

Theoretical Mathematics & Applications, vol.4, no.2, 2014, 31-44
ISSN: 1792-9687 (print), 1792-9709 (online)
Scienpress Ltd, 2014

Multiple Positive solutions of Sturm-Liouville problems for second order singular and impulsive differential equations

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Abstract

In this paper, we study the existence of multiple positive solutions of nonlinear singular two-point boundary value problems for second-order impulsive differential equations. The proof is based on the theory of fixed point index in cones.

Mathematics Subject Classification: 34B15

Keywords: Multiple positive solutions; Singular two-point boundary value problem; Second-order impulsive differential equations; Fixed point index in cones.

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Project supported by Heilongjiang province education department natural science research item, China. (12541076).

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1 Introduction

Impulsive differential equations play a very important role in modern applied mathematics due to their deep physical background and broad application. In this paper, we consider the existence of multiple positive solutions of two-point boundary value problems for nonlinear second-order singular and impulsive differential equations:

$$\begin{cases} -Lu = h(x)g(x, u), & x \in I', \\ -\Delta(pu')|_{x=x_k} = I_k(u(x_k)), & k = 1, 2, \dots, m, \\ \Delta(pu)|_{x=x_k} = \bar{I}_k(u(x_k)), & k = 1, 2, \dots, m, \\ R_1(u) = \alpha u(0) - \beta u'(0) = 0, \\ R_2(u) = \gamma u(1) + \delta u'(1) = 0, \end{cases} \quad (1.1)$$

here $Lu = (p(x)u')' + q(x)u$ is sturm-liouville operator, $I = [0, 1]$, $I' = I \setminus \{x_1, x_2, \dots, x_m\}$ and $0 < x_1 < x_2 < \dots < x_m < 1$ are given, $R^+ = [0, +\infty)$, $g \in C(I \times R^+, R^+)$, $I_k, \bar{I}_k \in C(R^+, R^+)$, $\Delta(pu')|_{x=x_k} = p(x_k)u'(x_k^+) - p(x_k)u'(x_k^-)$, $\Delta(pu)|_{x=x_k} = p(x_k)u(x_k^+) - p(x_k)u(x_k^-)$, $u'(x_k^+), u(x_k^+)$ ($u'(x_k^-), u(x_k^-)$) denote the right limit (left limit) of $u'(x)$ and $u(x)$ at $x = x_k$ respectively, $h(x) \in C(I, R^+)$ and may be singular at $x = 0$ or $x = 1$.

Throughout this paper, we always suppose that

$$(S_1) \quad p(x) \in C^1([0, 1], R), p(x) > 0, q(x) \in C([0, 1], R), q(x) \leq 0, \alpha, \beta, \gamma, \delta \geq 0, \rho = \beta\gamma + \alpha\gamma + \alpha\delta > 0.$$

It is well known that there are abundant results about the existence of positive solutions of boundary value problems for second order impulsive differential equations. Some works can be found in [1, 3, 6 – 9] and references therein. They, mainly investigated the case $p(x) = 1$ and $q(x) = 0$. In this paper, we will consider the case $p(x) \neq 1$, and $q(x) \neq 0$. Here we also mention that second order dynamic inclusions on time scales with impulses has been studied in [2]. We obtain the existence results of positive solutions, by means of the fixed point index theorem in cones under some conditions on $g(x, u)$ concerning the first eigenvalue corresponding to the relevant linear operator.

To conclude the introduction, we introduce the following notation:

$$g_0 = \liminf_{u \rightarrow 0^+} \min_{x \in [a, b]} \frac{g(x, u)}{u}, \quad I_0(k) = \liminf_{u \rightarrow 0^+} \frac{I_k(u)}{u}, \quad \bar{I}_0(k) = \liminf_{u \rightarrow 0^+} \frac{\bar{I}_k(u)}{u}$$

$$g_\infty = \liminf_{u \rightarrow +\infty} \min_{x \in [a, b]} \frac{g(x, u)}{u}, \quad I_\infty(k) = \liminf_{u \rightarrow +\infty} \frac{I_k(u)}{u}, \quad \bar{I}_\infty(k) = \liminf_{u \rightarrow +\infty} \frac{\bar{I}_k(u)}{u};$$

Moreover, for the simplicity in the following discussion, we introduce the following hypotheses.

(H₁) :

$$g_0 + \frac{\sigma \sum_{k=1}^m (I_0(k)\phi_1(x_k) + \bar{I}_0(k)\phi_1'(x_k))}{\int_a^b \phi_1(x)h(x)dx} > \lambda_1,$$

$$g_\infty + \frac{\sigma \sum_{k=1}^m (I_\infty(k)\phi_1(x_k) + \bar{I}_\infty(k)\phi_1'(x_k))}{\int_a^b \phi_1(x)h(x)dx} > \lambda_1.$$

here $\sigma = \min\{\frac{m(a)}{m(1)}, \frac{n(b)}{n(0)}\}$ (see section 2), and $\phi_1(x)$ is the eigenfunction related to the smallest eigenvalue λ_1 of the eigenvalue problem $-L\phi = \lambda\phi h$, $R_1(\phi) = R_2(\phi) = 0$.

(H₂) : There is a $p > 0$ such that $0 \leq u \leq p$ and $0 \leq x \leq 1$ implies

$$g(x, u) \leq \eta p, \quad I_k(u) \leq \eta_k p, \quad \bar{I}_k(u) \leq \bar{\eta}_k p$$

here $\eta, \eta_k, \bar{\eta}_k \geq 0, \eta + \sum_{k=1}^m (\eta_k + \bar{\eta}_k) > 0, \eta \int_0^1 G(y, y)h(y)dy + \sum_{k=1}^m G(x_k, x_k)(\eta_k + \bar{\eta}_k) < 1$ and $G(x, y)$ is the Green's function of boundary value problem $-Lu = 0, R_1(u) = R_2(u) = 0$ (see section 2).

(H₃) : $0 < \int_1^0 G(y, y)h(y)dy < +\infty$

2 Preliminary Notes

In this paper, we shall consider the following space

$PC(I, R) = \{u \in C(I, R); u|_{(x_k, x_{k+1})} \in C(x_k, x_{k+1}), u(x_k^-) = u(x_k), \exists u(x_k^+), k = 1, 2, \dots, m\}$
 $PC'(I, R) = \{u \in C(I, R); u'|_{(x_k, x_{k+1})} \in C(x_k, x_{k+1}), u'(x_k^-) = u'(x_k), \exists u'(x_k^+), k = 1, 2, \dots, m\}$ with the norm $\|u\|_{PC} = \sup_{x \in [0, 1]} |u(x)|$,
 $\|u\|_{PC'} = \max\{\|u\|_{PC}, \|u'\|_{PC}\}$, Then $PC(I, R), PC'(I, R)$ are Banach spaces.

Definition 2.1. A function $u \in PC'(I, R) \cap C^2(I', R)$ is a solution of (1.1), if it satisfies the differential equation

$$Lu + h(x)g(x, u) = 0, \quad x \in I'$$

and the function u satisfies conditions $\Delta(pu')|_{x=x_k} = -I_k(u(x_k)), \Delta(pu)|_{x=x_k} = \bar{I}_k(u(x_k))$ and $R_1(u) = R_2(u) = 0$.

Let $Q = I \times I$ and $Q_1 = \{(x, y) \in Q | 0 \leq x \leq y \leq 1\}$, $Q_2 = \{(x, y) \in Q | 0 \leq y \leq x \leq 1\}$. Let $G(x, y)$ is the Green's function of the boundary value problem

$$-Lu = 0, \quad R_1(u) = R_2(u) = 0.$$

Following from [4], $G(x, y)$ can be written by

$$G(x, y) := \begin{cases} \frac{m(x)n(y)}{\omega}, & (x, y) \in Q_1, \\ \frac{m(y)n(x)}{\omega}, & (x, y) \in Q_2. \end{cases} \quad (2.1)$$

Lemma 2.2. *Suppose that (S_1) holds, then the Green's function $G(x, y)$, defined by (2.1), possesses the following properties:*

- (i): $m(x) \in C^2(I, R)$ is increasing and $m(x) > 0$, $x \in (0, 1]$.
- (ii): $n(x) \in C^2(I, R)$ is decreasing and $n(x) > 0$, $x \in [0, 1)$.
- (iii): $(Lm)(x) \equiv 0$, $m(0) = \beta$, $m'(0) = \alpha$.
- (iv): $(Ln)(x) \equiv 0$, $n(1) = \delta$, $n'(1) = -\gamma$.
- (v): ω is a positive constant. Moreover, $p(x)(m'(x)n(x) - m(x)n'(x)) \equiv \omega$.
- (vi): $G(x, y)$ is continuous and symmetrical over Q .
- (vii): $G(x, y)$ has continuously partial derivative over Q_1, Q_2 .
- (viii): For each fixed $y \in I$, $G(x, y)$ satisfies $LG(x, y) = 0$ for $x \neq y$, $x \in I$.
Moreover, $R_1(G) = R_2(G) = 0$ for $y \in (0, 1)$.
- (viii): G'_x has discontinuous point of the first kind at $x = y$ and

$$G'_x(y+0, y) - G'_x(y-0, y) = -\frac{1}{p(y)}, \quad y \in (0, 1).$$

Following from Lemma 2.2, it is easy to see that:

$$\begin{aligned} G(x, y) &\leq G(y, y) = \frac{m(y)n(y)}{\omega}, \quad x, y \in [0, 1] \\ G(x, y) &\geq \sigma G(y, y), \quad x \in [a, b], y \in [0, 1], \end{aligned} \quad (2.2)$$

where $a \in (0, t_1]$, $b \in [t_m, 1)$, $0 < \sigma = \min\{\frac{m(a)}{m(1)}, \frac{n(b)}{n(0)}\} < 1$

Consider the linear Sturm-Liouville problem

$$-(Lu)(x) = \lambda u(x)h(x), \quad R_1(u) = R_2(u) = 0.$$

By the Sturm-Liouville theory of ordinary differential equations, we know that there exists an eigenfunction $\phi_1(x)$ with respect to the first eigenvalue $\lambda_1 > 0$ such that $\phi_1(x) > 0$ for $x \in (0, 1)$.

Let E be a Banach space and $K \subset E$ be a closed convex cone in E . For $r > 0$, let $K_r = \{u \in K : \|u\| < r\}$ and $\partial K_r = \{u \in K : \|u\| = r\}$. The following two Lemmas are needed in our argument.

Lemma 2.3. *Let $\Phi : K \rightarrow K$ be a continuous and completely continuous mapping and $\Phi u \neq u$ for $u \in \partial K_r$. Thus the following conclusions hold:*

- (i) If $\|u\| \leq \|\Phi u\|$ for $u \in \partial K_r$, then $i(\Phi, K_r, K) = 0$;
- (ii) If $\|u\| \geq \|\Phi u\|$ for $u \in \partial K_r$, then $i(\Phi, K_r, K) = 1$.

Lemma 2.4. *Let $\Phi : K \rightarrow K$ be a continuous and completely continuous mapping. Suppose that the following two conditions are satisfied:*

- (i) $\inf_{u \in \partial K_r} \|\Phi u\| > 0$;
- (ii) $\mu \Phi u \neq u$ for every $u \in \partial K_r$ and $\mu \geq 1$.

Then, $i(\Phi, K_r, K) = 0$.

In applications below, we take $E = C(I, R)$ and define

$$K = \{u \in C(I, R) : u(x) \geq \sigma \|u\|, x \in [a, b]\}.$$

One may readily verify that K is a cone in E .

Define an operator $\Phi : K \rightarrow K$ by

$$(\Phi u)(x) = \int_0^1 G(x, y)h(y)g(y, u(y))dy + \sum_{0 < x_k < x} G(x, x_k)(I_k(u(x_k)) + \bar{I}_k(u(x_k))), \quad x \in I.$$

It follows from (H_3) that Φ is well defined.

Lemma 2.5. *If (H_3) is satisfied, then $\Phi : K \rightarrow K$ is continuous and completely continuous. Moreover, $\Phi(K) \subset K$.*

Proof By the property of continuous of $g(x, u)$, $I_k(x)$, $\bar{I}_k(x)$, it is easy to see that $\Phi : K \rightarrow K$ is continuous and completely continuous. Thus we only need to show $\Phi(K) \subset K$. In fact, for $u \in K$, by using inequalities (2.2) and (H_3) , we have that

$$\|\Phi u\| \leq \int_0^1 G(y, y)h(y)g(y, u(y))dy + \sum_{0 < x_k < x} G(x_k, x_k)(I_k(u(x_k)) + \bar{I}_k(u(x_k))) < +\infty$$

On the other hand ,for any $x \in [a, b]$,by (2.2) we obtain

$$\begin{aligned}
(\Phi u)(x) &= \int_0^1 G(x, y)h(y)g(y, u(y))dy + \sum_{0 < x_k < x} G(x, x_k)(I_k(u(x_k)) + \bar{I}_k(u(x_k))) \\
&\geq \int_b^a G(x, y)h(y)g(y, u(y))dy + \sum_{0 < x_k < x} G(x, x_k)(I_k(u(x_k)) + \bar{I}_k(u(x_k))) \\
&\geq \sigma \left(\int_1^0 G(y, y)h(y)g(y, u(y))dy + \sum_{0 < x_k < x} G(x_k, x_k)(I_k(u(x_k)) + \bar{I}_k(u(x_k))) \right) \\
&\geq \sigma \|\Phi u\|
\end{aligned}$$

Thus, $\Phi(K) \subset K$.

Lemma 2.6. *If u is a fixed point of the operator Φ , then u is a solution of problem (1.1).*

3 Main Results

Lemma 3.1. *If (H_2) is satisfied, then $i(\Phi, K_p, K) = 1$.*

Proof Let $u \in K$ with $\|u\| = p$. It follows from (H_2) that

$$\begin{aligned}
\|\Phi u\| &\leq \int_0^1 G(y, y)h(y)g(y, u(y))dy + \sum_{k=1}^m G(x_k, x_k)(I_k(u(x_k)) + \bar{I}_k(u(x_k))) \\
&\leq p[\eta \int_0^1 G(y, y)h(y)dy + \sum_{k=1}^m G(x_k, x_k)(\eta_k + \bar{\eta}_k)] < p = \|u\|.
\end{aligned}$$

Thus

$$\|\Phi u\| < \|u\|, \quad \forall u \in \partial K_p.$$

It is obvious that $\Phi u \neq u$ for $u \in \partial K_p$. Therefore, $i(\Phi, K_p, K) = 1$, here we use Lemma2.3.

Theorem 3.2. *Assume that $(H_1) - (H_3)$ are satisfied. Then problem (1.1) has at least two positive solutions u_1 and u_2 with*

$$0 < \|u_1\| < p < \|u_2\|.$$

Proof According to Lemma 3.1, we have that

$$i(\Phi, K_p, K) = 1. \quad (3.1)$$

Since (H_1) holds, then there exists $0 < \varepsilon < 1$ such that

$$(1 - \varepsilon) \left[g_0 + \frac{\sigma \sum_{k=1}^m (I_0(k)\phi_1(x_k) + \bar{I}_0(k)\phi_1'(x_k))}{\int_a^b \phi_1(x)h(x)dx} \right] > \lambda_1,$$

$$(1 - \varepsilon) \left[g_\infty + \frac{\sigma \sum_{k=1}^m (I_\infty(k)\phi_1(x_k) + \bar{I}_\infty(k)\phi_1'(x_k))}{\int_a^b \phi_1(x)h(x)dx} \right] > \lambda_1. \quad (3.2)$$

By the definitions of g_0 , I_0 , one can find $0 < r_0 < p$ such that

$$g(x, u) \geq g_0(1 - \varepsilon)u, \quad I_k(u) \geq I_0(k)(1 - \varepsilon)u, \quad \bar{I}_k(u) \geq \bar{I}_0(k)(1 - \varepsilon)u \quad \forall x \in [a, b], \quad 0 < u < r_0.$$

Let $r \in (0, r_0)$, then for $u \in \partial K_r$, we have

$$u(x) \geq \sigma \|u\| = \sigma r > 0. \quad x \in [a, b]$$

Thus

$$\begin{aligned} (\Phi u)\left(\frac{1}{2}\right) &= \int_0^1 G\left(\frac{1}{2}, y\right)h(y)g(y, u(y))dy + \sum_{0 < x_k < \frac{1}{2}} G\left(\frac{1}{2}, x_k\right)(I_k(u(x_k)) + \bar{I}_k(u(x_k))) \\ &\geq \int_a^b G\left(\frac{1}{2}, y\right)h(y)g(y, u(y))dy + \sum_{0 < x_k < \frac{1}{2}} G\left(\frac{1}{2}, x_k\right)(I_k(u(x_k)) + \bar{I}_k(u(x_k))) \\ &\geq g_0(1 - \varepsilon) \int_a^b G\left(\frac{1}{2}, y\right)h(y)u(y)dy + (1 - \varepsilon) \sum_{0 < x_k < \frac{1}{2}} G\left(\frac{1}{2}, x_k\right)I_0(k)u(x_k) \\ &\quad + (1 - \varepsilon) \sum_{0 < x_k < \frac{1}{2}} G\left(\frac{1}{2}, x_k\right)\bar{I}_0(k)u(x_k) \\ &\geq (1 - \varepsilon)\sigma r \left(g_0 \int_a^b G\left(\frac{1}{2}, y\right)h(x)dy + \sum_{0 < x_k < \frac{1}{2}} G\left(\frac{1}{2}, x_k\right)(I_0(k) + \bar{I}_0(k)) \right) > 0 \end{aligned}$$

from which we see that $\inf_{u \in \partial K_r} \|\Phi u\| > 0$, namely, hypothesis (i) of Lemma 2.4 holds. Next we show that $\mu \Phi u \neq u$ for any $u \in \partial K_r$ and $\mu \geq 1$.

If this is not true, then there exist $u_0 \in \partial K_r$ and $\mu_0 \geq 1$ such that $\mu_0 \Phi u_0 = u_0$. Note that $u_0(x)$ satisfies

$$\begin{cases} Lu_0 + \mu_0 h(x)g(x, u_0(x)) = 0, & x \in I', \\ -\Delta(pu'_0)|_{x=x_k} = \mu_0 I_k(u_0(x_k)), & k = 1, 2, \dots, m, \\ \Delta(pu_0)|_{x=x_k} = \mu_0 \bar{I}_k(u_0(x_k)), & k = 1, 2, \dots, m, \\ \alpha u_0(0) - \beta u'_0(0) = 0, \\ \gamma u_0(1) + \delta u'_0(1) = 0. \end{cases} \quad (3.3)$$

Multiply equation (3.3) by $\phi_1(x)$ and integrate from a to b, note that

$$\begin{aligned} & \int_a^b \phi_1(x)[(p(x)u'_0(x))' + q(x)u_0(x)]dx = \int_a^{x_1} \phi_1(x)[(p(x)u'_0(x))' + q(x)u_0(x)]dx \\ & + \sum_{k=1}^{m-1} \int_{x_k}^{x_{k+1}} \phi_1(x)[(p(x)u'_0(x))' + q(x)u_0(x)]dx \\ & + \int_{x_m}^b \phi_1(x)[(p(x)u'_0(x))' + q(x)u_0(x)]dx \\ & = \phi_1(x_1)p(x_1)u'_0(x_1 - 0) - \int_a^{x_1} p(x)u'_0(x)\phi'_1(x)dx + \int_a^{x_1} q(x)u_0(x)\phi_1(x)dx \\ & + \sum_{k=1}^{m-1} [\phi_1(x_{k+1})p(x_{k+1})u'_0(x_{k+1} - 0) - \phi_1(x_k)p(x_k)u'_0(x_k + 0) \\ & - \int_{x_k}^{x_{k+1}} p(x)u'_0(x)\phi'_1(x)dx + \int_{x_k}^{x_{k+1}} q(x)u_0(x)\phi_1(x)dx] \\ & - \phi_1(x_m)p(x_m)u'_0(x_m + 0) - \int_{x_m}^b p(x)u'_0(x)\phi'_1(x)dx + \int_{x_m}^b q(x)u_0(x)\phi_1(x)dx \\ & = - \sum_{k=1}^m \Delta(p(x_k)u'_0(x_k))\phi_1(x_k) - \int_a^b p(x)\phi'_1(x)u'_0(x)dx + \int_a^b q(x)\phi_1(x)u_0(x)dx \end{aligned}$$

Also note that

$$\begin{aligned} & \int_a^b p(x)\phi'_1(x)u'_0(x)dx = \int_a^{x_1} p(x)\phi'_1(x)du_0(x) + \sum_{k=1}^{m-1} \int_{x_k}^{x_{k+1}} p(x)\phi'_1(x)du_0(x) \\ & + \int_{x_m}^b p(x)\phi'_1(x)du_0(x) \\ & = - \sum_{k=1}^m \Delta(p(x_k)u_0(x_k))\phi'_1(x_k) - \int_a^b u_0(x)(p(x)\phi'_1(x))'dx \\ & = - \sum_{k=1}^m \Delta(p(x_k)u_0(x_k))\phi'_1(x_k) + \int_a^b u_0(x)q(x)\phi_1(x)dx + \lambda_1 \int_a^b h(x)\phi_1(x)u_0(x)dx \end{aligned}$$

$$\begin{aligned}
& \int_a^b \phi_1(x)[(p(x)u_0'(x))' + q(x)u_0(x)]dx = - \sum_{k=1}^m \Delta(p(x_k)u_0'(x_k))\phi_1(x_k) \\
& + \sum_{k=1}^m \Delta(p(x_k)u_0(x_k))\phi_1'(x_k) - \lambda_1 \int_a^b h(x)\phi_1(x)u_0(x)dx \\
& = \sum_{k=1}^m \mu_0(I_k(u_0(x_k))\phi_1(x_k) + \bar{I}_k(u_0(x_k))\phi_1'(x_k)) - \lambda_1 \int_a^b u_0(x)h(x)\phi_1(x)dx
\end{aligned}$$

So we obtain

$$\begin{aligned}
\lambda_1 \int_a^b u_0(x)h(x)\phi_1(x)dx &= \sum_{k=1}^m \mu_0(I_k(u_0(x_k))\phi_1(x_k) + \bar{I}_k(u_0(x_k))\phi_1'(x_k)) \\
&+ \mu_0 \int_a^b \phi_1(x)h(x)g(x, u_0(x))dx \\
&\geq (1 - \varepsilon) \sum_{k=1}^m u_0(x_k)(I_0(k)\phi_1(x_k) + \bar{I}_0(k)\phi_1'(x_k)) \\
&+ (1 - \varepsilon)g_0 \int_a^b \phi_1(x)u_0(x)h(x)dx
\end{aligned}$$

Since $u_0(x) \geq \sigma \|u_0\| = \sigma r$, we have $\int_a^b \phi_1(x)u_0(x)h(x)dx > 0$ and $\sum_{k=1}^m u_0(x_k)(I_0(k)\phi_1(x_k) + \bar{I}_0(k)\phi_1'(x_k)) > 0$. So from the above inequality we see that $\lambda_1 > (1 - \varepsilon)g_0$.

Thus

$$\begin{aligned}
[\lambda_1 - (1 - \varepsilon)g_0] \int_a^b u_0(x)h(x)\phi_1(x)dx &\geq (1 - \varepsilon) \sum_{k=1}^m (I_0(k)\phi_1(x_k) + \bar{I}_0(k)\phi_1'(x_k))u_0(x_k) \\
&\geq (1 - \varepsilon)\sigma r \sum_{k=1}^m (I_0(k)\phi_1(x_k) + \bar{I}_0(k)\phi_1'(x_k)).
\end{aligned}$$

Since $\int_a^b u_0(x)h(x)\phi_1(x)dx \leq r \int_a^b \phi_1(x)h(x)dx$, we have

$$[\lambda_1 - (1 - \varepsilon)g_0] \int_a^b h(x)\phi_1(x)dx \geq (1 - \varepsilon)\sigma \sum_{k=1}^m (I_0(k)\phi_1(x_k) + \bar{I}_0(k)\phi_1'(x_k)),$$

which contradicts (3.2) again. Hence Φ satisfies the hypotheses of Lemma 2.4 in K_r . Thus

$$i(\Phi, K_r, K) = 0. \tag{3.4}$$

On the other hand, from (H_1) , there exists $H > p$ such that for any $x \in [a, b]$, $u \geq H$.

$$g(x, u) \geq g_\infty(1 - \varepsilon)u, \quad I_k(u) \geq I_\infty(k)(1 - \varepsilon)u, \quad \bar{I}_k(u) \geq \bar{I}_\infty(k)(1 - \varepsilon)u, \quad (3.5)$$

Let

$$C = \max_{0 \leq u \leq H} \max_{a \leq x \leq b} |g(x, u) - g_\infty(1 - \varepsilon)u| + \sum_{k=1}^m \max_{0 \leq u \leq H} |I_k(u) - I_\infty(k)(1 - \varepsilon)u| + \sum_{k=1}^m \max_{0 \leq u \leq H} |\bar{I}_k(u) - \bar{I}_\infty(k)(1 - \varepsilon)u|. \quad \text{It is clear that for any } x \in [a, b], u \geq 0.$$

$$g(x, u) \geq g_\infty(1 - \varepsilon)u - C, \quad I_k(u) \geq I_\infty(k)(1 - \varepsilon)u - C, \quad \bar{I}_k(u) \geq \bar{I}_\infty(k)(1 - \varepsilon)u - C, \quad (3.6)$$

Choose $R > R_0 := \max\{\frac{H}{\sigma}, p\}$ and let $u \in \partial K_R$. Since $u(x) \geq \sigma||u|| = \sigma R > H$ for $x \in [a, b]$, from (3.5) we see that

$$g(x, u(x)) \geq g_\infty(1 - \varepsilon)u(x) \geq \sigma g_\infty(1 - \varepsilon)R, \quad \forall x \in [a, b].$$

$$I_k(u(x_k)) \geq \sigma I_\infty(k)(1 - \varepsilon)R, \quad \bar{I}_k(u(x_k)) \geq \sigma \bar{I}_\infty(k)(1 - \varepsilon)R.$$

Essentially the same reasoning as above yields $\inf_{u \in \partial K_R} ||\Phi u|| > 0$. Next we show that if R is large enough, then $\mu \Phi u \neq u$ for any $u \in \partial K_R$ and $\mu \geq 1$. In fact, if there exist $u_0 \in \partial K_R$ and $\mu_0 \geq 1$ such that $\mu_0 \Phi u_0 = u_0$, then $u_0(x)$ satisfies equation (3.3).

Multiply equation (3.3) by $\phi_1(x)$ and integrate from a to b , using integration by parts in the left side to obtain

$$\begin{aligned} & \lambda_1 \int_a^b u_0(x)h(x)\phi_1(x)dx = \sum_{k=1}^m \mu_0 (I_k(u_0(x_k))\phi_1(x_k) + \bar{I}_k(u_0(x_k))\phi_1'(x_k)) \\ & + \mu_0 \int_a^b \phi_1(x)h(x)g(x, u_0(x))dx \\ & \geq (1 - \varepsilon) \sum_{k=1}^m (I_\infty(k)\phi_1(x_k) + \bar{I}_\infty(k)\phi_1'(x_k))u_0(x_k) + (1 - \varepsilon)g_\infty \int_a^b u_0(x)\phi_1(x)h(x)dx \\ & - C \left(\sum_{k=1}^m (\phi_1(x_k) + \phi_1'(x_k)) + \int_a^b \phi_1(x)h(x)dx \right). \end{aligned}$$

If $g_\infty \leq \lambda_1$, then we have

$$\begin{aligned} & [\lambda_1 - (1 - \varepsilon)g_\infty] \int_a^b u_0(x)h(x)\phi_1(x)dx + C \left(\sum_{k=1}^m (\phi_1(x_k) + \phi_1'(x_k)) + \int_a^b \phi_1(x)h(x)dx \right) \\ & \geq (1 - \varepsilon) \sum_{k=1}^m (I_\infty(k)\phi_1(x_k) + \bar{I}_\infty(k)\phi_1'(x_k))u_0(x_k). \end{aligned}$$

thus

$$\begin{aligned} & \|u_0\| [\lambda_1 - (1 - \varepsilon)g_\infty] \int_a^b h(x)\phi_1(x)dx + C \left(\sum_{k=1}^m (\phi_1(x_k) + \phi_1'(x_k)) + \int_a^b \phi_1(x)h(x)dx \right) \\ & \geq (1 - \varepsilon)\sigma \|u_0\| \sum_{k=1}^m (I_\infty(k)\phi_1(x_k) + \bar{I}_\infty(k)\phi_1'(x_k)). \end{aligned}$$

and

$$\begin{aligned} \|u_0\| & \leq \frac{C \left(\sum_{k=1}^m (\phi_1(x_k) + \phi_1'(x_k)) + \int_a^b \phi_1(x)h(x)dx \right)}{(1 - \varepsilon)\sigma \sum_{k=1}^m (I_\infty(k)\phi_1(x_k) + \bar{I}_\infty(k)\phi_1'(x_k)) - [\lambda_1 - (1 - \varepsilon)g_\infty] \int_a^b \phi_1(x)h(x)dx} \\ & =: \bar{R}. \end{aligned} \tag{3.7a}$$

If $g_\infty > \lambda_1$, we can choose $\varepsilon > 0$ such that $(1 - \varepsilon)g_\infty > \lambda_1$, then we have

$$\begin{aligned} C \left(\sum_{k=1}^m (\phi_1(x_k) + \phi_1'(x_k)) + \int_a^b \phi_1(x)h(x)dx \right) & \geq [(1 - \varepsilon)g_\infty - \lambda_1] \int_a^b \phi_1(x)u_0(x)h(x)dx \\ & \geq [(1 - \varepsilon)g_\infty - \lambda_1]\sigma \|u_0\| \int_a^b \phi_1(x)h(x)dx. \end{aligned}$$

Thus

$$\|u_0\| \leq \frac{C \left(\sum_{k=1}^m (\phi_1(x_k) + \phi_1'(x_k)) + \int_a^b \phi_1(x)h(x)dx \right)}{[(1 - \varepsilon)g_\infty - \lambda_1]\sigma \int_a^b \phi_1(x)h(x)dx} =: \bar{R}. \tag{3.7b}$$

Let $R > \max\{p, \bar{R}\}$, then for any $u \in \partial K_R$ and $\mu \geq 1$, we have $\mu\Phi u \neq u$. Hence hypothesis (ii) of Lemma 2.4 is satisfied and

$$i(\Phi, K_R, K) = 0. \tag{3.8}$$

In view of (3.1), (3.4) and (3.8), we obtain

$$i(\Phi, K_R \setminus \bar{K}_p, K) = -1, \quad i(\Phi, K_p \setminus \bar{K}_r, K) = 1.$$

Then Φ has fixed points u_1 and u_2 in $K_p \setminus \bar{K}_r$ and $K_R \setminus \bar{K}_p$, respectively, which means $u_1(x)$ and $u_2(x)$ are positive solution of the problem (1.1) and $0 < \|u_1\| < p < \|u_2\|$.

Corollary 3.3. *The conclusion of Theorem 3.2 is valid if (H_1) is replaced by*

$$(H_1^*) \quad g_0 = \infty \text{ or } \sum_{k=1}^m I_0(k)\phi_1(x_k) = \infty \text{ or } \sum_{k=1}^m \bar{I}_0(k)\phi_1'(x_k) = \infty;$$

and

$$g_\infty = \infty \text{ or } \sum_{k=1}^m I_\infty(k)\phi_1(x_k) = \infty \text{ or } \sum_{k=1}^m \bar{I}_\infty(k)\phi_1'(x_k) = \infty.$$

Example 3.4.

$$\begin{cases} Lu + \frac{1}{2}(u^{\frac{1}{3}} + u^3) = 0, & x \in I', \\ -\Delta(pu')|_{x=x_k} = c_k u(x_k), & c_k \geq 0, \\ \Delta(pu)|_{x=x_k} = d_k u(x_k), & d_k \geq 0, \\ R_1(u) = \alpha u(0) - \beta u'(0) = 0, \\ R_2(u) = \gamma u(1) + \delta u'(1) = 0, \end{cases} \quad (3.9)$$

here $Lu = (p(x)u')' + q(x)u$. Assume that (S_1) is satisfied. Then problem (3.9) has at least two positive solutions u_1 and u_2 with

$$0 < \|u_1\| < 1 < \|u_2\|$$

provided

$$1 < \frac{1}{d} \left(1 - \sum_{k=1}^m G(x_k, x_k)(c_k + d_k) \right), \quad d = \int_0^1 G(y, y)dy. \quad (3.10)$$

Proof To see this we will apply Theorem 3.2 (or Corollary 3.3).

By (3.10), $\eta > 0$ is chosen such that

$$1 < \eta < \frac{1}{d} \left(1 - \sum_{k=1}^m G(x_k, x_k)(c_k + d_k) \right).$$

Set

$$g(x, u) = \frac{1}{2}(u^{\frac{1}{3}} + u^3).$$

Note

$$g_0 = \infty, \quad g_\infty = \infty,$$

so (H_1) (or (H_1^*)) holds.

Let $\eta_k = c_k, \bar{\eta}_k = d_k$, then $\eta, \eta_k, \bar{\eta}_k$ satisfy

$$\eta \int_0^1 G(y, y) dy + \sum_{k=1}^m G(x_k, x_k)(\eta_k + \bar{\eta}_k) < 1.$$

Let $p = 1$, then for $0 \leq u \leq p$, we have

$$g(x, u) = \frac{1}{2}(u^{\frac{1}{3}} + u^3) \leq \frac{1}{2} + \frac{1}{2} < \eta p = \eta,$$

and

$$I_k(u) = c_k u = \eta_k u \leq \eta_k p, \quad \bar{I}_k(u) = d_k u = \bar{\eta}_k u \leq \bar{\eta}_k p$$

thus (H_2) holds. The result now follows from Theorem 3.2 (or Corollary 3.3).

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