Statistics of level crossing intervals

Nobuko Fuchikami[a] and Shunya Ishioka[b]

[a]Department of Physics, Tokyo Metropolitan University
Minami-Ohsawa, Hachioji, Tokyo 192-0397, Japan;
[b]Department of Information Science, Kanagawa University at Hiratsuka
Hiratsuka, Kanagawa 259-1293, Japan

ABSTRACT

We present an analytic relation between the correlation function of dichotomous (taking two values, ± 1) noise and the probability density function (PDF) of the zero crossing interval. The relation is exact if the values of the zero crossing interval τ are uncorrelated. It is proved that when the PDF has an asymptotic form $L(\tau) = 1/\tau^c$, the power spectrum density (PSD) of the dichotomous noise becomes $S(f) = 1/f^\beta$ where $\beta = 3 - c$. On the other hand it has recently been found that the PSD of the dichotomous transform of Gaussian $1/f^\alpha$ noise has the form $1/f^\beta$ with the exponent β given by $\beta = \alpha$ for $0 < \alpha < 1$ and $\beta = (\alpha + 1)/2$ for $1 < \alpha < 2$. Noting that the zero crossing interval of any time series is equal to that of its dichotomous transform, we conclude that the PDF of level-crossing intervals of Gaussian $1/f^\alpha$ noise should be given by $L(\tau) = 1/\tau^c$, where $c = 3 - \alpha$ for $0 < \alpha < 1$ and $c = (5 - \alpha)/2$ for $1 < \alpha < 2$. Recent experimental results seem to agree with the present theory when the exponent α is in the range $0.7 \lesssim \alpha < 2$ but disagrees for $0 < \alpha \lesssim 0.7$. The disagreement between the analytic and the numerical results will be discussed.

Keywords: Power law, $1/f^\alpha$ noise, probability density function, zero crossing interval, dichotomous noise.

1. INTRODUCTION

Statistcal characterization of the level crossing of stochastic variable is in general a difficult problem.[1, 2] Although the study dates back to 1940's, there are few rigorius solutions on this subject presented until now. So far, the average density of the level crossing time point t_n has been obtained, but the probability density function (PDF) of the level crossing interval: $\tau_n \equiv t_n - t_{n-1}$ is still unknown.

In this paper, we consider the simplest case of the level crossing, i.e., the zero crossing statistics. As far as we know, except rather simple extreme cases, i.e., white noise and $1/f^2$ noise (random work), any example has not been found in which the PDF of the zero crossing interval can be obtained explicitly.[6]

We shall take a new approach to the problem, which will reveal an essential relation between the PDF of the zero crossing interval and the power spectrum (or equivalently the correlation function) of the noise: If we assume

1)the noise is stationary Gaussian with zero mean,

2) the zero crossing intervals are uncorrelated with each other,

then the PDF of the zero crossing interval is expressed in terms of the correlation functon.

This holds in general and if applied to $1/f^\alpha$ noise, the PDF can be given in an explicit form based on our recent study in which it was proved that the power spectrum density (PSD) of the dichotomous transform of Gaussian $1/f^\alpha$ noise is of power law type: $\sim 1/f^\beta$ and β is obtained from α.[3–5]

Recently, Mingesz et al. observed the PDF $L(\tau)$ of the zero crossing interval τ for Gaussian $1/f^\alpha$ noise and obtained α-dependence of $L(\tau)$.[6] In this paper, we shall show analytically that $L(\tau)$ obeys the power law as $\sim 1/\tau^c$ for the Gaussian $1/f^\alpha$ noise, and present α-dependence of the exponent c. Our result will be compared with Mingesz et al.'s experiments.

Further author information: (Send correspondence to N.F.)
N.F.: E-mail: fuchi@kuramae.ne.jp
S.I.: E-mail: ishioka@info.kanagawa-u.ac.jp

2. DICHOTOMOUS TRANSFORMATION

Let us define the dichotomous transformation of stochastic time series data $x(t)$ by

$$y(t) = \mathrm{sgn}\,(x(t)) \equiv \begin{cases} 1 & \text{for } x(t) \geq 0 \\ -1 & \text{for } x(t) < 0 \end{cases}. \tag{1}$$

It is only assumed in this section that $x(t)$ is stationary and symmetrically distributed, although Gaussian $1/f^{\alpha}$ noise will be considered later. Since the dichotomous transform $y(t)$ is also stationary, the correlation function of $y(t)$ is given by

$$\varphi_y(\tau) \equiv \langle y(t)y(t+\tau)\rangle = \langle y(0)y(\tau)\rangle = \mathrm{Prob}\,\{y(0)y(\tau) > 0\} - \mathrm{Prob}\,\{y(0)y(\tau) < 0\}\,. \tag{2}$$

Let us denote by $P_n(\tau)$ the probability that n zero crossings of $x(t)$ occur during the time interval between $t = 0$ and τ, where

$$\sum_{n=0}^{\infty} P_n(\tau) = 1. \tag{3}$$

Because the zero crossing interval of $x(t)$ is equal to the interval during which the dichotomous transform $y(t)$ keeps the same value 1 or -1, the correlation function in Eq. (2) can be expressed in terms of $P_n(\tau)$ as

$$\varphi_y(\tau) = \sum_{n=0}^{\infty} (-1)^n P_n(\tau). \tag{4}$$

We assume in the present paper that the time series of the zero crossing intervals, $\tau_1, \tau_2, \cdots$ is independent identically distributed (i.i.d.).* Then an analytic relation between the PDF $L(\tau)$ of the zero crossing interval τ and the correlation function $\varphi_y(\tau)$ will be obtained.

Suppose we start counting the zero crossing number from $t = 0$, and let $I_n(t)dt$ ($n \geq 1$) be the probability that the nth zero crossing occurs in the time interval between t and $t + dt$. For $n \geq 2$, $I_n(t)$ satisfies the following recursion formula:

$$I_n(t) = \int_0^t I_{n-1}(t')L(t-t')dt', \qquad n \geq 2. \tag{5}$$

This formula assumes that the zero crossing intervals are uncorrelated. $I_1(t)$ is obtained as

$$I_1(t) = \frac{1}{\langle\tau\rangle}\int_t^{\infty} L(\tau)d\tau, \tag{6}$$

$$\langle\tau\rangle \equiv \int_0^{\infty} \tau L(\tau)d\tau, \tag{7}$$

which is shown in Appendix A.

Let us define the Laplace transform of function $F(t)$ by

$$\mathcal{L}(F)[\lambda] \equiv F[\lambda] = \int_0^{\infty} F(t)e^{-\lambda t}dt. \tag{8}$$

The Laplace transformation of Eqs. (5) and (6) yields

$$I_n[\lambda] = L[\lambda]I_{n-1}[\lambda], \qquad n \geq 2, \tag{9}$$

$$I_1[\lambda] = \frac{1}{\langle\tau\rangle\,\lambda}\left(1 - L[\lambda]\right), \tag{10}$$

*This does not necessary mean that $x(t)$ is i.i.d. We will assume in the next section that $x(t)$ is correlated Gaussian with a power law spectrum.

and then

$$I_n[\lambda] = L[\lambda]^{n-1} I_1[\lambda] = \frac{1}{\langle\tau\rangle\,\lambda} L[\lambda]^{n-1} \left(1 - L[\lambda]\right), \qquad n \geq 1. \tag{11}$$

The probability that the number of zero crossing during the interval between $t = 0$ and τ is equal to or more than n is given by

$$\int_0^\tau I_n(t')dt' = \sum_{j=n}^{\infty} P_j(\tau) \qquad \text{for} \quad n \geq 1, \tag{12}$$

which leads to

$$P_n(\tau) = \int_0^\tau I_n(t')dt' - \int_0^\tau I_{n+1}(t')dt', \qquad n \geq 1. \tag{13}$$

For $n = 0$, we obtain

$$P_0(\tau) = 1 - \int_0^\tau I_1(t')dt'. \tag{14}$$

The Laplace transforms of Eqs. (13) and (14) are calculated as

$$P_n[\lambda] = \frac{1}{\lambda}\left(I_n[\lambda] - I_{n+1}[\lambda]\right) = \frac{1}{\langle\tau\rangle\,\lambda^2} L[\lambda]^{n-1}\left(1 - L[\lambda]\right)^2 \quad \text{for} \quad n \geq 1, \tag{15}$$

and

$$P_0[\lambda] = \frac{1}{\lambda}\left(1 - I_1[\lambda]\right) = \frac{1}{\lambda} - \frac{1}{\langle\tau\rangle\,\lambda^2}\left(1 - L[\lambda]\right). \tag{16}$$

Substitution of Eqs. (15) and (16) into the Laplace transform of Eq. (4) results in

$$\varphi_y[\lambda] = \frac{1}{\lambda} - \frac{2}{\langle\tau\rangle\,\lambda^2}\frac{1 - L[\lambda]}{1 + L[\lambda]}. \tag{17}$$

The above equation gives the relation between the PDF of the zero crossing interval of $x(t)$ and the correlation function of the dichotomous transform $y(t)$. If $L(\tau)$ is given, using $L[\lambda]$ together with Eq. (7), we can obtain $\varphi_y[\lambda]$ and so $\varphi_y(\tau)$. Inversely, if $\varphi_y(\tau)$ is known, we should calculate

$$L[\lambda] = \frac{1 - \frac{\langle\tau\rangle}{2}\lambda\left(1 - \lambda\varphi_y[\lambda]\right)}{1 + \frac{\langle\tau\rangle}{2}\lambda\left(1 - \lambda\varphi_y[\lambda]\right)} \tag{18}$$

to obtain $L(\tau)$. Note that Eq. (7) cannnot be used for $\langle\tau\rangle$ because $L(\tau)$ is unknown. In fact, it can be shown in Apendix B that $\langle\tau\rangle$ is available from $\varphi_y(\tau)$.[†]

3. ZERO CROSSING OF THE NOISE WITH POWER LAW SPECTRUM

It will turn out that the PDF $L(\tau)$ of Gaussian $1/f^\alpha$ noise exhibits the power-law dependence $\sim 1/\tau^c$ in a range $\tau \gg 1$. We are going to derive the relation between two exponents α and c in this section.

Since our interest is in the asymptotic form of $L(\tau)$, we may assume $\lambda \ll 1$ in its Laplace transform $L[\lambda]$, which corresponds to $\tau \gg 1$. Noting that the Laplace transform satisfies $L[0] = \int_0^\infty L(\tau)d\tau = 1$ and defining

$$l[\lambda] \equiv 1 - L[\lambda], \tag{19}$$

[†]Equation (18), yielding $L(\tau)$ expressed in terms of the dichotomous transform, only assumes that $x(t)$ is stationary and symmetrically distributed. It is shown in Appendix B that a further assumption that $x(t)$ is Gaussian makes it possible to express $L(\tau)$ directly, i.e., in terms of the correlation function of $x(t)$ itself.

we can approximate $l(\lambda) \ll 1$ when $\lambda \ll 1$. Substituting Eq. (19) into Eq. (17), the Laplace transform of the correlation function $\varphi_y(\tau)$ is obtained as

$$\varphi_y[\lambda] = \frac{1}{\lambda} - \frac{2}{\langle\tau\rangle \lambda^2} \frac{l[\lambda]}{2 - l[\lambda]} = \frac{1}{\lambda} - \frac{l[\lambda]}{\langle\tau\rangle \lambda^2} \left(1 + \frac{l[\lambda]}{2} + \cdots\right), \tag{20}$$

$$\sim \frac{1}{\lambda} - \frac{l[\lambda]}{\langle\tau\rangle \lambda^2} \quad \text{for } \lambda \ll 1. \tag{21}$$

We consider two cases:

Case A) When $2 < c < 3$,

$$L(\tau) = \frac{A}{(1+\tau)^c}, \qquad A = c - 1, \tag{22}$$

$$\sim \frac{1}{\tau^c} \qquad \text{for } \tau \gg 1. \tag{23}$$

Equation (22) leads to the average of τ as

$$\langle\tau\rangle = \int_0^\infty \tau L(\tau) d\tau = \frac{1}{c-2}. \tag{24}$$

Case B) When $1 < c < 2$,

$$L(\tau) = \frac{Be^{-\delta\tau}}{(1+\tau)^c}, \qquad B = \frac{c-1}{1 - e^{\delta}\delta^{c-1}\Gamma(2-c,\delta)}, \tag{25}$$

$$\sim \frac{1}{\tau^c} \quad \text{for } 1/\delta \gg \tau \gg 1, \tag{26}$$

where $\Gamma(z,p)$ is the incomplete gamma function:

$$\Gamma(z,p) \equiv \Gamma(z) - \int_0^p e^{-x} x^{z-1} dx, \qquad z > 0, \tag{27}$$

$$\Gamma(z) = \Gamma(z,0) = \int_0^\infty e^{-x} x^{z-1} dx, \qquad z > 0. \tag{28}$$

The convergence factor $e^{-\delta\tau}$ with a small positive value of δ is necessary so that the average of τ has a finite value:

$$\langle\tau\rangle = \frac{(c+\delta-1)e^{\delta}\delta^{-(2-c)}\Gamma(2-c,\delta) - 1}{1 - e^{\delta}\delta^{c-1}\Gamma(2-c,\delta)}. \tag{29}$$

We do not consider the case of $c < 1$, because the integral $\int_0^\infty L(\tau)d\tau$ does not converge.

Case A)

The Laplace transform of Eq. (22) is given by ‡

$$L[\lambda] = 1 - \frac{\lambda}{c-2} + \frac{e^{\lambda}\lambda^{c-1}}{c-2}\Gamma(3-c,\lambda), \qquad 3 > c > 2, \tag{30}$$

‡The expression including incomplete gamma functions is not unique because $\Gamma(z+1,p)$ can be expressed in terms of $\Gamma(z,p)$ for $z > 0$. Here, $\Gamma(z,p)$ is chosen such that the argument z is the smallest positive number.

which is derived in Appendix C. Substituting Eqs. (19), (24) and (30) into the lowest order approximation of $\varphi_y[\lambda]$: Eq. (21), we obtain

$$\varphi_y[\lambda] = e^{\lambda}\lambda^{c-3}\Gamma(3-c,\lambda), \qquad 3 > c > 2, \tag{31}$$

which yields the inverse Laplace transform as

$$\begin{aligned} \varphi_y(\tau) &= \mathcal{L}^{-1}\left(e^{\lambda}\lambda^{c-3}\Gamma(3-c,\lambda)\right)(\tau) = \frac{1}{(1+\tau)^{c-2}}, \qquad 3 > c > 2, && (32)\\ &\sim \frac{1}{\tau^{c-2}} \equiv \frac{1}{\tau^{b}} \quad \text{for } \tau \gg 1, && (33) \end{aligned}$$

where

$$b = c-2, \qquad 0 < b < 1. \tag{34}$$

Case B)

The Laplace transform of Eq. (25) is given by

$$L[\lambda] = \frac{1-e^{\lambda+\delta}(\lambda+\delta)^{c-1}\Gamma(2-c,\lambda+\delta)}{1-e^{\delta}\delta^{c-1}\Gamma(2-c,\delta)}, \qquad 2 > c > 1. \tag{35}$$

Substituting Eq. (35) into (19) and assuming $1 \gg \lambda \gg \delta$ (which corresponds to $1 \ll \tau \ll 1/\delta$ in τ-space), we obtain

$$\begin{aligned} l[\lambda] &= \frac{e^{\lambda+\delta}(\lambda+\delta)^{c-1}\Gamma(2-c,\lambda+\delta)-e^{\delta}\delta^{c-1}\Gamma(2-c,\delta)}{1-e^{\delta}\delta^{c-1}\Gamma(2-c,\delta)}, \qquad 2 > c > 1, && (36)\\ &\sim \frac{e^{\lambda}\lambda^{c-1}\Gamma(2-c,\lambda)}{1-e^{\delta}\delta^{c-1}\Gamma(2-c,\delta)} \quad \text{for } 1 \gg \lambda \gg \delta, \qquad 2 > c > 1. && (37) \end{aligned}$$

Substitution of Eq. (37) into Eq. (21) yields

$$\varphi_y[\lambda] = \frac{1}{\lambda} - \frac{1}{\langle\tau\rangle}\frac{e^{\lambda}\lambda^{-(3-c)}\Gamma(2-c,\lambda)}{1-e^{\delta}\delta^{c-1}\Gamma(2-c,\delta)}, \qquad 2 > c > 1, \tag{38}$$

and its inverse Laplace transform is calculated as

$$\varphi_y(\tau) = \mathcal{L}^{-1}\left(\frac{1}{\lambda}\right)(\tau) - \frac{1}{\langle\tau\rangle}\frac{1}{1-e^{\delta}\delta^{(c-1)}\Gamma(2-c,\delta)}\mathcal{L}^{-1}\left(e^{\lambda}\lambda^{c-3}\Gamma(2-c,\lambda)\right)(\tau) \qquad 2 > c > 1. \tag{39}$$

which has a final form

$$\begin{aligned} \varphi_y(\tau) &= 1 + B'\left(1-(1+\tau)^{2-c}\right), \qquad 2 > c > 1, && (40)\\ &\sim 1 - B'\tau^{2-c} \equiv 1 - \frac{B'}{\tau^{b}} \quad \text{for } 1 \ll \tau \ll 1/\delta, && (41) \end{aligned}$$

with

$$b = c-2, \qquad -1 < b < 0, \tag{42}$$

and

$$\begin{aligned} B' &= \frac{1}{2-c}\frac{1}{(c+\delta-1)e^{\delta}\delta^{-(2-c)}\Gamma(2-c,\delta)-1}, \qquad 2 > c > 1, && (43)\\ &\sim \frac{\delta^{2-c}}{(c-1)(2-c)\Gamma(2-c)} \qquad 2 > c > 1, && (44) \end{aligned}$$

where Eq. (29) has been substituted.§ Equations (34) and (42) show that the exponent b of the correlation function of the dichotomous transform relates to the exponent c of the PDF of the zero crossing interval as

$$b = c - 2 \quad \text{for } 1 < c < 3. \tag{45}$$

When the correlation function obeys the power law like Eqs. (34) or (42), the corresponding power spectrum also obeys the power law : $\sim 1/f^{\beta}$ and the two exponents relate to each other as[7]

$$b = 1 - \beta \quad \text{for } 0 < \beta < 2. \tag{46}$$

Equations (45) and (46) yield

$$c = b + 2 = 3 - \beta \quad \text{for } 0 < \beta < 2. \tag{47}$$

This is the relation between the PDF of the zero crossing interval of $x(t)$ and the power spectrum of its dichotomous transform $y(t)$.

Now, let us assume that $x(t)$ is Gaussian $1/f^{\alpha}$ noise, i.e., the power spectrum is $S_x(f) \sim 1/f^{\alpha}$. It has been proved[3] that the power spectrum of its dichotomous transform $y(t)$ defined by Eq. (1) is also power-law type: $S_y(f) \sim 1/f^{\beta}$. The relation between these two exponents is

$$\beta = \begin{cases} \alpha, & 0 < \alpha < 1 \\ \frac{\alpha+1}{2}, & 1 < \alpha < 2 \end{cases} . \tag{48}$$

Equations (47) and (48) yield the final result which we want:

$$c = \begin{cases} 3 - \alpha, & 0 < \alpha < 1 \\ \frac{5-\alpha}{2}, & 1 < \alpha < 2 \end{cases} . \tag{49}$$

4. SUMMARY AND DISCUSSIONS

We obtained the asymptotic form for the distribution of the zero crossing interval of Gaussian $1/f^{\alpha}$ noise, by assuming that the time series of the zero crossing intervals is i.i.d. The relation among the varous exponents in the power law behavior is presented in Fig. 1.

It should be noted that the result derived in the previous section does not include the case of white Gaussian noise: $\alpha = 0$, which is a singular point in the present treatment and should be considered separately.

When $x(t)$ is white Gaussian, the spectrum of the dichotomous transform $y(t)$ is also white: $\beta = \alpha = 0$. The correlation function of $y(t)$ can be given by a Debye type relaxation function with a short relaxation time τ_0:

$$\varphi_y(\tau) = e^{-|\tau|/\tau_0}. \tag{50}$$

The PSD of $y(t)$ becomes white in the limit of $\tau_0 \to 0$:

$$S_y(f) = \int_0^{\infty} e^{-\tau/\tau_0} \cos(2\pi f \tau) d\tau \propto \frac{1}{1 + (2\pi f \tau_0)^2} \sim \text{const.} \quad \text{for } f\tau_0 \ll 1. \tag{51}$$

Substituting the Laplace transform of Eq. (50): $\varphi_y[\lambda] = 1/(\lambda + \tau_0)$ into Eq. (17), we obtain $L[\lambda]$ and its inverse Laplace transform as

$$L(\tau) = \frac{1}{2\tau_0} e^{-\tau/2\tau_0}. \tag{52}$$

The limit $\alpha \to 0$ in $S_x(f) \sim 1/f^{\alpha}$ corresponds to $\beta \to 0$ in $S_y(f) \sim 1/f^{\beta}$. In the previous section, it was shown that this limit corresponds to $b \to 1$ in $\varphi_y(\tau) \sim 1/\tau^b$, and $c \to 3$ in $L(\tau) \sim 1/\tau^c$, which does not tend to the above exponential form.

§The expressions (40) and (41) are correct only when $\varphi_y(\tau) > 0$. This is guaranteed for $1 \gg \tau \gg 1/\delta$ and this range can be large if the value of δ is chosen small enough so that B' is small.

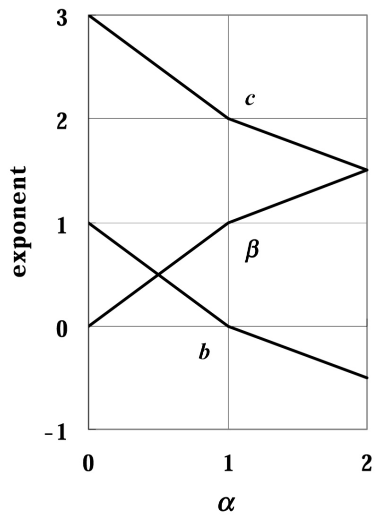

Figure 1. Relation among the vatious expoments in power law behavior. When $S_x(f) \sim 1/f^\alpha$, we obtain $S_y(f) \sim 1/f^\beta$, $\varphi_y(\tau) \sim 1/\tau^b$ (for $0 < b < 1$) or $\varphi_y(\tau) \sim 1 - B'/\tau^b$ (for $1 < b < 2$), and $L(\tau) \sim 1/\tau^c$.

The result can be compared with the recent experiment by Mingesz et al.[6] The experimental results seem to agree with our analytic relation: Eq. (49) for $0.7 \lesssim \alpha < 2$, but not for $0 < \alpha \lesssim 0.7$. When $\alpha = 0$, Eq. (52) is applicable to the experimental results.

As mentioned already, the present theory has assumed that the zero crossing intervals are uncorrelated, on which one might blame the origin of the discrepancy between the theory and experiment. However, the correlation of τ is reported to be rather small[6] especially for $\alpha \ll 1$ and $2 - \alpha \ll 1$. Moreover the correlation is of the same order in both ranges $0 < \alpha \lesssim 0.7$ and $0.7 \lesssim \alpha < 2$, in the latter the agreement is good. We therefore cannot conclude that the disagreement arises from the correlation. To invoke another candidate, we should remember[4] that "$1/f^\alpha$" noise cannot be of $1/f^\alpha$ type in the whole range of f. In other words, there should be a high frequency cut off for $0 < \alpha < 1$. We are thus suggested that the comparison between the theory and experiment should be done by taking account of this point.

APPENDIX A. DERIVATION OF EQ. (6)

$I_1(t)$ is obtained as follows: Let $P_L(\tau)d\tau$ be the probability that the length of the time interval which includes a specified point, say, $t = 0$ is in between τ and $\tau + d\tau$. This probability is proportional to the length τ of the interval and to $L(\tau)d\tau$ which is the probability that the length of the interval is between τ and $\tau + d\tau$, and is given as

$$P_L(\tau)d\tau = \frac{\tau}{\langle\tau\rangle} L(\tau)d\tau, \tag{53}$$

where

$$\langle\tau\rangle \equiv \int_0^\infty \tau L(\tau)d\tau, \tag{54}$$

so that

$$\int_0^\infty P_L(\tau)d\tau = 1. \tag{55}$$

Hence the probability that the first zero crossing occurs between t and $t + dt$ is

$$I_1(t)dt = \int_t^\infty \frac{dt}{\tau} P_L(\tau)d\tau = \frac{1}{\langle\tau\rangle} \int_t^\infty L(\tau)d\tau dt, \tag{56}$$

and we obtain

$$I_1(t) = \frac{1}{\langle\tau\rangle}\int_t^\infty L(\tau)d\tau. \tag{57}$$

APPENDIX B. PDF OF ZERO CROSSING INTERVALS FOR GAUSSIAN NOISE

In Sect. 2, the correlation function of dichotomous noise $\varphi_y(\tau)$ was derived when the PDF of the zero crossing interval $L(\tau)$ is given. Our subject is, however, the inverse problem of it. Here, we show how to obtain $L(\tau)$ from the correlation function of the original noise $x(t)$ by assuming that $x(t)$ is stationary Gaussian. Remember that the correlation function

$$\varphi_x(\tau) \equiv \langle x(t)x(t+\tau)\rangle = \langle x(0)x(\tau)\rangle \tag{58}$$

fully characterizes the statistics of $x(t)$ because $x(t)$ is Gaussian, i.e., the information of the zero crossing statistics should be derived only from $\varphi_x(\tau)$.

The correlation function of the dichotomous transform $y(t) \equiv \mathrm{sgn}\,(x(t))$ is given by[3,4]

$$\varphi_y(\tau) = \frac{2}{\pi}\arcsin\left[\frac{\varphi_x(\tau)}{\varphi_x(0)}\right]. \tag{59}$$

Hence the Laplace transform in Eq. (18) is obtained as

$$\varphi_y[\lambda] = \int_0^\infty e^{-\lambda\tau}\frac{2}{\pi}\arcsin\left[\frac{\varphi_x(\tau)}{\varphi_x(0)}\right]d\tau. \tag{60}$$

The quantity $\langle\tau\rangle$: the average interval of zero crossing defined by Eq. (7) is estimated as follows: First we note that $1/\langle\tau\rangle$ is the average rate of the zero crossing. Consider $P_n(\tau)$, the probability that n zero crossings occur during $(0,\tau]$. When τ is small, $P_n(\tau) = O(\tau^n)$. Specifically we have

$$P_0(\tau) = 1 - \frac{\tau}{\langle\tau\rangle} + O(\tau^2), \tag{61}$$

$$P_1(\tau) = \frac{\tau}{\langle\tau\rangle} + O(\tau^2). \tag{62}$$

Substituting Eqs. (62) and (61) into Eq. (4), we obtain

$$\varphi_y(\tau) = 1 - \frac{2\tau}{\langle\tau\rangle} + O(\tau^2) \tag{63}$$

for small τ. Differenciation of Eq. (63) yields

$$\frac{1}{\langle\tau\rangle} = -\frac{\varphi_y'(0)}{2} \tag{64}$$

From Eq. (59), we have

$$\frac{\varphi_x(\tau)}{\varphi_x(0)} = \sin\left[\frac{\pi\varphi_y(\tau)}{2}\right]. \tag{65}$$

Differenciation of the above equation twice leads to the well-known formula:[1,2]

$$\frac{1}{\langle\tau\rangle} = \frac{1}{\pi}\sqrt{\frac{-\varphi_x''(0)}{\varphi_x(0)}}. \tag{66}$$

The PDF $L(\tau)$ is determined by the inverse transformation of Eq. (18):

$$L(\tau) = \mathcal{L}^{-1}(L[\lambda])(\tau), \tag{67}$$

where $\langle\tau\rangle$ of Eq. (64) or (66) is substituted. $L(\tau)$ thus obtained gives rise to the same average $\langle\tau\rangle$ because $L[\lambda]$ defined in Eq. (18) satisfies $L'[0] = -\langle\tau\rangle$.

APPENDIX C. DERIVATION OF EQ. (30)

Let us consider the Laplace transformation of the following function:

$$L(\tau) = \frac{c-1}{(1+\tau)^c}, \qquad 2 < c < 3. \tag{68}$$

The transform is calculated as

$$\begin{aligned} L[\lambda] &= (c-1)\int_0^\infty e^{-\lambda\tau}\frac{1}{(1+\tau)^c}d\tau \qquad 2 < c < 3 &(69)\\ &= (c-1)e^\lambda\lambda^{c-1}\int_\lambda^\infty e^{-x}x^{-c}dx, \quad \text{for } \lambda > 0, &(70)\end{aligned}$$

which is not equal to the incomplete gamma function $\Gamma(1-c,\lambda)$ because $1-c$ is negative. By applying the following recurtion formula

$$\int_\lambda^\infty e^{-x}x^{z-1}dx = -\frac{1}{z}e^{-\lambda}\lambda^z + \frac{1}{z}\int_\lambda^\infty e^{-x}x^z dx \quad (z \neq 0, \lambda > 0) \tag{71}$$

to Eq. (70) twice, we obtain Eq. (30).

REFERENCES

1. A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, McGraw-Hill, London, 1965.
2. T. Munakata, "Some unsolved problems on the level crossing of random processes" in *Unsolved Problems of Noise*, C. R. Doering, L. B. Kiss and M. F. Shlesinger eds., pp. 213–222, World Scientific, Singapore, 1997.
3. S. Ishioka, Z. Gingl, D. Choi and N. Fuchikami, "Amplitude truncation of Gaussian $1/f^\alpha$ noises," *Phys. Lett. A* **269**, pp. 7–12, 2000.
4. D. Choi and N. Fuchikami, "Generalized amplitude truncation of Gaussian $1/f^\alpha$ noise," *J. Phys. Soc. Japan* **70**, pp. 297-304, 2001.
5. Z. Gingl, S. Ishioka, D. Choi and N. Fuchikami, "Amplitude truncation of Gaussian $1/f^\alpha$ noises: Results and problems," *Chaos* **11**, pp. 619–623, 2001.
6. R. Mingesz, Z. Gingl, and P. Makra, "Level crossing time statistics of Gaussian $1/f^\alpha$ noises," in *Fluctuation and Noise in Biological, Biophysical, and Biomedical Systems*, S. M. Bezrukov, H. Frauenfelder and F. Moss eds., Proc. SPIE **5110**, pp. 312–319, 2003.
7. H. E. Stanley, S. V. Buldyrev, A. L. Goldberger, Z. D. Goldberger, S. Havlin, R. N. Mantegna, S. M. Ossadnik, C.-K. Peng and M. Simons, "Statistical mechanics in biology: how ubiquitous are long-range correlations? "*Physica A* 2̲05, pp.214–253, 1994.