Proceedings of WTC2005
World Tribology Congress III
September 12-16, 2005, Washington, D.C., USA

WTC2005-63888

A GENERALIZED ANALYTICAL MODEL FOR THE ELASTIC DEFORMATION OF AN ADHESIVE CONTACT BETWEEN A SPHERE AND A FLAT SURFACE

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ABSTRACT

A new method to calculate the elastic deformation of a sphere on a flat surface is presented. The model considers the influence of short-range as well as long-range attractive forces both inside and outside the actual contact area. In contrast to earlier models, this theory describes the nature of these deformations in the intermediate regime between the so-called JKR and DMT limits by simple analytic expressions. Equations for the calculation of the contact radius, the deformation, and the pressure distribution are given. In all equations, the critical force that might vary between the limiting values found in the DMT and the JKR model acts as transition parameter.

1. INTRODUCTION

The problem of the elastic deformation of a contact between a sphere and a flat surface (or two spheres, respectively) has been evolved in many regards as the "classical" contact mechanical problem ever since it was first addressed by Hertz in 1881 [1]. Hertz's approach, however, does not include adhesion, which might significantly influence the contact's behavior, especially if the contact's dimensions shrink to the nanometer scale. It took almost a century before Johnson, Kendall, and Roberts (JKR) presented in 1971 a theory that included the effect of adhesion [2]. Unfortunately, the approximations used in the derivation restrict its range of validity to large, soft contacts (the so-called JKR limit). In contrast, a model developed by Derjaguin, Muller, and Toporov (DMT) in 1975 [3], which was later simplified by Maugis [4] (referred to as the "DMT-M" [5] or "Hertz-plus-offset" model), applies only to the limit of small, hard contacts (DMT limit). For both models, simple analytic descriptions exist, while the general theoretical description of the mechanical behavior of models covering the intermediate regime between the JKR and the DMT limit so far required numerical approaches [4-6].

In this paper, we will present a new theory [7] that covers the full parameter range for the adhesive sphere-plane contact, but nevertheless results in simple equations describing the contact area, the vertical deformation, and the pressure distribution as a function of the externally applied load. The theory is based on a model interaction force that includes both short-range and long-range components. In all equations, the critical force that might vary between the limiting values found in the DMT and the JKR model acts as transition parameter.

2. DEVELOPMENT OF THE NEW MODEL

The theories currently used most for the description of the adhesive sphere-on-plane contact are all based on the same set of assumptions:

1. Deformations are supposed to be purely elastical.
2. The contacting materials are elastically isotropic.
3. Neither Young's modulus E nor Poisson's ratio ν change under load.
4. The atomic structure is not taken into account.
5. The contact radius a is small compared to the radius R of the sphere.
6. The curvature of the sphere in the contact area is described by a paraboloid.

Differences between the various models occur, however, in the nature of the forces that are supposed to act between sphere and flat. Figure 1 illustrates the issue, where the interaction forces (thick grey lines) used in the various models (b-f) are plotted in comparison to a realistic interaction [(a) and black lines in (b-f)]. Attractive forces are not included in the Hertz model (b) [1], whereas they are considered by a delta function in the JKR model (c) [2], by a long-range force in the DMT model (d) [3], and by a step function (Dugdale force) in the Maugis-Dugdale (MD) model (e) [4]. The MD model is the most popular theory so far approximating the full behavior of a sphere-plane contact even in the JKR-DMT transition regime, since it features a set two coupled analytical equations, which are comparatively easy to solve numerically. All model interactions feature hard-wall repulsion if intimate contact is established at $z = z_0$.

It is essential to recognize that all valid model forces that include adhesion have to possess an area between the force curve and the distance axis that is equivalent to the work of adhesion γ if plotted in units of force per unit area (shaded areas in a). Therefore, the total areas within the delta function in c) as

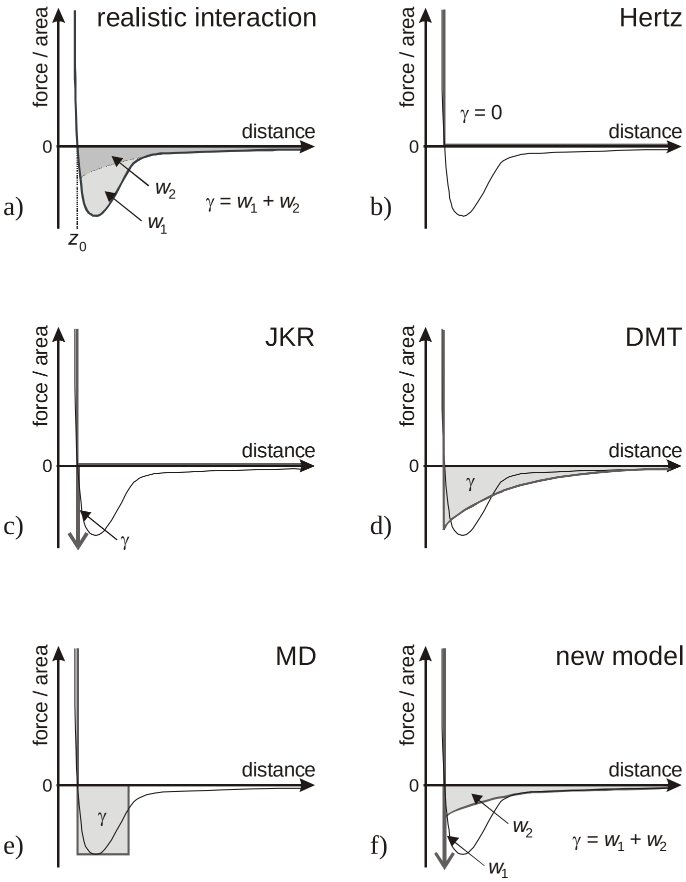

Figure 1. The interaction forces used in the various models (see text for details).

well as within the long-range potential in d) and the step function potential in d) have to be adjusted accordingly. As a consequence, the JKR model tends to overestimate short-range forces, while the DMT model overstates long-range forces.

To derive the new model introduced here, we formally split the area representing γ in the realistic interaction into two parts: Part 1, denoted as w_1, reflects the work of adhesion due to short-range forces, while the second part w_2 includes the work of adhesion due to long-range interactions. w_1 and w_2 are chosen such that $w_1 + w_2 = \gamma$. This allows us to create a model force for the new theory that consists out of a combination between a delta function as in the JKR model and a long-range interaction similar to the one in the DMT model (see Fig. 1f). If the two curves are adjusted such as the area within the delta peak is equivalent to w_2 and the area of the long-range potential equivalent to w_2, we obtain a valid model force since $w_1 + w_2 = \gamma$. By combining mathematical procedures developed for the derivation of the JKR and the DMT models, the contact mechanical problem can be solved, resulting in the following equations for the contact radius a, the deformation δ, and the pressure distribution p as a function of the distance r from the center of the contact circle:

$$a = \left(\frac{R}{K}\right)^{1/3}\left(\sqrt{3F_\mathrm{c} + 6\pi R\gamma} \pm \sqrt{F_\mathrm{l} - F_\mathrm{c}}\right)^{2/3} \quad ,$$

$$\delta = \frac{a^2}{R} - 4\sqrt{\frac{\pi a}{3K}\left(\frac{F_\mathrm{c}}{\pi R} + 2\gamma\right)} \quad ,$$

$$p(r) = \frac{3Ka}{2\pi R}\sqrt{1-\left(\frac{r}{a}\right)^2} - \sqrt{\frac{3K}{\pi a}\left(\frac{F_\mathrm{c}}{\pi R} + 2\gamma\right)}\left(1-\left(\frac{r}{a}\right)^2\right)^{-1/2}$$

Here, F_l denotes the externally applied load and K the effective elastic modulus, which combines the elastic properties of sphere and plane., F_c represents the critical force that has to be applied to separate sphere and plane, which found to be $F_\mathrm{c} = -1.5Rw_1 - 2Rw_2$. A detailed description of the derivation of the above equations including an in-depth discussion of the theoretical background can be found in Ref. 7. In the same work, it has also been shown that the equation for the contact radius a fits very well to results computed using the MD model, demonstrating the validity of the new approach.

From the way we constructed the new model force, we are free to choose the relative fraction with which w_1 and w_2 contribute to the total work of adhesion γ. For $w_1/\gamma = 1$, e.g., we obtain the JKR model, while $w_1/\gamma = 0$ leads to the DMT solution. If, however, the ratio w_1/γ is somewhere between 0 and 1, we are in the transition region from JKR to DMT. With other words, the relative strength of long-range compared to short-range forces can be continuously adjusted by changing the ratio w_1/γ between 0 and 1, thereby covering the JKR-DMT transition. This is introduced indirectly in the above equations via the critical force F_c, which changes from $F_\mathrm{c} = -1.5R\gamma$ for $w_1/\gamma = 1$ to $F_\mathrm{c} = -2R\gamma$ for $w_1/\gamma = 0$. Thus, the critical force F_c effectively acts as transition parameter, which determines how "JKR-like" or "DMT-like" the contact appears to be.

In conclusion, we presented a new model that describes the elastic behavior of an adhesive contact between a sphere and a plane in the full parameter range between the JKR and DMT limits, but still results in simple analytical equations.

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