(r\Omega t/2)$ relation to this time variant object for the fundamental parametric resonance gives in Nyquist plane *parametric circle* (stationary phase-dependent transfer function) with center a_0 and radius a_r by:

$$W_{rp}(j\varphi)=a_0-a_r e^{-j\varphi}\,, \tag{3}$$

where $\varphi=\Omega\tau$. Different types of time variant element can be referred in literature [8]. Thus, the characteristic equation of the time variant parameter and linear time invariant elements can be written by:

$$1+W(j\omega)W_{rp}(j\varphi)=0, \tag{4}$$

where $W(j\omega)$ is the frequency response function (FRF) of linear time invariant elements. Conditions of fundamental parametric resonance for the given system can be depicted in Nyquist plane as shown in Fig. 1.

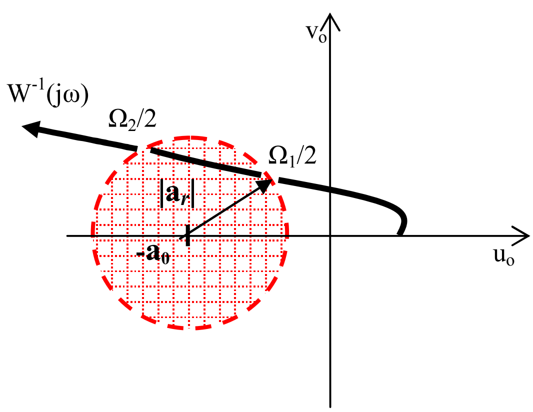

Fig. 1 Conditions of fundamental parametric resonance in complex plane.

It is apparent that the necessary condition of the parametric resonance is the existence of cross points for inverse Nyquist diagram with parametric circle. If the cross points exist, it is easy to derive unstable frequency regions from it (shaded regions in Fig.1). Similarly, conditions of parametric resonance in the systems with distributed parameters are also valid and will have more cross points as shown in Fig. 2.

Figure 2 shows the characteristic of parametric resonances by combining parametric circle with frequency responses of systems with distributed parameters as the distributed parameter systems have many frequency resonance characteristics in frequency domain.

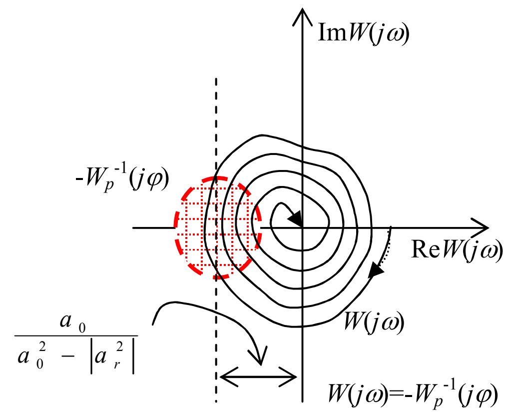

Fig. 2 Condition of parametric resonances for periodical time variant systems with distributed parameters in complex plane.

In case of the systems with distributed parameters it is more convenient to use central parametric circle than current non-central parametric circle owing to many cross points of frequency contours of the system. When the periodical time variant element $a(t)$ consists of mean value a_0 and periodical time variant signal $a_1(t)$,

$$a(t)=a_0+a_1(t). \tag{5}$$

Parametric circle can be positioned in the center of the complex plane by moving mean value of the signal to the time invariant element. Then, the regions of instability can be easily found and experimentally obtained FRFs can be also included. In this case modified transfer function of time invariant elements becomes:

$$W_M(p)=\frac{W(p)}{1+a_0W(p)}\,. \tag{6}$$

Therefore, the first parametric resonance of the system can be found by simple relations;

$$\left|W_M^{-1}\left(j\frac{r\Omega}{2}\right)\right|=\left|a_r\right| \text{ or } \left|W_M\left(j\frac{r\Omega}{2}\right)\right|=\left|a_r^{-1}\right|. \tag{7}$$

Instability regions from equation (7) can be shaded in both complex and frequency-magnitude planes as shown in Fig. 3. The great advantage of the modified circle is to use frequency-magnitude graph which is a standard for engineers and scientists in theory and practice. Because of the modified transfer function, the regions of parametric instability lie on the outer space of centralized parametric circle in Nyquist plane and the upper part of instability line in frequency-magnitude graph other than the non-centralized parametric circle as shown in Fig. 1.

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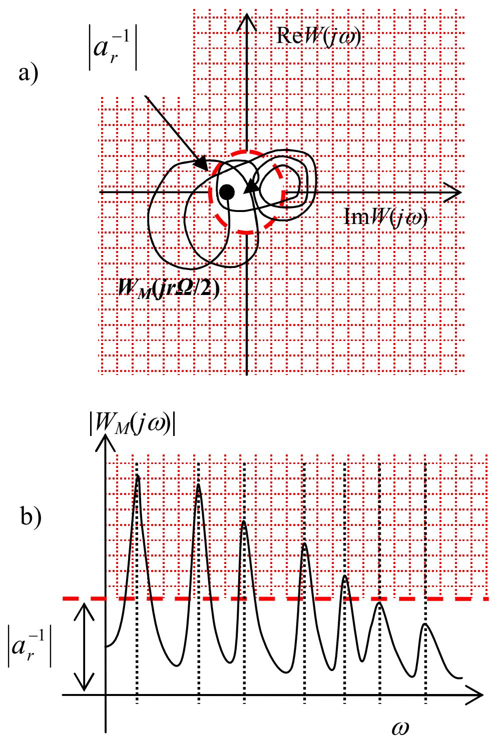

Fig. 3 Central parametric circle with frequency response of systems with distributed parameters in complex plane (a) and frequency-magnitude plane (b).

The main idea for the calculation of parametric resonance regions in systems which include linear time invariant distributed parameters is to separate time variant elements from linear time invariant elements for the calculation of FRFs and combine them in complex plane in order to find regions of parametric resonance in frequency domain. This kind of approach can make engineers intuitively and quickly find regions of parametric resonance in a very complicated system, especially a system with distributed parameters. Following two examples shall be helpful to understand the idea and procedure for the calculation of parametric resonance regions in distributed parameter systems.

3. NUMERICAL EXAMPLE I

As an example of time-periodic distributed parameter system, consider a coordinate control manipulator system as shown in Fig. 4:

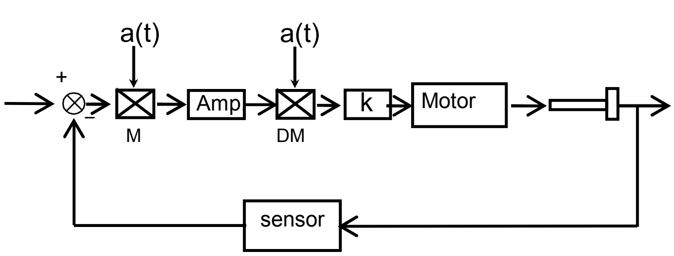

Fig. 4 Robot-manipulator system

The system consists of a shaft, a distributed element as itself, with lumped mass stuck at the end of the arm and modulator - demodulator (MDM) amplifier. The feedback is required for measuring gripper position by optic sensor. Then, the transfer function of motor can be written:

$$W_M(p) = \frac{\Phi(p)}{U(p)} = \frac{K_M p}{T_L T_M p^2 + T_M p + 1}, \quad (8)$$

where K_M, T_L, and T_M are constants from motor characteristics and p is Laplace operator. The modulator-demodulator amplifier transforms input signal $u_c(t)$ into motor torque signal $u(t)$ with relation

$$u(t) = a(t)\cdot u_c(t) = (\sqrt{2}\sin\omega_M t)^2 u_c(t)$$
$$= (1-\cos 2\omega_M t)u_c(t).$$

Therefore, amplifier is a time-variable parameter which has double modulation

$$\Omega = 2\omega_M. \quad (9)$$

For compensating the shaft, controller has been chosen as PD (Proportional-plus-Derivative) compensator with transfer function

$$W_c(p) = K_p + K_d p, \quad (10)$$

where K_p and K_d imply proportional and derivative constants respectively and are chosen mainly in order to compensate for the lumped inertia of the system.

The sensor is considered as an inertialess element and its gain K_s, as well as motor gain K_M, controller gain K_c, and amplifier gain K_a is included in a total gain K:

$$K = K_s K_M K_a K_c. \quad (11)$$

Thus frequency response of the open loop time invariant system is

$$W(j\omega) = K\frac{(K_p + K_d j\omega)(j\omega)}{(T_L T_M (j\omega)^2 + T_M(j\omega)+1)} W_d(j\omega), \quad (12)$$

where $W_d(j\omega)$ – frequency response function of the shaft

Frequency response of total system with the shaft is calculated from equation (12) with following physical numerical data : inertia (I_o) = 0.09 *kg.m*^{2}, density (ρ) = 7.8 * 10^3 *kg*/m^3, radius of the shaft (r) = 0.01 *m*., length (l) = 0.36 *m*., damping coefficient (b) = 0.0005 *N.s/m*, and shear modulus (G) = 40 10^9*N/m*^{2}.

As the system with a lumped element is unstable without compensators, following compensator is chosen: $W_c = 0.01*(j\omega)+1$. Frequency response function of the motor with numerical data is given as

$$W_{motor} = \frac{70*(j\omega)}{8*10^{-5}*(j\omega)^2 + 0.081*(j\omega)+1}. \quad (13)$$

Impulse response of the lumped system in use of a lumped element shaft damps out by time as shown in Fig. 5

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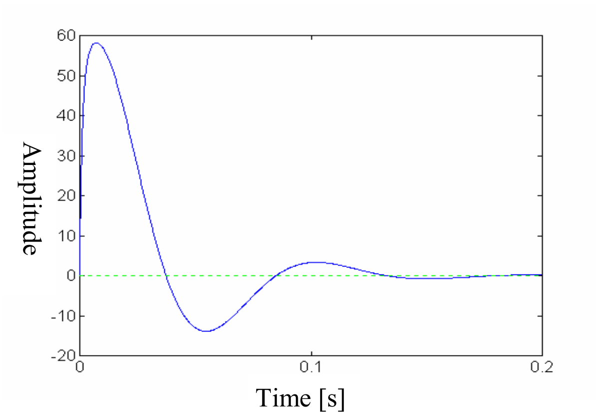

Fig. 5. Transient response of stationary lumped system with compensation

However, real servo system gives frequency response which is vulnerable to parametric resonance as to the characteristic of the distributed parameters as shown in Fig. 6.

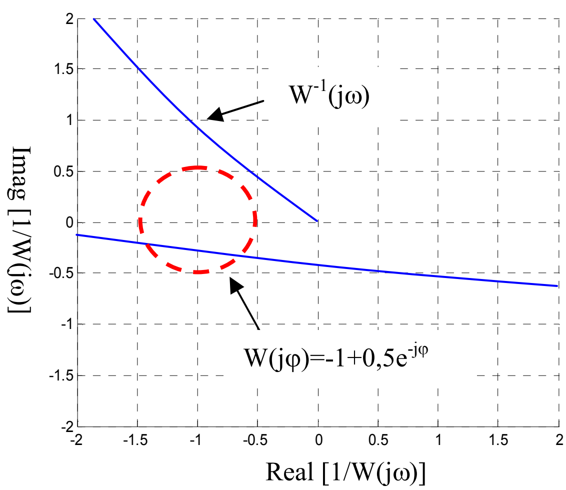

Fig. 6 Inverse Nyquist plot of the servo system with distributed parameters and parametric circumference

It is obvious that the inverse frequency response of the given servo system intersects with the first parametric resonance circle. It means that time variable system has parametric resonance and unstable at $\omega=\Omega/2$.

Frequency models and transfer function of the shaft for the numerical calculation is achieved by Finite Element Method (FEM). Numerical modeling of the total system is implemented by commercial software «SIMULINK» as shown in Fig. 7.

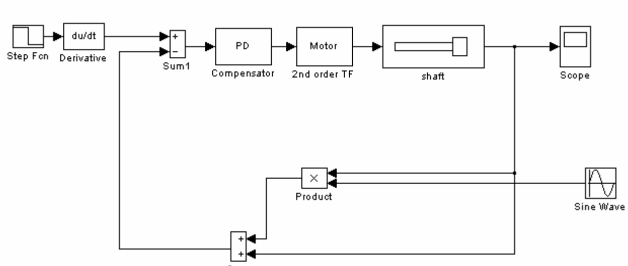

Fig. 7 Schematic diagram for the numerical modeling of the total system with a time-varying element

The numerical experiment shows the occurrence of the first parametric resonance (see Fig. 8 (b)) which does not exist in the absence of time-periodic parameter (Fig. 8 (a)).

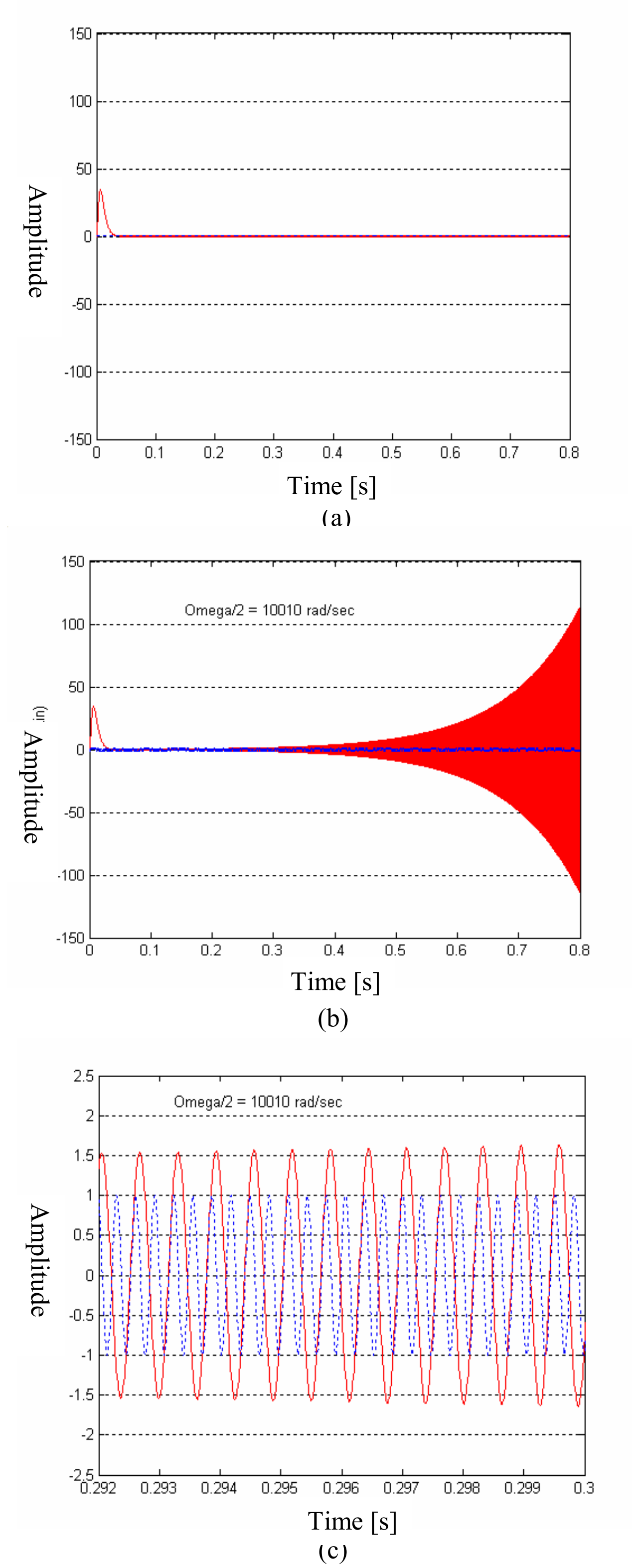

Fig. 8 Time histories of the shaft (a) transient signal in the absence of time-varying parameter, (b) general view of transient signal in the presence of time-varying parameter and (c) Zoomed view of (b)

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Therefore, conditions of parametric resonance excitation are illustrated in a servo system with distributed parameters of the shaft. It is shown that modulation frequencies of the compensator estimated as time varying elements can induce parametric instability of the system when it is continuous with distributed parameters.

To identify the occurrence of parametric resonance in other systems with distributed parameters, consider a following electromechanical system which is excited by the mutual inductance between rotor and stator windings of the generator.

4. NUMERICAL EXAMPLE II

M.L.Levinshtein[13] performed an experimental research of parametric resonance on a synchronous generator with capacitance loading, but he considered existence of a parametric resonance in a generator only. In the present paper, “motor-generator” system with long electrical line shall be considered. In the frame of single-phased synchronous generator as shown in Fig. 9, it is not difficult to write the equation of motion with general variables:

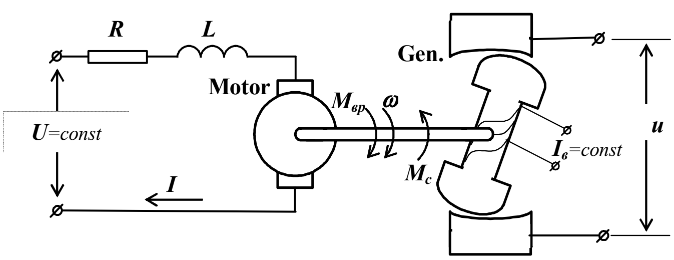

Fig. 9 Motor-Generator electrical transformer system

$$J\ddot{\alpha} = C_M I - M_c, U = RI + L\dot{I} + C_e\omega,$$
$$M_c = \frac{\partial M}{\partial \alpha} i I_в, \frac{\partial M}{\partial t} I_в = Z_n i \quad , \tag{14}$$

Here are constants and variables used in this section:

C_e - electrical constant of motor;
C_M - electromechanical constant of motor;
C_U – maximum value of the mutual inductance
$I_в$ – excitational current of generator rotor;
i - current of generator stator;
J - moment of inertia for «motor-generator» rotors;
M - mutual inductance of stator windings and generator rotor;
$M_{вр}$ and M_c – rotational moment of motor and resist moment of generator to the shaft
p- Laplace variable
R and L – active resistance and inductance of constant
u – voltage in loading circuit.
U and I- voltage and current in circuit of constant current motor;
$W_n(p) = 1/Z_n$ - transfer function from voltage to the current of the loading circuit.
Z_n - complex resistance of loading (impedance);
α - angle of rotational shaft;
ω - angular rotational speed of motor and generator;
ξ and γ – some parameters depending on operator p

Alternation of the mutual position between rotor and stator of the generator reflects that the inductance of the generator M can be changed by time. Ideal law of alternation M and $\partial M/\partial t$ during constant shaft rotation is generally adapted [14] as shown in Fig. 10.

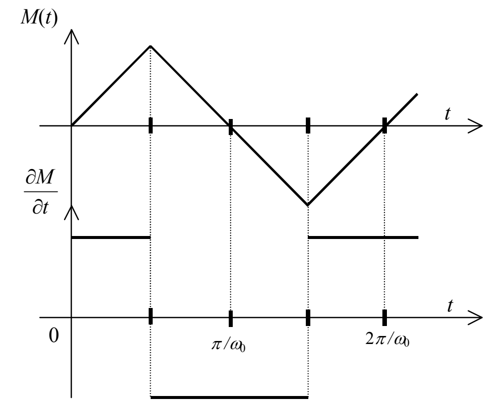

Fig. 10 Alternating character of the mutual inductance according to time during shaft roation

Smooth variation of this dependence is applied in this work:

$$M(t) = C_U \sin(\alpha) = C_U \sin(\omega_0 t). \tag{15}$$

Obviously, the inductance is time-periodic with a basic frequency ω_0 and the system becomes time-variant. Applying Laplace transform to the system equation (14) and arranging the equation by the relative angle of rotational shaft α, following nonlinear equation can be obtained;

$$T_a T_M p^3\alpha + T_M p^2\alpha + p\alpha + k_1 W_n(p)\left(\frac{\partial M}{\partial \alpha}\right)^2 p\alpha$$
$$+ T_a k_1 W_n(p)\left\{2\frac{\partial M}{\partial \alpha}\left(\frac{\partial^2 M}{\partial \alpha^2}\right)p^2\alpha^2 + \left(\frac{\partial M}{\partial \alpha}\right)^2 p^2\alpha\right\} = \frac{U}{C_e} \quad , \tag{16}$$

where $k_1 = T_M I_в^2 / J$, $T_a = L/R$, $T_M = RJ/C_e C_M$ and p is Laplace operator.

As the mutual inductance M is the function of rotation angle α , equation (16) is nonlinear. For the purpose of stability analysis of the rotation let us introduce small motion perturbation $\Delta\alpha$: $\alpha = \alpha^* + \Delta\alpha$ from unperturbed motion, namely, rotation of the rotor with constant speed ω_0, $\alpha^* = \omega_0 t$. The equation (16) is rewritten in terms of $\Delta\alpha$ as follows

$$T_a T_M p^3\Delta\alpha + T_M p^2\Delta\alpha + p\Delta\alpha + \left\{2k_1 W_n(p)\left(\frac{\partial M}{\partial \alpha}\right)\left(\frac{\partial^2 M}{\partial \alpha^2}\right)\omega_0\right\}\Delta\alpha + \left\{k_1 W_n(p)\left(\frac{\partial M}{\partial \alpha}\right)^2 p\right\}\Delta\alpha +$$
$$+ T_a k_1 W_n(p)\left[2\left\{\left(\frac{\partial^2 M}{\partial \alpha^2}\right)^2 \omega_0{}^2 + \left(\frac{\partial M}{\partial \alpha}\right)\left(\frac{\partial^3 M}{\partial \alpha^3}\right)\omega_0{}^2 + \left(\frac{\partial M}{\partial \alpha}\right)\left(\frac{\partial^2 M}{\partial \alpha^2}\right)(2\omega_0)p\right\} + \left(\frac{\partial M}{\partial \alpha}\right)^2 p^2\right]\Delta\alpha = 0. \tag{17}$$

Since we assumed α^* to be periodic, its derivatives are periodic, therefore equation (17) is periodically time variant. Applying stationarization [8,15] to equation (17) we can obtain approximate excitation condition of the first parametric resonance.

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$$T_a T_M p^3 + \left(T_M + \frac{1}{2}T_a q\right)p^2 + \left(1 + \frac{1}{2}q\right)p$$
$$+ \left\{\frac{1}{2}qT_a p^2 + \left(\frac{1}{2}q + p(2T_a)q\right)p + pq\right\}\left(-\frac{1}{2}e^{-j\varphi}\right) = 0 \quad , \tag{18}$$

where $q = k_1 W_n(p) C_u^{\ 2}$, and $p=j\omega_0=j\Omega/2$, where Ω - alternating frequency of the parameter (in this study we assumed single pole machine, therefore, $\Omega = 2\omega_0$).

Equation (18) can be considered as a characteristic polynomial with exponential multiplier of some linear stationary systems. Equality condition of left part is analogous to the searching condition on the boundary of system stability with feedback connection which gives following transfer function:

$$W(j\Omega) = \frac{2.5T_a q(j\Omega/2) + 1.5q}{T_a T_M (j\Omega/2)^2 + (T_M + T_a q/2)(j\Omega/2) + (1 + q/2)} \tag{19}$$

$$W_1(j\varphi) = \left(-0.5e^{-j\varphi}\right). \tag{20}$$

Hence, in accordance with Nyquist criterion, uniform rotation of the system (motor-generator) rotor with speed ω_0 can be unstable in condition of loading circuit $W_n(p)$ and alternating mutual inductance $M(t) = C_U \sin\omega_0 t$ if Nyquist diagram of the closed-loop transfer function $W(p)W_1(j\varphi)$ encloses point $(-1;j0)$ on imaginary plane:

$$W(j\Omega/2)W_1(j\varphi) = -1 \text{ or } W_1(j\varphi) = -W^{-1}(j\Omega/2). \tag{21}$$

Therefore, critical frequencies and amplitudes of the changing parameter which cause parametric resonance excitation can be found.

Let us consider electromechanical transformer on long (250m) electrical line with resistance, inductance and capacitance values per unit length: r_L=1e-6 Ohm/m, l_L=1e-5 H/m, c_L=1e-8 F/m. Lumped loading inductance l_n=0.032 H is assumed to be attached at the end of the line. Transfer function of the long line with reactive loading inductance can be written as:

$$W_n(p) = \frac{1}{(\sinh\xi/\gamma)\sinh\xi N + \cosh\xi N \cdot R(p)} \tag{22}$$

,where

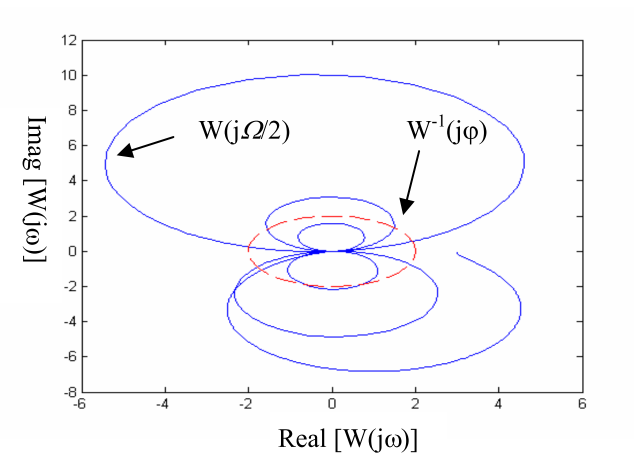
$$\gamma = \frac{\left(\frac{4}{h}\right)c_L p}{\left(\frac{2}{h}\right)^2 - c_L p(r_L + l_L p)}, \quad \cosh\xi = \left[\frac{(2/h)^2 + c_L p(r_L + l_L p)}{(2/h)^2 - c_L p(r_L + l_L p)}\right],$$

$R(p) = l_n p$ and N indicates the number of finite elements with discrete step length h.

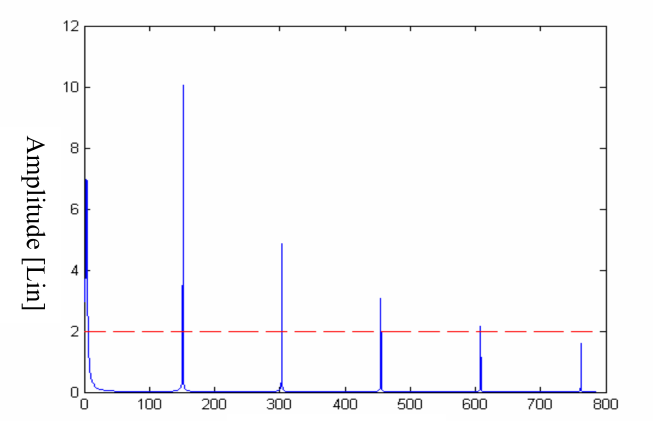
Nyquist diagram and amplitude-frequency characteristic of the given system with distributed long electrical line loading is shown in Fig. 11. Calculation shows that the first parametric resonance can occur at the following frequency outskirts: till 5.91; from 151.06 to 151.49; from 302.99 to 303.19; from 455.20 to 455.30 rad/sec. The most interesting resonant frequency is the third one as it is closely located near the nominal frequency of the rotation 300 rad/sec.

As shown in Fig. 11, third frequency mode of the distributed parameters crosses the absolute radius of the parametric resonance circle. It means that the rotating motor-generator systems with 300 rad/sec rotational speed may have parametric resonance when the length of the electrical line extends to 250 meters.

Since the frequency response of the given system changes according to the length of the electrical line, we can imagine that the regions of the parametric resonance changes by the length of the line. Figure 12 shows the conditions of parametric excitation according to the length of the electrical line. As the regions of instability for each mode are not wide, each mode was depicted by the square and circle marks. Here, we can find that the parametric resonances may occur with the following lengths of the line: 170m(2nd mode), 250m (3rd mode), 310m (4th mode), 360m (5th mode), and so on.



(a)



(b)

Fig. 11 Conditions of first parametric resonance excitement on electromechanical transformer with long line (250m) loading; (a) Nyquist diagram and (b) magnitude versus frequency plot

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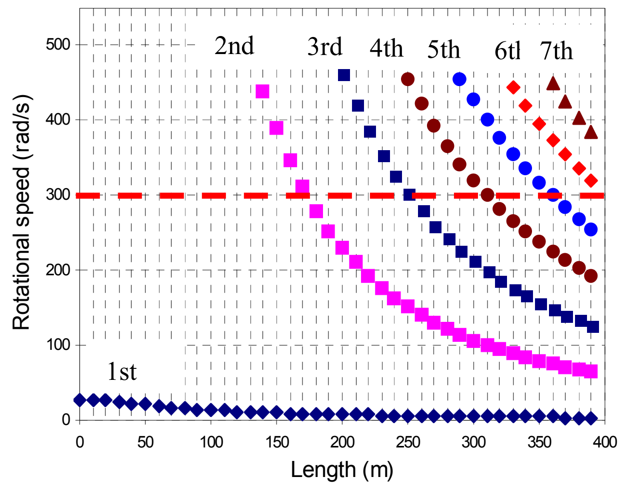

Fig. 12 Parametric excitation according to the length of electrical line: dotted line – rotation speed of the shaft (300 rad/sec)

Numerical experiment was performed with a commercial software Simulink (Fig. 13) with equivalent transfer functions $W_1(p)$ and $W_3(p)$:

$$W_1(p) = \frac{\frac{1}{2}k_1 C_U^2 (T_a p + 1)}{T_a T_M p^2 + T_M p + 1} \quad ,(23)$$

$$W_3(p) = \frac{\frac{1}{2}k_1 C_U^2 (5T_a p + 3)}{T_a T_M p^2 + T_M p + 1} \text{ ,and } W_2(p) = W_n(p) \text{ .}$$

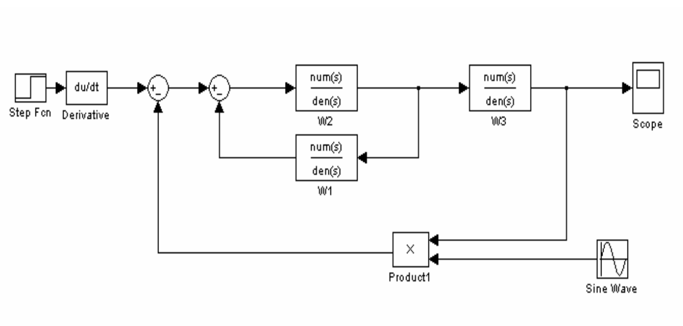

Fig. 13 Block scheme of Simulink for numerical experiment

It is easy to prove that in this case whole transfer function

$$W(p) = \frac{W_2(p)}{1 + W_1(p) W_2(p)} W_3(p) \quad (24)$$

coincides with equation (19).

In case that the transfer function of the loading is hyperbolic it is not possible to use a standard simulation tool like Simulink. However for the observation of first parametric resonance it is not always necessary to use exact transfer function of equation (22), but enough to use its approximation around interesting frequency range.

Experiment shows the presence of the first parametric resonance (Fig. 14).

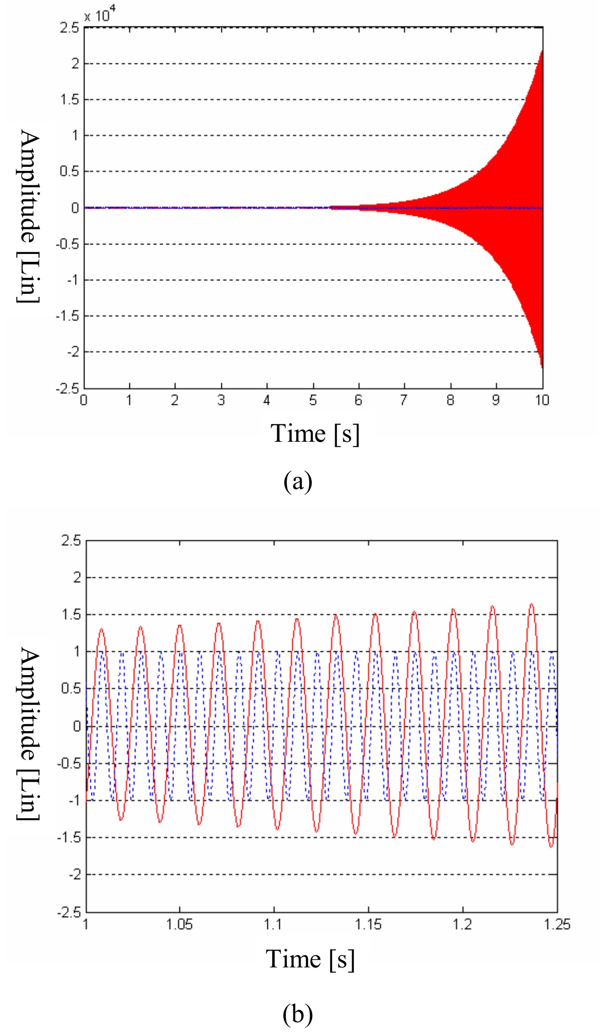

Fig. 14 Oscillation growth for the electromechanical transformer by numerical experiment. (a) general view, (b) fragment of (a) (dotted line – vibration of parameter, solid line – vibration in the system).

5. CONCLUDING REMARKS

In the present paper we have considered the problem of parametric resonance in systems with distributed parameters whose equation of motion is described by partial differential equation. In this article we have explored two examples of such systems and found it useful for ones to use modified parametric circle to determine instability regions of parametric resonance. Moreover it allows ones to use experimental FRFs for the calculation of parametric instability regions.

Occurrence of parametric instability in systems with distributed parameters such as robot servo system and motor-generator has been shown. It has been demonstrated that neglecting distributed parameters in robot servo system can cause loss of stability owing to the lack of high frequency

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modes from the characteristics of continuous systems. Stability analysis for an electromechanical transformer system, which consists of constant current motor and synchronous generator with long electrical lines as a distributed loading, was performed for another example of parametric resonance. In this example we have shown that the mutual inductance between the rotor and stator windings and electrical line loading can incur the loss of stability.

Some future areas of research are to explore a wider range of parametric space and higher parametric resonance frequencies. Also, a comparison of experimental and theoretical results shall be more interesting in those distributed parameter systems. Usability of parametric circle in order to make an unstable system stable can be also very interesting.

6. REFERENCES

[1] L.D. Landau and E.M. Lifshitz, “Mechanics”, Pergamon press, Oxford, (1976).

[2] N.N.Bogolubov and Y.A.Mitropolsky, “Asymptotic Methods in the Theory of Non-linear Oscillations”, Gordon and Breach, New York, (1961).

[3] M.R.Jovanovic, B.Bamieh, and M.Grebeck, “Parametric resonance in spatially distributed systems”, Proc. of the ACC, pp. 119-124, IEEE (2003).

[4] C. Joas, M.E.Raikh, and F.V. Oppen, “Parametric resonance of a two-dimensional electron gas under bichromatic irradiation:”, Phys. Rev. B 70, 235302(2004).

[5] F.J.Poulin and M.Scott, “Stochastic parametric resonance in shear flows”, Nonlinear Processes in Geophysics, 12, 871-876, (2005).

[6] E.Khain and B.Meerson, “Parametric autoresonance”, Phys. Rev. Vol. 64, 036619(2001).

[7] D. Jeon, M.Bai, C.M.Chu, X.Kang, S.Y.Lee, A.Riabko, and X.Zhao, “Role of parametric resonances in global chaos”, Phys. Rev. E Vol. 54, No.4, (1996).

[8] Chechurin S. L., “Parametric Oscillations and Stability of Periodical Motion. (in Russian)”, LGU Publication, (1983).

[9] Seyranian A. P., Solem F., Pedersen P., “Stability Analysis for Multiparameter Linear Periodic Systems” Archive Appl. Mech., 69(3), pp. 160-180, (1999).

[10] Yakubovich V. A. and Strazhinskii V. M., “Parametric Resonance in Linear Systems.”, Nauka, (1987).

[11] Chechurin S.L., Chechurin L.S. “Physical fundamentals of Oscillation theory” (in Russian), SPbSPU, 258p, (2005).

[12] K. Ogata, “ Discrete-Time Control Systems ”, Prentice Hall, (1987)

[13] Levinshtein M. L., “Phenomenon of Parametric Resonance in Works of Synchronous Machines on Capacitance Loading. (in Russian)”, Transactions of LPI named after Kalinina, 3, pp 13–41, (1948).

[14] Vazhnov A. I., “Electrical Machines. (in Russian)”, Leningrad, Energiya, (1969).

[15] Lee Y. K., “Frequency models of periodically non-stationary systems with distributed parameters. (in Russian)”, St.Petersburg State Polytechnic Univ. Ph.D dissertation, (2004)

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