

PATH PLANNING OF MULTIPLE UAVS IN AN ENVIRONMENT OF RESTRICTED REGIONS

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ABSTRACT

This paper describes a novel idea of path planning for multiple UAVs (Unmanned Aerial Vehicles). The path planning ensures safe and simultaneous arrival of the UAVs to the target while meeting curvature and safety constraints. Pythagorean Hodograph (PH) curve is used for path planning. The PH curve provides continuous curvature of the paths. The offset curves of the PH paths define safety margins around and along each flight path. The simultaneous arrival is satisfied by generation of paths of equal lengths. This paper highlights the mathematical property – changing path-shape and path-length by manipulating the curvature and utilises this to achieve the following constraints: (i) Generation of paths of equal length, (ii) Achieving maximum bound on curvature, and, (iii) Meeting the safety constraints by offset paths.

INTRODUCTION

Path planning is a guidance algorithm that generates instantaneous configuration of a moving vehicle. A path can be con-

sidered as a sequence of configurations. The configuration is a set of states which comprises the position and direction of the vehicle. Thus, each point on the path corresponds to a specific configuration of the moving vehicle. Conversely, for a particular type of path (for example, circular or polynomial) the configurations along the path are fixed. In other words, the position and the direction of a moving object can be predetermined at any time of its travel if the type of path is known in advance. Thus providing configuration at any instant of travel, path-planning can be considered as a guidance-algorithm to autonomous moving vehicle.

In addition to providing the state (the terms, “state” and “configuration” are used interchangeably) of a system, the kinematic parameters of the vehicle can be derived from the curve properties. For example, the curvature of the path decides whether a vehicle can take a turn on the given path or can not. The kinematic properties coupled with the inertial parameters provide the dynamic variables of the vehicle in motion. Thus, path planning is a guidance algorithm which can be used to determine the dynamic state of a vehicle at any instant.

While employing a group of UAVs for a mission, it is impor-

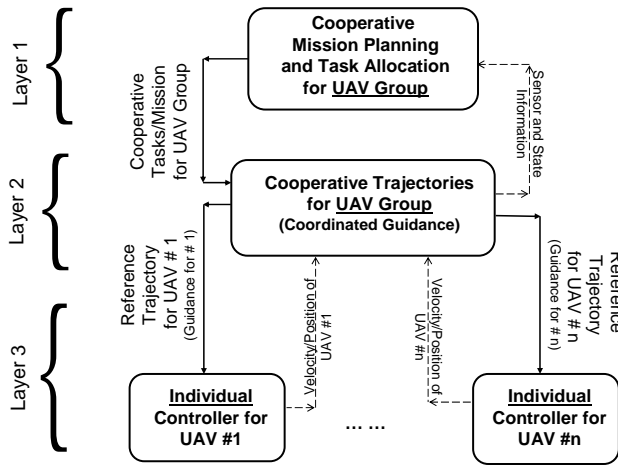


Figure 1. HIERARCHY OF MISSION PLANNING

tant for each UAV to cooperate among themselves and with the environment. In such instances, the path planning for a swarm of UAVs can be considered as coordinated-guidance. Figure 1 shows the schematic sketch of various levels of coordinated closed-loop guidance- and-control. The mission objective and task allocation of a group of UAVs are decided in Layer-1. The Layer-2 produces n co-operating trajectories, Each i^{th} reference trajectory (guidance demand) generated in the Layer-2 is followed by the individual controller of the i^{th} UAV in the Layer-3. Thus the hierarchial architecture provides a Closed Loop Guidance system.

This paper focusses path planning (also called coordinated guidance algorithm) of a swarm of Unmanned Aerial Vehicles (UAVs). A good coordinated guidance algorithm must possess several important attributes. Firstly, it must compute a stealthy path in order to avoid enemy's radar location. Secondly, it must generate paths of minimal length. Thirdly, the path should be a feasible one. Fourthly, The coordinated-guidance-algorithm must be compatible with the cooperative nature envisioned for the UAVs. Finally, the algorithms are expected to be coded in the software that runs on an airborne processor.

Mathematically, a path is the geometric evolution of a curve in free space. Such path can be represented in parametric-form. As the parameter varies, the path is generated. Path planning can be defined as generating a feasible path between a pair of configuration. In other words, for a given initial configuration, $P_i(x_i, y_i, \theta_i)$ and a final configuration, $P_f(x_f, y_f, \theta_f)$, generating a feasible curve, $\mathbf{r}(t)$ is path planning. If the parametric form of $\mathbf{r}(t)$ is known, the configuration of the vehicle can be determined at any instant of its travel. But in real world, the path

planning has to satisfy constraints from the vehicle such as maximum bound on curvature and other imposed constraints such as mission objectives and shall cope with unforeseen environments. The section that follows traces the prior work on path planning of UAVs.

Prior Work

The path planning of multiple UAVs an active research area in recent times. The objectives and approaches may differ depending on the applications, such as surveillance, search and kill, rescue mission. The constraint on length of the path is one of the criteria for path planning. The shortest path provides the advantages on long durability, less fuel consumption and minimum time travel. A comprehensive review of types of paths used for path planning can be seen in [1]. The physical constraints of UAV and other imposed constraints such as threat avoidance, cooperation between other UAVs demand for an optimal path to achieve the mission objective. The coordinated trajectory generation for rendezvous application is dealt in [2–4]. An optimal path generation to avoid radar exposure is studied by [5]. The author describes the analytical and discrete optimization approaches with the constraint on the path length. The path planning problem is treated as a search problem in the partitioned cells in [6]. An overview of coordinated control of UAVs can be found in [7], where path planning using Voronoi diagrams is described.

In this paper, we focus on path planning of multiple UAVs. The mission objective is simultaneous arrival of UAVs to the target in an environment with restricted zones. Flying over these zones is not allowed. The UAVs have a maximum-bound on the curvature, κ_{max} or minimum turning radius, ρ_{min} .

The features of our solution approach are as follows.

1. Specific type of path is used for path planning, which eliminates the search of polygonal paths using Voronoi diagram. Shanmugavel et al [8] solved the simultaneous arrival using Dubins set of paths for a group of UAVs.
2. As a path of continuous curvature is used for path planning, path-smoothing step is eliminated.
3. The offset curves used for defining the safety margins of flight-path. The rational form of the offset curves provides the equal length of safe-distance all along the flight-path.

The paper is organized as follows. The next section explains the scenario adopted for the purposes of this paper. In the two sections that follow this, the properties of the Pythagorean Hodograph curve and our methodology for framing the path planning problem are explained. In the next section, our approach as well as implementation is discussed in detail. The penultimate section details the simulation results. The paper ends with conclusions and further work.

MISSION OBJECTIVE

Figure 2 shows the schematic of the mission plan. The mission objective is simultaneous arrival of UAVs to the target. The UAVs are leaving from the base at the same time. The time taken by each UAV to reach the target has to be same. The configurations at the base and at the target are defined for each UAV. The environment has some restricted zones and flying over these zones is prohibited. The location of these regions are known a priori. The UAVs are assumed flying at constant altitudes with constant speed.

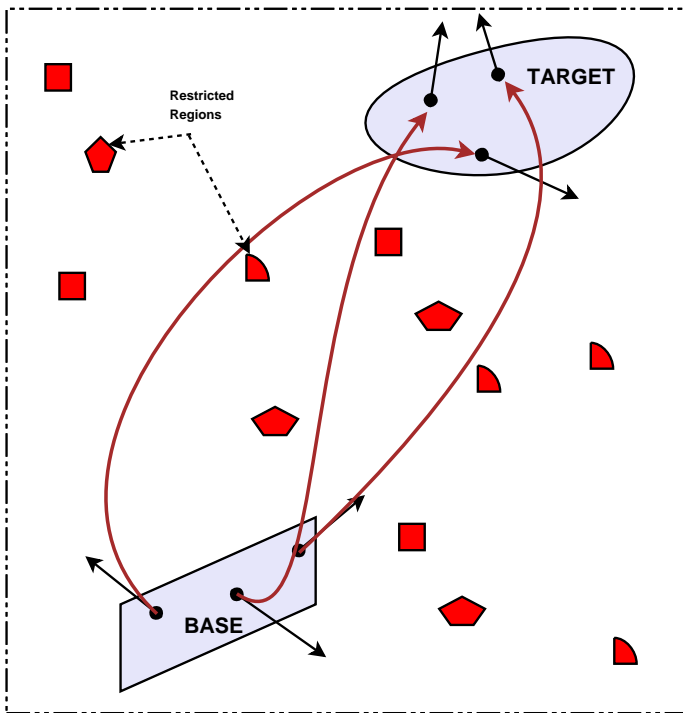


Figure 2. MISSION PLANNING

PYTHAGOREAN HODOGRAPH CURVE

The pythagorean hodograph (PH) curve is introduced by Farouki [9]. The detailed work can be found in the publications by Farouki et al ([10]). The speciality of a Pythagorean Hodograph curve is that its arc-length is a polynomial of its parameter and its curvature and offset curve are in rational form. This section describes a short introduction to Pythagorean Hodograph and its properties.

A Pythagorean Hodograph curve, $\mathbf{r}(t) = (x(t), y(t))$ is a polynomial curve whose tangents, $\dot{x}(t)$ and $\dot{y}(t)$ satisfies

$$\dot{x}^2(t) + \dot{y}^2(t) = \sigma^2(t) \quad (1)$$

for some polynomial, $\sigma(t)$. where $\dot{x}(t) = \frac{dx}{dt}$ and $\dot{y}(t) = \frac{dy}{dt}$.

From the principles of differential geometry, the path length of a parametric curve is given by

$$s = \int_{t_1}^{t_2} \|\dot{\mathbf{r}}(t)\| dt \quad (2)$$

$$= \int_{t_1}^{t_2} \sqrt{\dot{x}^2(t) + \dot{y}^2(t)} dt \quad t \in [t_1, t_2] \quad (3)$$

where, $\dot{x}(t)$ and $\dot{y}(t)$ are hodographs (tangents) of the curve $\mathbf{r}(t)$ and t is a parameter.

In Eqn. (2), if the sum of square of the tangents $\dot{x}(t)$ and $\dot{y}(t)$, could be represented by perfect square of a single polynomial, it leads to two advantages: 1. The radical form for calculating the path length is eliminated and 2. The parametric speed of the curve is a polynomial function of the parameter, t . This is achieved by forming the hodograph of the curve, $\mathbf{r}(t)$ of the form:

$$\dot{x}(t) = w(t)[u^2(t) - v^2(t)] \quad (4a)$$

$$\dot{y}(t) = 2w(t)u(t)v(t) \quad (4b)$$

Equation (1) becomes

$$\sqrt{\dot{x}^2(t) + \dot{y}^2(t)} = w(t)[u^2(t) + v^2(t)] = \sigma(t)$$

where $w(t)$, $u(t)$, $v(t)$ and $\sigma(t)$ are non-zero real polynomials, satisfying $\gcd(u(t), v(t)) = 1$.

Now, the Eqn. (2). is reduced to the simple integration of the polynomial

$$s = \int_{t_1}^{t_2} \sigma(t) dt \quad t \in [t_1, t_2] \quad (5)$$

Depending on order of the polynomials, $w(t)$, $u(t)$ and $v(t)$, the PH curve can be either cubic, quartic or quintic.

The expressions for unit tangent, \mathbf{T} , unit normal \mathbf{N} , and curvature κ of a PH curve are

$$\mathbf{T} = \frac{(u^2 - v^2, 2uv)}{(u^2 + v^2)} \quad (6)$$

$$\mathbf{N} = \frac{(2uv, v^2 - u^2)}{(u^2 + v^2)} \quad (7)$$

$$\kappa = \frac{2(u\dot{v} - \dot{u}v)}{w(u^2 + v^2)} \quad (8)$$

where $w = w(t)$, $u = u(t)$, $v = v(t)$, $\dot{u} = \frac{du(t)}{dt}$ and $\dot{v} = \frac{dv(t)}{dt}$.

The offset curve $\mathbf{r}_o(t)$ of the curve $\mathbf{r}(t)$ at a distance $\pm d$ is

$$\mathbf{r}_o(t) = \mathbf{r}(t) \pm d\mathbf{N}(t) \quad (9)$$

The unit tangent, \mathbf{T} , unit normal, \mathbf{N} , curvature, κ , and the offset path, $\mathbf{r}_o(t)$ are all of rational form.

Quintic PH Path for Path Planning

A quintic PH path is a fifth order polynomial curve, whose tangents satisfy the pythagorean condition (Eqn. (1)). The quintic PH is the lowest order curve in its class having inflection points ([9]). (Hereafter, PH curve or curve or path implies quintic PH curve). The following paragraphs explains the description of constructing the quintic PH path. The detailed derivation of the PH quintic curve can be found in papers by Farouki [9] and [10]. However, his work details the general formulation and solution of the curve.

For the purposes of path planning, we have had to move considerably from the general formulation towards a particular formulation. The initial starting position and pose as well as the approach position and angle are known. For the purposes of constructing the Bernstein polynomial [11], this is very effective. This leaves us with enough leeway in the form of two more control points that help us in manipulating path length and path shape. Intuitively, this is akin to tying an elastic band to drawing pins and stretching the middle of the band, once we have fixed two pins at the start and the end. In a more formal manner, this can be mathematically represented by using path planning terms, as follows:

A fifth order polynomial curve in Bernstein-Bézier form is

$$\mathbf{r}(t) = \sum_{k=0}^5 b_k \binom{5}{k} (1-t)^{5-k} t^k; \quad t \in [0, 1] \quad (10)$$

where $b_k = (x_k, y_k)$ are control points, whose vertices define the control polygon or Bézier polygon, t is a parameter and $k = 1 \dots 5$.

Let the initial and final configurations be (x_i, y_i, θ_i) and (x_f, y_f, θ_f) respectively. The four control points of the Bézier polygons are calculated by first order Hermite interpolation as follows:

$$b_0 = (x_i, y_i) \quad (11a)$$

$$b_5 = (x_f, y_f) \quad (11b)$$

$$d_0 = (\cos(\theta_i), \sin(\theta_i)) \quad (11c)$$

$$d_5 = (\cos(\theta_f), \sin(\theta_f)) \quad (11d)$$

$$b_1 = P_0 + (1/5) * d_0 \quad (11e)$$

$$b_4 = P_5 - (1/5) * d_5 \quad (11f)$$

where (x_i, y_i) is initial position, (x_f, y_f) is final position, θ_i is initial orientation and θ_f is final orientation. From Eqn.(11), the control points (b_0, b_1, b_4, b_5) are fixed. Now the problem is reduced to finding the control points, b_2 and b_3 .

The two polynomial curves $u(t)$ and $v(t)$ in Eqn. (4) are given by

$$w(t) = 1 \quad (12a)$$

$$u(t) = \sum_{j=0}^2 u_j \binom{2}{j} (1-t)^{(2-j)} t^j \\ = u_0(1-t)^2 + 2u_1(1-t)t + u_2t^2 \quad t \in [0, 1] \quad (12b)$$

$$v(t) = \sum_{j=0}^2 v_j \binom{2}{j} (1-t)^{(2-j)} t^j \\ = v_0(1-t)^2 + 2v_1(1-t)t + v_2t^2 \quad t \in [0, 1] \quad (12c)$$

where (u_0, u_1, u_2) and (v_1, v_2, v_3) are control points.

Substituting Eqn.(12) in

$$\int \binom{n}{k} (1-t)^{(n-k)} t^k dt = \frac{1}{n+1} \sum_{i=k+1}^{n+1} (1-t)^{(n+1-i)} t^i \quad (13)$$

and equating the coefficients in Eqn.(10) results in the set of equations to be solved for the control points b_2 and b_3 in terms of (u_0, u_1, u_2) and (v_1, v_2, v_3) . There are four solutions results in four PH paths. Among the four paths, only one path has acceptable shape. The path of acceptable shape is found out by calculating the bending energy of the paths. The bending energy

of the PH path is having a closed form solution [12] [12]. The bending energy ε of path is

$$\varepsilon = \int_{t_i}^{t_f} \kappa^2(t) \sigma(t) dt \quad (14)$$

The path which is having minimum bending energy is used for path planning. The path of minimum energy p_ε of q^{th} path is

$$p_\varepsilon = \min(\varepsilon_q), \quad q = 1, \dots, 4. \quad (15)$$

Figure 5 shows the paths of minimum energies for a given pair of configurations.

SOLUTION TO SIMULTANEOUS ARRIVAL

The path obtained from the Eqn. (15), are further refined to meet the curvature constraint and safety constraints. This involves following steps:

1. Meeting curvature constraint
2. Meeting safety constraints
3. Paths of equal length

MEETING CURVATURE CONSTRAINT

The path of minimum energy p_ε obtained from Eqn. (15) has acceptable shape, i.e., without any cusps or twists. However, the path does not meet the maximum curvature bound. The curvature of the path has to meet the maximum curvature bound κ_{max} of each UAV.

$$\kappa_i < \kappa_{max} \quad (16)$$

where κ_i is the curvature of i^{th} path and κ_{max} is the maximum curvature bound of the UAV.

The shape of a Bézier curve is determined by the locations of its control points. The positions of the control points decide the curvature of the path. Thus meeting the curvature constraint of the path is essentially relocation of its control points. The control points b_0 and b_5 are fixed by the boundary conditions (refer Eqn. (11)). The remaining control points b_1 , b_2 , b_3 and b_4 are to be relocated so that the path shall have curvature less than κ_{max} everywhere along the path. An iterative method adopted by Bruyninckx [13] is used in this paper as there is no closed form solution to calculate the control points meeting the bound on curvature. The curvature bound is met by varying the boundary tangent values iteratively. The method is described below.

The equations Eqn. (11c) and Eqn. (11d) are the boundary tangents. These equations are written in the following form.

$$d_0 = c_0(\cos(\theta_i), \sin(\theta_i)) \quad (17a)$$

$$d_5 = c_5(\cos(\theta_f), \sin(\theta_f)) \quad (17b)$$

where c_0 and c_5 are numerical constants which are used to vary the tangent values at the end points of the path.

The change in c_0 and c_5 in Eqn. (17a) and Eqn. (17b), relocate the the control points b_1 and b_4 in the equations Eqn. (11e) and Eqn. (11e) respectively. As the ratio of components of the tangents are maintained, the initial and final orientations are preserved. The remaining two control points b_2 and b_3 are calculated by satisfying the PH condition in Eqn. (4). The values of c_0 and c_5 are find out by iteratively until the path meets the curvature bound. Figure 6 shows the paths meeting the curvature constraint.

The curvature optimization ensure that the curvature variation of path is within the curvature limit of each UAV. But it does not guarantee the flying over the restricted zones. Hence these paths further to be refined to meet the safety constraints.

MEETING SAFETY CONSTRAINTS

The safety of the UAVs are measured by how far they fly away from the restricted zones. This can be measured and verified by two methods:

1. Measuring the distance from any point of the curve to the boundary of the restricted zones. This can be measured along normal vectors, \mathbf{N} of the curve, $\mathbf{r}(t)$ (Refer Fig. 3). This method is best suited to the real time or online path planning. Because, this method requires the distance from the curve to the boundary is to be calculated for every point on the curve. Triangulation method is preferred for this calculation [14].
2. Non-intersection of the path with the boundary of the restricted zones, (Fig. 4). This method is chosen for our application. The safe flight path is ensured by non-intersection of the path with the boundary. This is verified by empty set of intersection of points of the offset-path with the boundary of the restricted zones. The intersection is checked numerically. If the non-empty set results, the values c_0 and c_5 are adjusted until the path meets the safety constraints.

The safe flight paths of UAVs is calculated by maintaining the safety margin of each UAV. The safety margin, d_s is the safe distance of the UAV from the restricted zone. The minimum safety margin must be greater than the minimum turning radius of each UAV, ρ_{UAV} .

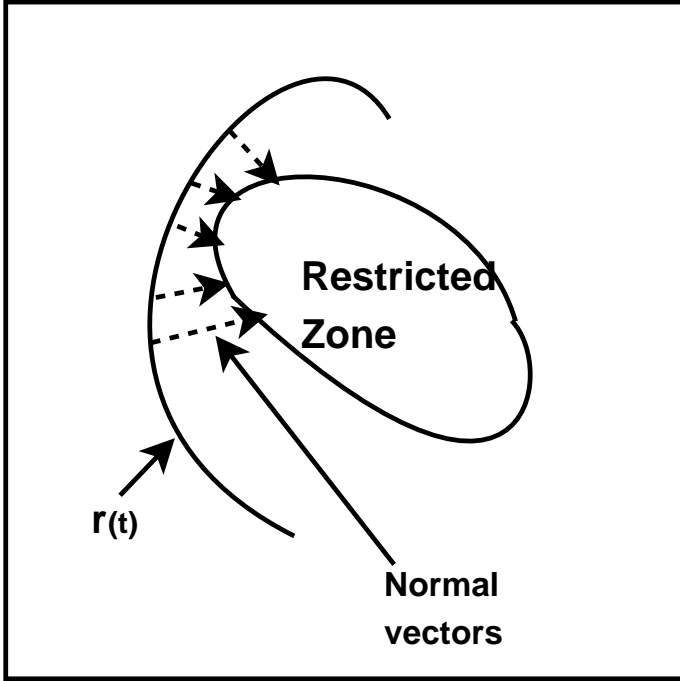


Figure 3. CALCULATION OF SAFETY MARGIN USING NORMAL VECTORS

$$d_s > \rho_{UAV} \quad (18)$$

The measure of safety-margin of UAVs is quantified by offset paths of the flight path. The offset paths can also be used to define the size of the UAVs and flight formation. The procedure for calculating the measure of safety-margin is as follows.

1. Generate the offset curves \mathbf{r}_o of the flight path with offset distance greater than the minimum turning radius of the UAVs. Eqn. (9) is used to generate the safety margin for the UAVs along their path. The offset curves in their component form is:

$$x_o(t) = x(t) + \frac{(\pm d_s) \frac{dy}{dt}}{\sigma(t)} \quad (19)$$

$$y_o(t) = y(t) - \frac{(\pm d_s) \frac{dx}{dt}}{\sigma(t)} \quad (20)$$

2. Check for intersection of offset curves with the boundaries of the restricted zones.
3. In the case of intersection adjust the values of c_0 and c_5 to meet the safety constraints.

In simulation, c_0 and c_5 are iteratively varied until the paths meet the curvature and safety constraints simultaneously.

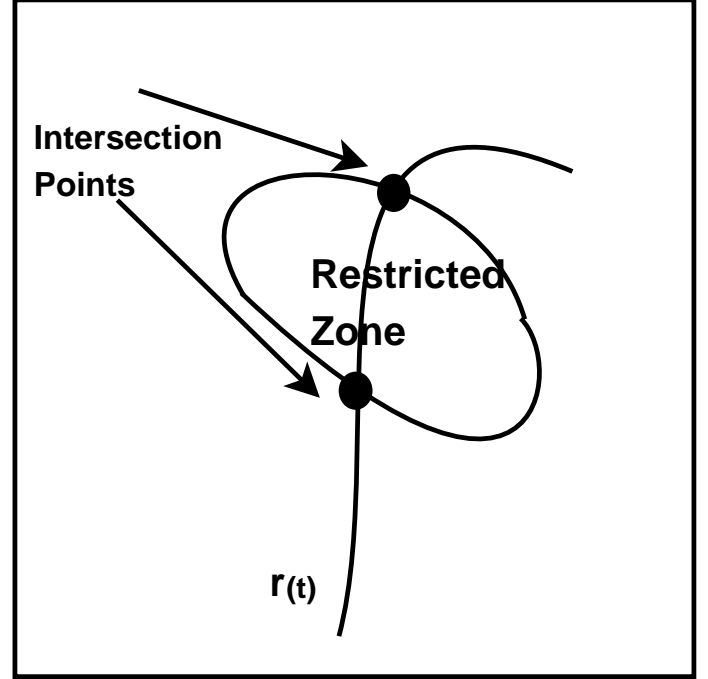


Figure 4. CALCULATION OF SAFETY MARGIN BY INTERSECTION METHOD

PATHS OF EQUAL LENGTH

The generation of paths of equal length involves two steps:

1. Identification of reference path (path of maximum length) and
2. Increasing the lengths of other paths equal to that of the reference path. The length of the paths are calculated from algorithm provided in [12] [12]. The reference path, \mathbf{r}_{ref} is the longest of all paths of UAVs. The length of reference path is:

$$s_{ref} = \max(s_i), \quad i = 1 \dots N \quad (21)$$

where N is the number of UAVs.

The lengths of shorter paths are increased to be equal to that of the reference path by adjusting the values of c_0 and c_5 . Thus all the paths are of equal in length meeting the curvature constraint and safety constraint.

SIMULATIONS

Two UAVs are considered for simulation. The UAVs are assumed to be homogenous in their physical capabilities. The

UAVs are flying at constant speeds at constant altitudes. Figure ?? shows the PH paths of the UAVs prior to curvature optimization. P_i and P_f are the initial and final configuration respectively. The patches observed on the figure are the restricted zones. It is seen that the UAVs are flying over the restricted zones. The crossing of the paths on the tangent circles (whose radius is taken as 3 units) shows that the paths are not meeting the constraint of the maximum curvature bound. Thus it is required that the paths are to be optimized for safe flight path following optimization for their curvature. The path lengths of UAV1 is: 35.96 units and that of UAV2 is 35.92 units.

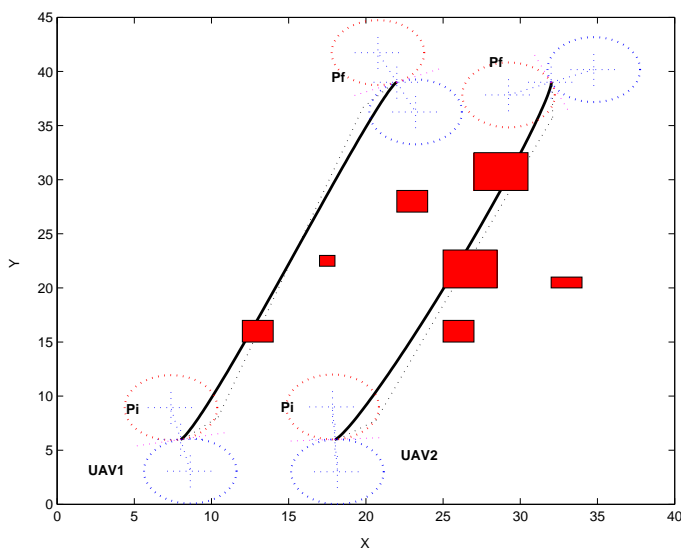


Figure 5. PATHS OF UAVs PRIOR TO CURVATURE OPTIMIZATION

Figure 6 shows the UAV paths optimized for their curvatures. The tangent circles are not crossed by the paths. However, the paths do not satisfy the safe flight path with minimum safety margin greater than 3 units. The path of UAV1 is directly passing over the restricted zone and the path of UAV2 is not meeting the minimum safety margin. Both the paths need further change in their curvature and in turn their lengths. The path lengths are: UAV1 40.69 units and UAV2 37.13 units.

Figure 7 shows that each UAV is provided with the safety margin. The offset curves (dashed lines) with an offset distance of ± 3.01 unit ensure the safety of UAVs. The path-length of UAV1 is 42.57 units and that of UAV2 is 41.12 units. The paths are not of equal lengths. The path-length of UAV1 is greater than that of UAV2. So, path of UAV1 is the reference path. The path length of UAV2 has to be increased to that of UAV1 for generating paths of equal length for simultaneous arrival. Figure 8 shows the paths of UAV1 and UAV2 having equal path length of 42.57 units. Thus, achieving the mission objective of simultane-

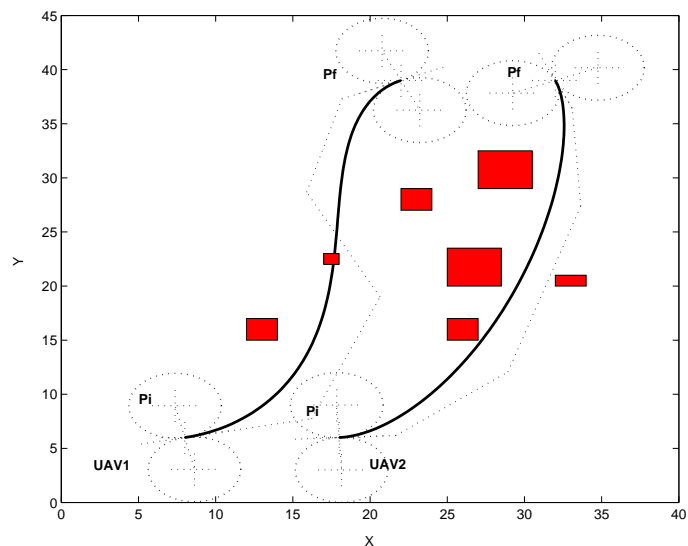


Figure 6. PATHS OF UAVs AFTER CURVATURE OPTIMIZATION

ous arrival to the target in an environment with restricted zones.

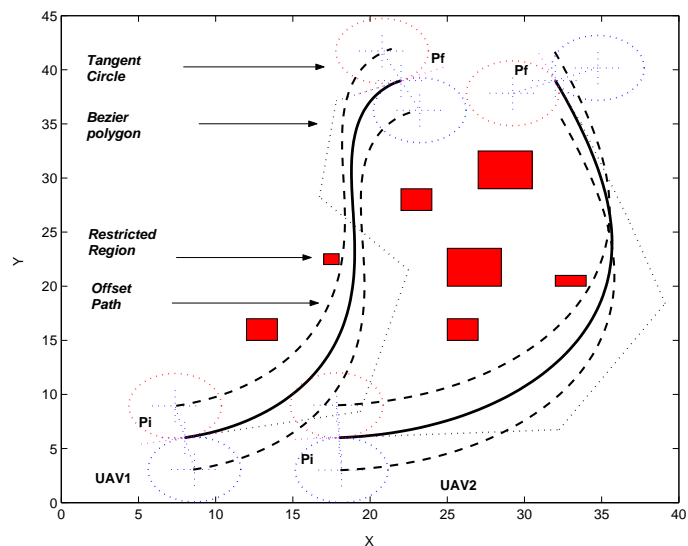


Figure 7. PATHS OPTIMIZED FOR SAFE FLIGHT PATH

CONCLUDING REMARKS

The path planning solution to simultaneous arrival of multiple UAVs in an environment of restricted zones is discussed. A specific type of path (Pythagorean Hodograph) is used for the path planning. This can be used to predict the configurations

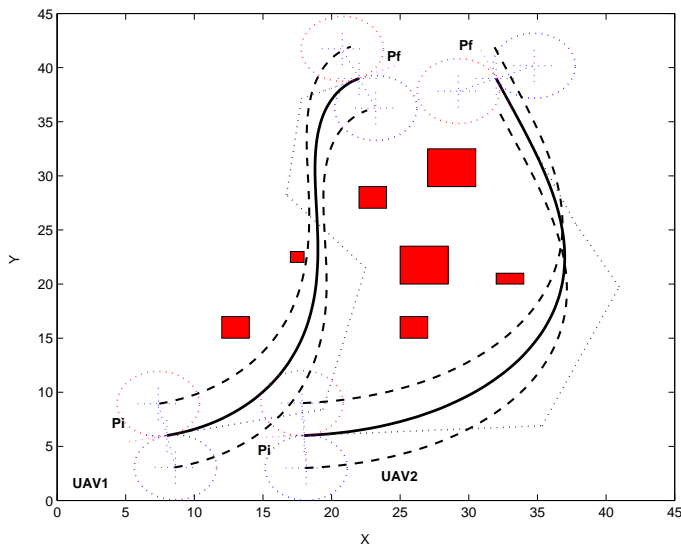


Figure 8. PATHS MEETING MISSION OBJECTIVE

of UAVs. The path planning is defined for a pair of configurations and this eliminates the polygonal path approximation. The curvature-continuity of the PH curve eliminates the smoothing of the curves for flyable paths. The rational offset curves of the PH path defines the safety margin. The paths are made equal by adjusting the lengths of the paths to the longest path in the set of curves. The ideas and concepts are demonstrated by numerical simulations.

FUTURE WORK

It is intended to extend the work as follows: 1. Controlling the length of the path with respect to its intrinsic properties, curvature and torsion (in three dimension), 2. Developing algorithm for online path planning using the PH curve, and, 3. Developing algorithms for measuring distance between the PH paths and an arbitrary curve.

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