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Generalization of Titchmarsh's Theorem for the Generalized Fourier-Bessel Transform in the Space $L^2_{\alpha,n}$

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Abstract. In this paper, using a generalized translation operator, we prove the estimates for the generalized Fourier-Bessel transform in the space $L^2_{\alpha,n}$, on certain classes of functions.

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1. Introduction and preliminaries

Integral transforms and their inverses the Bessel transform are widely used to solve various problems in calculus, mechanics, mathematical physics, and computational mathematics (see [3,7]).
In [5], E. C. Titchmarsh's characterizes the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy-Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transform, namely we have

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THEOREM 1.1 *Let $\alpha \in (0,1)$ and assume that $f \in L^2(\mathbb{R})$. Then the following are equivalents*
(a) $\|f(t+h) - f(t)\| = O(h^\alpha)$, *as* $h \to 0$,
(b) $\int_{|\lambda|\geqslant r} |\widehat{f}(\lambda)|^2 d\lambda = O(r^{-2\alpha})$ *as* $r \to \infty$,
where $\widehat{f}$ stands for the Fourier transform of f.

In this paper, we consider a second-order singular differential operator $\mathcal{B}$ on the half line which generalizes the Bessel operator $\mathcal{B}_\alpha$. we prove an analog of theorem 1.1 in the generalized Fourier-Bessel transform associated to $\mathcal{B}$ in $L^2_{\alpha,n}$. For this purpose, we use a generalized translation operator. We point out that similar results have been established in the context of non compact rank one Riemannian symetric spaces [8].
We briefly overview the theory of generalized Fourier-Bessel transform and related harmonic analysis (see[1,6]):
Consider the second-order singular differential operator on the half line defined by

$$\mathcal{B}f(x) = \frac{d^2 f(x)}{dx^2} + \frac{(2\alpha+1)}{x}\frac{df(x)}{dx} - \frac{4n(\alpha+n)}{x^2}f(x),$$

where $\alpha > \frac{-1}{2}$ and $n = 0, 1, 2, \ldots$. For $n = 0$, we obtain the classical Bessel operator

$$\mathcal{B}_\alpha f(x) = \frac{d^2 f(x)}{dx^2} + \frac{(2\alpha+1)}{x}\frac{df(x)}{dx}.$$

Let M be the map defined by

$$Mf(x) = x^{2n} f(x), \quad n = 0, 1, \ldots.$$

Let $L^p_{\alpha,n}$, $1 \leqslant p < \infty$, be the class of measurable functions f on $[0, \infty[$ for which

$$\|f\|_{p,\alpha,n} = \|M^{-1}f\|_{p,\alpha+2n} < \infty,$$

where

$$\|f\|_{p,\alpha} = \left(\int_0^\infty |f(x)|^p x^{2\alpha+1} dx\right)^{1/p}.$$

If $p = 2$, then we have $L^2_{\alpha,n} = L^2([0,\infty[, x^{2\alpha+1})$.
For $\alpha \geqslant \frac{-1}{2}$, we introduce the Bessel normalized function of the first kind j_α defined by

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n}}{n!\Gamma(n+\alpha+1)}, \quad z \in \mathbb{C}, \tag{1}$$

where $\Gamma(x)$ is the gamma-function (see [4]). The function $y = j_\alpha(z)$ satisfies the differential equation

$$\mathcal{B}_\alpha y + y = 0,$$

with the initial condition $y(0) = 0$ and $y'(0) = 0$. The function $j_\alpha(z)$ is infinitely differentiable, even, and, moreover entire analytic.

From (1) we see that

$$\lim_{z\to 0}\frac{j_\alpha(z)-1}{z^2}\neq 0,$$

hence, there exists $c>0$ and $\eta>0$ satisfying

$$|z|\leqslant\eta\Rightarrow|j_\alpha(z)-1|\geqslant c|z|^2 \tag{2}$$

From [2], we have

$$|j_\alpha(x)|\leqslant 1 \tag{3}$$
$$1-j_\alpha(x)=O(x^2),\quad 0\leqslant x\leqslant 1. \tag{4}$$

For $\lambda\in\mathbb{C}$, and $x\in\mathbb{R}$, put

$$\varphi_\lambda(x)=x^{2n}j_{\alpha+2n}(\lambda x). \tag{5}$$

From [1,6] recall the following properties.

Proposition 1.2
(c) φ_λ satisfies the differential equation

$$\mathcal{B}\varphi_\lambda=-\lambda^2\varphi_\lambda.$$

(d) For all $\lambda\in\mathbb{C}$, and $x\in\mathbb{R}$

$$|\varphi_\lambda(x)|\leqslant x^{2n}e^{|Im\lambda||x|}.$$

The generalized Fourier-Bessel transform that we call it the integral transform defined by

$$\mathcal{F}_\mathcal{B}f(\lambda)=\int_0^\infty f(x)\varphi_\lambda(x)x^{2\alpha+1}dx,\lambda\geqslant 0,f\in L^1_{\alpha,n},$$

(see [1]).
Let $f\in L^1_{\alpha,n}$ such that $\mathcal{F}_\mathcal{B}(f)\in L^1_{\alpha+2n}=L^1([0,\infty[,x^{2\alpha+4n+1}dx)$. Then the inverse generalized Fourier-Bessel transform is given by the formula (see [1])

$$f(x)=\int_0^\infty \mathcal{F}_\mathcal{B}f(\lambda)\varphi_\lambda(x)d\mu_{\alpha+2n}(\lambda),$$

where

$$d\mu_{\alpha+2n}(\lambda)=a_{\alpha+2n}\lambda^{2\alpha+4n+1}d\lambda,\quad a_\alpha=\frac{1}{4^\alpha(\Gamma(\alpha+1))^2}.$$

From [1,6], we have

Proposition 1.3

(e) For every $f \in L^1_{\alpha,n} \cap L^2_{\alpha,n}$ we have the Plancherel formula

$$\int_0^{+\infty} |f(x)|^2 x^{2\alpha+1} dx = \int_0^{+\infty} |\mathcal{F}_{\mathcal{B}} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

(f) The generalized Fourier-Bessel transform $\mathcal{F}_{\mathcal{B}}$ extends uniquely to an isometric isomorphism from $L^2_{\alpha,n}$ onto $L^2([0,+\infty[,\mu_{\alpha+2n})$.

Define the generalized translation operator T^h, $h \geqslant 0$ by the relation

$$T^h f(x) = (xh)^{2n} \tau^h_{\alpha+2n}(M^{-1}f)(x), x \geqslant 0,$$

where $\tau^h_{\alpha+2n}$ is the Bessel translation operator of order $\alpha + 2n$ defined by

$$\tau^h_\alpha f(x) = c_\alpha \int_0^\pi f(\sqrt{x^2 + h^2 - 2xh\cos t}) \sin^{2\alpha} t dt,$$

where

$$c_\alpha = \left(\int_0^\pi \sin^{2\alpha} t dt\right)^{-1} = \frac{\Gamma(\alpha+1)}{\Gamma(\pi)\Gamma(\alpha+\frac{1}{2})}.$$

For $f \in L^2_{\alpha,n}$, we have

$$\mathcal{F}_{\mathcal{B}}(T^h f)(\lambda) = \varphi_\lambda(h) \mathcal{F}_{\mathcal{B}}(f)(\lambda), \quad (6)$$

$$\mathcal{F}_{\mathcal{B}}(\mathcal{B} f)(\lambda) = -\lambda^2 \mathcal{F}_{\mathcal{B}}(f)(\lambda), \quad (7)$$

(see [1,6] for details).
Let $f \in L^2_{\alpha,n}$. We define the differences of the orders $k(k = 1, 2, ..)$ with a step $h > 0$ by

$$\Delta^k_h f(x) = (T^h - h^{2n} I)^k f(x), \quad (8)$$

where I is the unit operator in $L^2_{\alpha,n}$.
Let $W^k_{2,\alpha,n}$ be the Sobolev space constructed by the Bessel operator $\mathcal{B}$, i.e.,

$$W^k_{2,\alpha,n} = \left\{f \in L^2_{\alpha,n}, \mathcal{B}^m f \in L^2_{\alpha,n}, m = 1, 2, ..., k\right\}.$$

2. Main Result

LEMMA 2.1 *For $f \in W^k_{2,\alpha,n}$, we have*

$$\left(h^{4nk} \int_0^\infty \lambda^{4m} |j_{\alpha+2n}(\lambda h) - 1|^{2k} |\mathcal{F}_{\mathcal{B}} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right)^{\frac{1}{2}} = \|\Delta^k_h \mathcal{B}^m f\|_{2,\alpha,n},$$

where $m = 0, 1, ..., k$.

Proof From formula (7), we obtain

$$\mathcal{F}_{\mathcal{B}}(\mathcal{B}^m f)(\lambda) = (-1)^m \lambda^{2m} \mathcal{F}_{\mathcal{B}} f(\lambda); m = 0, 1, ... \quad (9)$$

By using the formulas (5), (6) and (9), we conclude that

$$\mathcal{F}_{\mathcal{B}}(T^h\mathcal{B}^m f)(\lambda) = (-1)^m h^{2n} j_{\alpha+2n}(\lambda h)\lambda^{2m}\mathcal{F}_{\mathcal{B}}f(\lambda). \tag{10}$$

From the definition of finite difference (8) and formula (10), the image $\Delta_h^k\mathcal{B}^r f(x)$ under the generalized Fourier-Bessel transform has the form

$$\mathcal{F}_{\mathcal{B}}(\Delta_h^k\mathcal{B}^m f)(\lambda) = (-1)^m h^{2nk}(j_{\alpha+2n}(\lambda h)-1)^k\lambda^{2m}\mathcal{F}_{\mathcal{B}}f(\lambda).$$

Now by proposition 1.3, we have the result. ■

Our main result is as follows.

Theorem 2.2 *Let $f \in W^k_{2,\alpha,n}$. Then the following are equivalents*

(i) $\|\Delta_h^k\mathcal{B}^m f\|_{2,\alpha,n} = O(h^{\delta+2nk}), \quad as \quad h \to 0, \quad 0<\delta<1.$
(ii) $\displaystyle\int_r^{\infty} \lambda^{4m}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O(r^{-2\delta}), \quad as \quad r\to\infty,$
where $m = 0,1,...,k$.

Proof *(i)* $\Rightarrow$ *(ii)*. Suppose that

$$\|\Delta_h^k\mathcal{B}^m f\|_{2,\alpha,n} = O(h^{\delta+2nk}), \quad h\to 0.$$

From Lemma 2.1, we have

$$\|\Delta_h^k\mathcal{B}^m f\|^2_{2,\alpha,n} = h^{4nk}\int_0^{\infty}\lambda^{4m}|j_{\alpha+2n}(\lambda h)-1|^{2k}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

By formula (2), we get

$$\int_{\frac{\eta}{2h}}^{\frac{\eta}{h}}\lambda^{4m}|j_{\alpha+2n}(\lambda h)-1|^{2k}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) \geqslant \frac{c^{2k}\eta^{4k}}{2^{4k}}\int_{\frac{\eta}{2h}}^{\frac{\eta}{h}}\lambda^{4m}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

Note that there exists then a positive constant C such that

$$\begin{aligned}\int_{\frac{\eta}{2h}}^{\frac{\eta}{h}}\lambda^{4m}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) &\leqslant C\int_{\frac{\eta}{2h}}^{\frac{\eta}{h}}\lambda^{4m}|j_{\alpha+2n}(\lambda h)-1|^{2k}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\\ &\leqslant \frac{C}{h^{4nk}}\|\Delta_h^k\mathcal{B}^m f\|^2_{2,\alpha,n}\\ &= O(h^{2\delta}).\end{aligned}$$

Then we have

$$\int_r^{2r}\lambda^{4m}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O(r^{-2\delta}), \quad r\to\infty.$$

Furthermore, we obtain

$$\int_r^\infty \lambda^{4m}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = \sum_{i=0}^\infty \int_{2^i r}^{2^{i+1}r} \lambda^{4m}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$
$$= \sum_{i=0}^\infty O((2^i r)^{-2\delta}).$$

This proves that

$$\int_r^\infty \lambda^{4m}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O(r^{-2\delta}), \quad as \quad r \to \infty.$$

$(ii) \Rightarrow (i)$. Suppose now that

$$\int_r^\infty \lambda^{4m}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O(r^{-2\delta}), \quad as \quad r \to \infty.$$

and write

$$\|\Delta_h^k \mathcal{B}^m f\|_{2,\alpha,n}^2 = h^{4nk}(I_1 + I_2),$$

where

$$I_1 = \int_0^{1/h} \lambda^{4m}|j_{\alpha+2n}(\lambda h) - 1|^{2k}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda),$$

and

$$I_2 = \int_{1/h}^\infty \lambda^{4m}|j_{\alpha+2n}(\lambda h) - 1|^{2k}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

Using the inequality (3), we get

$$I_2 \leqslant 4^k \int_{1/h}^\infty \lambda^{4m}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O(h^{2\delta}), \quad as \quad h \to 0.$$

Set

$$\phi(\lambda) = \int_\lambda^\infty x^{4m}|\mathcal{F}_{\mathcal{B}}f(x)|^2 d\mu_{\alpha+2n}(x).$$

From formula (4) and integration by parts, we have

$$\begin{aligned} I_1 &= -\int_0^{1/h} |j_{\alpha+2n}(\lambda h) - 1|^{2k} |\phi'(\lambda) d\lambda \\ &\leqslant -C_1 h^{4k} \int_0^{1/h} \lambda^{4k} \phi'(\lambda) d\lambda \\ &\leqslant -C_1 \phi(\frac{1}{h}) + 4C_1 k h^{4k} \int_0^{1/h} \lambda^{4k-1} \phi(\lambda) d\lambda \\ &\leqslant C_2 h^{4k} \int_0^{1/h} \lambda^{4k-1-2\delta} d\lambda \\ &\leqslant C_3 h^{2\delta}, \end{aligned}$$

where C_1, C_2 and C_3 are positive constants and this ends the proof. ■

COROLLARY 2.3 *Let* $f \in W^k_{2,\alpha,n}$ *and let*

$$\|\Delta_h^k \mathcal{B}^m f\|_{2,\alpha,n} = O(h^{\delta+2nk}), \quad as \quad h \to 0.$$

Then

$$\int_r^{\infty} |\mathcal{F}_{\mathcal{B}} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O(r^{-4m-2\delta}), \quad as \quad r \to \infty,$$

where $m = 0, 1, ..., k$.

3. Conclusions

In this work we have succeded to generalise the theorem in [5] for the generalized Fourier-Bessel transform in the Sobolev space $W^k_{2,\alpha,n}$ constructed by the singular differential operator $\mathcal{B}$. We proved that $\|\Delta_h^k \mathcal{B}^m f\|_{2,\alpha,n} = O(h^{\delta+2nk})$, as $h \to 0$, $0 < \delta < 1$ if and only if $\int_r^{\infty} \lambda^{4m} |\mathcal{F}_{\mathcal{B}} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O(r^{-2\delta})$, as $r \to \infty$.

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