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Further Developments in the Aerodynamic Analysis of Unsteady Supersonic Cascades

Part 1: The Unsteady Pressure Field

This paper presents, in two parts, a theoretical investigation of the aerodynamic response produced by an oscillating cascade placed in a supersonic stream with subsonic axial velocity component. Predictions are based on the successive solution of two linear boundary value problems which treat the velocity potential and the pressure, respectively, as basic dependent variables. A solution for the potential has been reported earlier and is used here to provide upper surface blade pressure distributions. This information serves as a boundary condition for the second problem. The solution for the unsteady pressure field, described in Part 1, is obtained by a construction procedure which parallels that used earlier to determine the potential. With the present procedure, blade pressure difference distributions and aerodynamic coefficients are accurately and efficiently determined for both subresonant and superresonant blade motions. Supersonic resonance phenomena and selected numerical results are discussed in Part 2 of the paper.

Introduction

Supersonic flutter in fan rotors is one of the most serious problems encountered in the development of advanced turbomachinery engines [1].[1] Since this type of flutter can occur at the design operating condition, it imposes a limit on high-speed operation of the machine. Hence it is important to have an efficient and accurate mathematical analysis for predicting the onset of supersonic flutter and determining the relative influence of the complex array of parameters which control blade response. Such capability is also required if the unsteady aerodynamic analysis is applied to the prediction of resonant blade stresses arising from the passage of a periodic distortion in the upstream flow through the blade row. To meet these needs the theoretical prediction of the aerodynamic characteristics of an oscillating cascade in a supersonic stream with subsonic axial velocity component (Fig. 1) has received considerable attention in recent years.

Existing analyses are based on a linear potential flow model for the flow past a two-dimensional oscillating cascade of flat plate airfoils. The blades are aligned parallel to a supersonic stream which has a subsonic velocity component normal to the cascade leading edge locus (axial direction), and they are assumed to perform identical harmonic motions of small amplitude with constant phase angle between the motion of adjacent blades. Since the free-stream velocity component in the axial direction is subsonic, unsteady disturbances exist infinitely far upstream of each blade passage and the flow adjacent to the lower surface of each blade is influenced by the blades and wakes below. These features have provided the major obstacles to obtaining a mathematical resolution of the unsteady flow field. The first analysis to provide useful design information applied finite difference solutions to flows past cascades consisting of a finite number of blades [2, 3]. More recently, analytic solutions to the infinite cascade problem have been determined and these are still undergoing further development. The authors of such solutions follow two distinct lines of approach. The first [4, 5] is to extend the Laplace transform results obtained by Miles [6, 7] and Lane [8] for simpler unsteady supersonic flow configurations, and the second [9, 10] is to determine a solution in terms of pressure dipoles distributed along the blades. The latter method had previously been successfully applied in the treatment of unsteady subsonic cascades [11–13].

Predictions of blade pressure distributions and aerodynamic forces

[1] Numbers in brackets designate References at end of paper.

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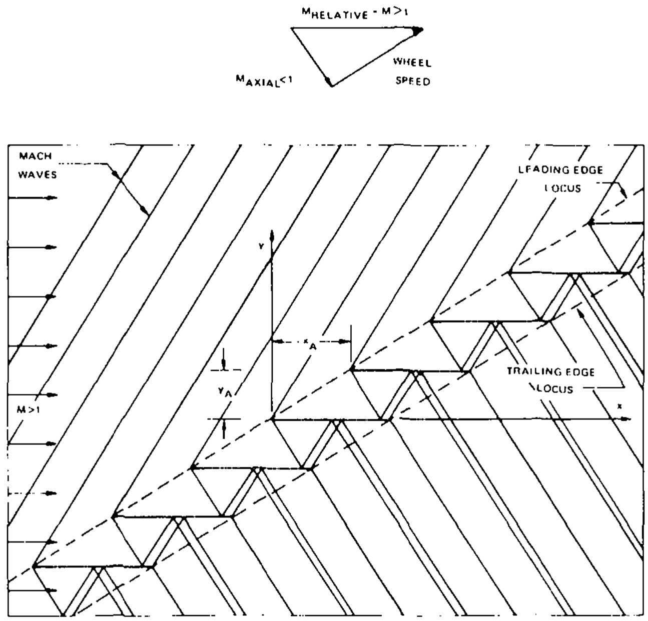

Fig. 1 Supersonic cascade in subsonic axial flow

and moments based on the foregoing supersonic analyses appear to be in very good agreement [2, 3, 5, 9, 14]. In addition, such predictions have been shown to be remarkably successful at predicting the observed flutter behavior of supersonic test fans [15, 16]. However, to date, meaningful analytical results have been reported only for subresonant blade motions. The distinction between subresonant and superresonant oscillations will be discussed in some detail in Part 2. At this point it is sufficient to note that superresonant motions occur over a range of negative interblade phase angles, i.e., the condition in which the motion of a given blade leads the motion of the blade above. In this case the elastic deflection wave of the cascade (or fan rotor) travels in the direction opposite to the direction of rotation and is thus classified as a backward-traveling wave. To have a complete flutter or resonant stress prediction system the capability of analyzing both forward- and backward-traveling deflection waves must be realized.

The determination of aerodynamic response parameters requires the solution of a boundary-value problem for a dependent fluid variable from which blade pressure distributions can be easily evaluated. Verdon and McCune [5] have formulated and solved a linear boundary-value problem which treats the velocity potential as the basic dependent flow variable. This solution permits the convenient calculation of the pressure distribution on the entire upper surface of a given blade and on the lower surface from the leading edge to the point of impingement of the trailing edge Mach wave from the adjacent blade below. Downstream of this point, disturbances produced by the infinite array of lower blades and wakes contribute to the pressures acting on the lower surface of the given blade. This contribution has proven to be difficult to evaluate using the velocity potential solution of [5], particularly for the case of superresonant oscillations. Therefore, in the present effort a second boundary-value problem which treats pressure as the basic dependent variable has been considered. The previous velocity potential solution provides the boundary information of a known pressure distribution on the upper surface of a given blade, and the solution for the pressure field is determined by a construction procedure similar to that used earlier to obtain the velocity potential. The procedure of solving two boundary-value problems successively has previously been applied to obtain an explicit expression for the wake upwash and a convenient means of computing the unsteady pressure field downstream of an isolated airfoil oscillating in a supersonic stream [17, 18]. For the cascade problem this approach facilitates the evaluation of the pressure distribution acting on the lower surface of a given blade. As a result, unsteady aerodynamic coefficients may be conveniently determined for both subresonant and superresonant blade motions.

Formulation

Since the present work is closely related to that of reference [5], symbol definitions used in this earlier study are retained herein. All parameters introduced in the following are dimensionless. Lengths have been scaled with respect to blade chord, time with respect to the blade chord divided by the undisturbed free-stream speed, and pressure with respect to the free-stream density multiplied by one-half the square of the free-stream speed.

Irrotational and isentropic flow, with negligible body forces, of an inviscid, nonconducting, perfect gas past a two-dimensional oscillating cascade is considered. The free-stream flow is supersonic with subsonic velocity component normal to the locus of blade leading edges (Fig. 1). The isentropic assumption implies that the entropy produced by shock waves is negligible. In addition, the shocks are assumed to be weak. They appear as lines in the flow field (Mach waves) across

Nomenclature

All physical parameters are dimensionless. Lengths have been scaled with respect to blade chord, time with respect to the blade chord divided by the undisturbed free-stream speed, and pressure with respect to the free-stream density multiplied by one-half of the square of the free-stream speed.

$A(x)$ = reference blade and wake velocity disturbance function

$A_n(x)$ = nth blade and wake velocity disturbance function

$B = x_A - \mu y_A$

$B(x)$ = reference blade and wake pressure disturbance function

$B_n(x)$ = nth blade and wake pressure disturbance function

$C = 1 - 2\mu y_A$

$D = x_A + \mu y_A$

$\mathcal{F}[B(x), x]$ = blade-disturbance functional, defined by equation (28)

$\mathcal{G}[B(x), x]$ = wake-disturbance functional, defined by equation (29)

$I_{m,n}{}^{\pm}(x, y)$ = velocity potential influence function, defined by equation (16)

$J_0(x)$ = Bessel function of the first kind of order zero

$J_1(x)$ = Bessel function of the first kind of order one

$K(x)$ = kernel function, defined by equation (12)

k = compressible reduced frequency, $\omega M \mu^{-2}$

$P(x, y)$ = modified relative pressure, defined by equation (2)

$P_i(x, y)$ = ith component of the modified relative pressure (cf. equations (17)–(21))

$p(x, y, t)$ = pressure

p_{∞} = free-stream pressure

$Q_{m,n}{}^{\pm}(x, y)$ = pressure influence function, defined by equation (16)

t = time

$U(x)$ = Heaviside or unit step function

$V(x, 0)$ = modified normal velocity on reference blade and wake

x, y = spatial coordinates

x_A = stagger distance

y_A = normal gap distance

$\Delta P(x)$ = modified pressure difference across the reference blade, defined by equation (26)

$\delta(x)$ = Dirac or unit impulse function

$\mu = (M^2 - 1)^{1/2}$

σ = interblade phase angle

$\psi(x, y)$ = modified velocity potential defined by equation (6)

$\Omega = \sigma + kMx_A$

ω = reduced frequency based on blade chord which the flow variables are discontinuous. The blades are thin, slightly cambered lifting surfaces with small mean angle of attack relative to the incoming stream. They are performing identical harmonic motions of small amplitude with constant phase angle between the motion of adjacent blades.

The foregoing restrictions permit a linear analysis of the disturbed flow in which the effects of thickness, camber, mean angle of attack, and the unsteady displacement of the blades may be determined separately and the results added to obtain their combined effect. Only the unsteady disturbances are of interest here, and consequently the blades are assumed to be flat plates aligned parallel to the stream with mean positions at $nx_A \leq x \leq nx_A + 1, y = ny_A, n = 0, \pm 1, \pm 2, \ldots,$ where x_A and y_A are the blade stagger and normal gap distances, respectively (Fig. 1). The unsteady wakes are thin vortex sheets which emanate from the trailing edges of the blades and extend infinitely far downstream. Their mean positions lie on the lines $1 + nx_A < x < \infty, y = ny_A, n = 0, \pm 1, \pm 2, \ldots$. The linear approximation permits boundary conditions at the blade and wake surfaces to be satisfied at the mean positions of these surfaces. In addition, the cascade geometry and the prescribed type of blade motion require that the unsteady flow exhibit blade-to-blade periodicity. Hence it is sufficient to determine the flow in a single extended blade passage region of the cascade (say the zeroth or reference passage defined by $|x| < \infty, 0 < y < y_A$) and to determine the pressure distribution acting on a single blade, i.e., the zeroth or reference blade.

Unsteady pressure fluctuations are governed by the differential equation

$$\frac{\partial^2 P}{\partial y^2} - \mu^2 \frac{\partial^2 P}{\partial x^2} - \mu^2 k^2 P = 0 \qquad (1)$$

where

$$P(x, y) = [p(x, y, t) - p_\infty] \exp [i(kMx - \omega t)] \qquad (2)$$

The dependent variable transformation, equation (2), has been introduced to remove time dependence from the basic equations and also to simplify the various mathematical relations describing the unsteady flow. The unsteady pressure in the extended reference blade passage will be determined as a solution of equation (1) subject to a given pressure distribution on the upper surface of the reference blade, i.e.,

$$P(x, 0^+) \quad \text{known for} \quad 0 \leq x \leq 1 \qquad (3)$$

and the following conditions. The modified pressure and its normal or y-derivative are continuous in y along the upstream extensions of the blade chord lines and along the unsteady wakes. Further, $\partial P/\partial y$ is continuous across blade surfaces. It therefore follows from the blade-to-blade periodicity of the unsteady flow that

$$P(x, 0^+) = P(x, 0^-) = P(x + x_A, y_A{}^-)e^{-i\Omega} \qquad x < 0 \text{ and } x > 1 \qquad (4)$$

and

$$\frac{\partial P}{\partial y}(x, 0^+) = \frac{\partial P}{\partial y}(x, 0^-) = \frac{\partial P}{\partial y}(x + x_A, y_A{}^-)e^{-i\Omega} \qquad |x| < \infty \qquad (5)$$

where $\Omega = \sigma + kMx_A$. In addition to the foregoing conditions at the upper and lower boundaries of the extended reference blade passage region, there can be no upstream propagation of disturbances in supersonic flow, pressure disturbances must be bounded at an infinite distance from their source, and disturbance waves impinging on blade surfaces must be reflected, while those impinging on wake surfaces must be transmitted through the wake.

The boundary value problem for the unsteady pressure field is similar to that previously formulated for the modified velocity potential, i.e.,

$$\psi(x, y) = \phi(x, y, t) \exp [i(kMx - \omega t)] \qquad (6)$$

where $\phi(x, y, t)$ is the velocity potential as defined in the usual manner. The only significant difference is that the pressure field will be determined in terms of a known pressure distribution on the upper surface of the reference blade, while the potential field was determined using the condition of flow tangency at the reference blade surface, i.e.,

$$\frac{\partial \psi}{\partial y}(x, 0^+) = V(x, 0) \qquad 0 \leq x \leq 1 \qquad (7)$$

where the modified normal velocity, $V(x, 0)$, is specified by the prescribed blade motion. The upper surface pressure distribution cannot be prescribed a priori, but must be evaluated from the pressure-potential relation

$$P(x, y) = -2 \left(\frac{\partial}{\partial x} - i\omega\mu^{-2}\right) \psi(x, y) \qquad (8)$$

and the solution for the potential.

Previous Related Results

The solution for the modified potential [5] was determined by generalizing Miles' solution [6] for an isolated airfoil oscillating in a supersonic stream and Lane's result [8] for a supersonic cascade in supersonic axial flow. An explicit representation for the potential in the reference passage was obtained in terms of "velocity disturbance functions," $A_n(x), n = 0, \pm 1, \pm 2, \ldots,$ distributed along the mean positions of the blades and wake surfaces. The nth velocity disturbance function describes the contribution to the unsteady upwash or normal velocity produced by the motion of the nth blade and its wake. The explicit result satisfies the governing differential equation, the far-field conditions, and the requirements of reflection of disturbance waves impinging on blade surfaces and transmission of disturbance waves impinging on wake surfaces. Blade-to-blade periodicity and the continuity conditions on ψ and $\partial\psi/\partial y$ are satisfied if the nth disturbance function is related to the reference blade and wake disturbance function as follows:

$$A_n(x + nx_A)e^{-in\Omega} = A_0(x) = A(x) \qquad n = 0, \pm 1, \pm 2, \ldots (9)$$

The foregoing relation permits the potential to be expressed functionally in terms of $A(x)$ and the spatial coordinates x and y, i.e.,

$$\psi(x, y) = \Psi[A(x), x, y] \qquad (10)$$

Finally, the conditions of flow tangency at the reference blade surface and continuity of pressure across the reference wake provide two integral relations which can be solved numerically to determine the disturbance function on the reference blade and its wake, respectively.

Two results from the velocity potential solution are required for the present pressure formulation. The first is the integral relation used to determine the velocity disturbance function distribution on the reference blade which follows from the explicit solution for the potential and the flow tangency condition, i.e.,

$$A(x) = V(x, 0) - \int_0^B A(\xi)K(x - \xi)d\xi \qquad 0 \leq x \leq 1 \quad (11)$$

where

$$K(x) = k\mu y_A \sum_{n=-\infty}^{-1} ne^{in\Omega} \frac{J_1\{k[(x - nx_A)^2 - (\mu ny_A)^2]^{1/2}\}}{[(x - nx_A)^2 - (\mu ny_A)^2]^{1/2}} \quad (12)$$

The second is the expression for the pressure distribution on the upper surface of the reference blade which follows from the explicit solution and the pressure-potential relation, i.e.,

$$\begin{aligned}\mu P(x, 0^+)/2 = &- \int_0^{B^-} A(\xi) \sum_{n=-\infty}^{-1} e^{in\Omega}K_{n,n}(x - \xi)d\xi \\ &- \int_0^1 A(\xi)K_{0,0}(x - \xi)d\xi - 2 \int_B^C A(\xi)K_{0,-2}(x - \xi)d\xi \\ &+ 2 \int_0^{1-D} A(\xi)e^{i\Omega}K_{1,-1}(x - \xi)d\xi \qquad 0 \leq x \leq 1 \quad (13)\end{aligned}$$

where

$$K_{m,n}(x) = -\left(\frac{\partial}{\partial x} - i\omega\mu^{-2}\right) I_{m,n}^{-}(x, 0) = [k(x - mx_A)$$

$$J_1\{k[(x - mx_A)^2 - (\mu n y_A)^2]^{1/2}\}/[(x - mx_A)^2 - (\mu n y_A)^2]^{1/2}$$
$$+ i\omega\mu^{-2}J_0\{k[(x - mx_A)^2 - (\mu n y_A)^2]^{1/2}\}]U(x - mx_A + \mu n y_A)$$
$$- J_0\{k[(x - mx_A)^2 - (\mu n y_A)^2]^{1/2}\}\delta(x - mx_A + \mu n y_A) \quad (14)$$

The integral term in equation (11) represents the influence of the normal velocity disturbances generated at the leading edge segments, $nx_A \leq x < nx_A + B$, $n = -1, -2, \ldots$, of the semi-infinite array of lower blades on the disturbances generated by the motion of the reference blade. For $0 \leq x < B$, equation (11) is a Fredholm integral equation with unknown function $A(x)$. Once $A(x)$ is determined in the interval $0 \leq x < B$, its value over the remainder of the blade is determined by straightforward integration. The pressure distribution on the upper surface of the reference blade is then obtained by performing the integrations indicated by equation (13). If $D = x_A + \mu y_A > 1$, the last two integral terms in equation (13) do not apply.

Some insight into the behavior of the velocity-disturbance function along blade wakes can be gained by considering supersonic flow past an isolated airfoil. For an oscillating flat plate airfoil with mean position at $y = 0$, $0 \leq x \leq 1$, the velocity-disturbance function is equal to the upwash or normal velocity on the airfoil and its wake. An explicit expression for the wake upwash has been derived previously [17] and has the form

$$A(x) = V(x, 0) = -\omega\mu^{-1}\psi(1, 0^+)\exp[i\omega\mu^{-2}(x - 1)]$$
$$- \mu k \int_0^x \psi(\xi, 0^+)(x - \xi)^{-1}J_1[k(x - \xi)]d\xi \qquad x > 1 \quad (15)$$

The harmonic term on the right-hand side of equation (15) results from the counter-vorticity shed from the trailing edge of the airfoil. Note that the wake disturbance function distribution depends on the airfoil loading. For the supersonic cascade in subsonic axial flow the blade loading is, in turn, dependent on the wake-disturbance function distribution. In this case the wake integral equation derived from the velocity-potential formulation contains an infinite series, each term of which involves the integral of the unknown function, $A(x)$, over a segment of the reference wake. The oscillatory behavior of the velocity-disturbance function along blade wakes has rendered it difficult to achieve a numerical resolution of this integral relation, and hence to determine lower surface blade pressures.

To overcome this problem an alternative representation for the cascade pressure field in terms of a "pressure-disturbance function" distribution has been determined. It will be seen that the pressure and the velocity-potential solutions are similar in form. The pressure-disturbance function, $B(x)$, measures the contribution to the unsteady pressure field due to the motion of the reference blade and wake of the cascade. For the isolated airfoil $B(x)$ equals the unsteady pressure $P(x, 0^+)$, acting at a point on the upper surface of the airfoil or its wake, and hence $B(x)$ is identically zero on the wake. For the cascade the pressure disturbance function rapidly attenuates along blade wakes. This behavior permits a simple truncation procedure to be applied to the wake infinite series and leads to a viable numerical algorithm for solving the wake integral relation derived from the pressure formulation. As a result the contribution to lower surface reference blade pressures due to disturbances produced by the lower blades and wakes may be conveniently evaluated.

Explicit Solution for the Unsteady Pressure Field

An explicit solution for the unsteady pressure field in the reference blade passage region ($0 < y < y_A$, $|x| < \infty$) will be constructed in terms of pressure disturbance functions, $B_n(x)$, defined on the mean positions of the blade and wake surfaces. Two flow configurations which depend on the free-stream Mach number and the cascade geometry are of current practical interest (Fig. 2). In the first, $D = x_A + \mu y_A > 1$ and the lower leading edge Mach wave from any blade passes behind the blades below. In the second case, $D < 1 < x_A + 3\mu y_A$ and the leading edge Mach waves are reflected once by the adjacent blades below. The points labeled 1, 2, and 3 in Fig. 2 denote the locations at which Mach waves impinge on and are reflected by the zeroth

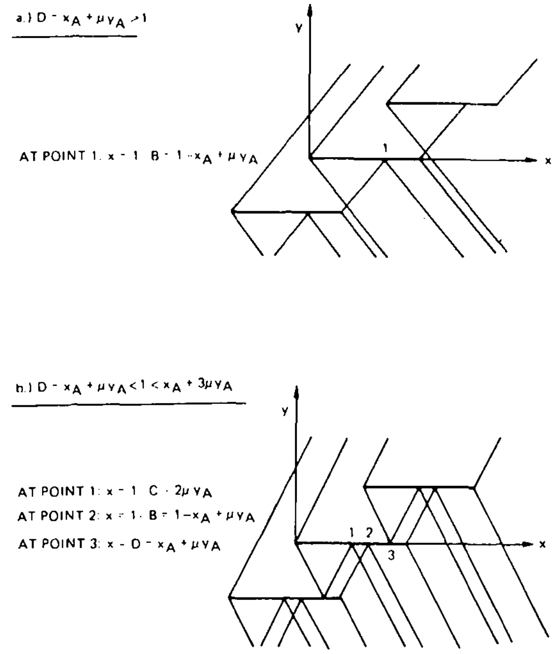

Fig. 2 Two supersonic cascade flow geometries

blade. Discontinuities in the zeroth or reference blade pressure distributions occur at these points. In the former case there is only one pressure discontinuity, which occurs on the lower surface of the reference blade at $x = 1 - B = 1 - x_A + \mu y_A$, while in the latter case pressure discontinuities occur at $x = D$ on the upper surface and at $x = 1 - C = 2\mu y_A$ and $x = 1 - B$ on the lower surface. The solution procedure is better illustrated by a development leading to an expression for the modified pressure for the second case. The result for the simpler flow geometry can then be obtained by simply neglecting the terms corresponding to the additional reflections. More complicated flow geometries can be treated by a straightforward extension of the concepts presented subsequently; however, such cases do not appear to be of interest for present applications.

The unsteady flow in the reference passage results from disturbances produced by the zeroth and first blades of the cascade and their wakes and, in addition, disturbances produced by their neighboring blades and wakes which propagate into the reference passage. The latter include upward propagating disturbances coming from below the line $y = 0$ and either upstream of the characteristic $x = \mu y$ or downstream of the characteristic $x = 1 + \mu y$, and downward propagating disturbances coming from above the line $y = y_A$ and downstream of the characteristic $x = 1 - \mu y + D$ (Fig. 3). In the following analysis component pressure terms, $P_i(x, y)$, which account for disturbances originating from different sources, are first determined. The complete solution for the unsteady pressure in the reference passage is then obtained by superposing the various component functions. The construction in terms of component pressures serves to clarify the illustration of the solution procedure. Each component is a solution of the governing differential equation and satisfies the far-field conditions and the reflection and transmission conditions at blade and wake surfaces, respectively. Influence functions which appear in the pressure solution have the form

$$Q_{m,n}^{\pm}(x, y) = \mu^{-1}\frac{\partial I_{m,n}^{+}}{\partial y}(x, y) = \mu^{-1}\frac{\partial}{\partial y}[J_0\{k[(x - mx_A)^2$$
$$- \mu^2(y - ny_A)^2]^{1/2}\}U[(x - mx_A) \pm \mu(y - ny_A)]]$$
$$= \pm\delta[x - mx_A \pm \mu(y - ny_A)]J_0\{k[(x - mx_A)^2 - \mu^2(y - ny_A)^2]^{1/2}\}$$
$$+ k\mu(y - ny_A)J_1\{k[(x - mx_A)^2 - \mu^2(y - ny_A)^2]^{1/2}\}$$
$$\times U[x - mx_A \pm \mu(y - ny_A)]/[(x - mx_A)^2 - \mu^2(y - ny_A)^2]^{1/2} \quad (16)$$

where $I_{m,n}{}^{\pm}(x - \xi, y)$ are influence functions for the velocity potential solution [5].

The unsteady flow in the reference passage upstream of the characteristic $x - \mu y = 0$ is caused by disturbances originating at the leading edge segments, $nx_A \leq x < nx_A + B$, $y = ny_A{}^+$, of the blades below, i.e., $n < 0$. The pressure due to these disturbances is obtained by generalizing the solution for the pressure field due to an oscillating isolated airfoil [18] and is given by

$$P_1(x, y) = -\sum_{n=-\infty}^{-1} \int_{nx_A}^{nx_A+B^-} B_n(\xi) Q_{0,n}{}^-(x - \xi, y) d\xi \qquad 0 < y < y_A \quad (17)$$

The negative superscript on the upper limit of each integral in equation (17) indicates that the range of integration does not include the point $\xi = nx_A + B$. This is important to note because of the presence of the impulse function in each integrand. In contrast to the isolated airfoil result, the disturbance function and upper surface blade pressure distributions are not equal to each other since the unsteady disturbances produced by the motion of the lower blades contribute to the reference blade pressure distribution. Disturbances originating at the nth blade surface ($n < 0$) at and downstream of the point $x = nx_A + B$ do not propagate directly into the reference passage, but are reflected by the adjacent upper blade.

Disturbances produced by the zeroth blade and its wake are accounted for in a similar manner; however, those originating on the interval $B \leq x \leq 1 + B$, $y = 0^+$ impinge on the first ($n = 1$) blade giving rise to a reflected disturbance which also contributes to the unsteady flow in the reference passage. If $D < 1$ there will be further downstream reflections. The pressure due to this disturbance wave system has the form

$$P_2(x, y) = -\int_0^\infty B_0(\xi) Q_{0,0}{}^-(x - \xi, y) d\xi + \int_B^{1+B} B_0(\xi) Q_{0,2}{}^+(x - \xi, y) d\xi - \int_B^C B_0(\xi) Q_{0,-2}{}^-(x - \xi, y) d\xi + \int_B^C B_0(\xi) Q_{0,4}{}^+(x - \xi, y) d\xi \qquad 0 < y < y_A \quad (18)$$

The first integral term in equation (18) accounts for disturbances originating at the zeroth blade and wake. The second term is due to the reflection of this disturbance wave system by the first blade. The third and fourth terms account for the subsequent reflections of the original disturbance wave by the zeroth and first blade, respectively. Note that incident and reflected waves produce equal contributions to the pressure acting on a blade surface, but their contributions to the normal derivative of the pressure, $\partial P/\partial y$, at a blade surface cancel. The upper limit on the integrals representing reflected disturbances are finite because the reflecting blade surfaces have finite length and the wakes are assumed to transmit impinging disturbance waves.

Similarly, the pressure due to disturbances originating at the first blade and its wake and their reflections off the zeroth and first blade is given by

$$P_3(x, y) = -\int_{x_A}^\infty B_1(\xi) Q_{0,1}{}^+(x - \xi, y) d\xi + \int_{x_A}^{1-\mu y_A} B_1(\xi) Q_{0,-1}{}^-(x - \xi, y) d\xi - \int_{x_A}^{1-\mu y_A} B_1(\xi) Q_{0,3}{}^+(x - \xi, y) d\xi \qquad 0 < y < y_A \quad (19)$$

The minus sign preceding the first integral term in equation (19) signifies that self-induced upper and lower surface blade pressures are equal in magnitude but opposite in sign.

The pressure component $P_4(x, y)$ accounts for disturbances produced by the blades and wakes below the zeroth blade which propagate upward and the reflections of these disturbances from the lower surface of the reference blade. This disturbance system propagates into the reference passage downstream of the zeroth blade trailing edge Mach wave, $x - \mu y = 1$, $y > 0$. The effect of unsteady disturbances originating in the nth passage ($n < 0$) on the flow in the reference passage is accounted for by the disturbance function distribution on the ($n + 1$)th wake downstream of the characteristic $x - \mu y = 1$. Thus the influence of the disturbances from the adjacent lower passage, $n = -1$, has already been included in equation (18) and it follows that

$$P_4(x, y) = -\sum_{n=-\infty}^{-1} \int_{1+\mu n y_A}^\infty B_n(\xi) Q_{0,n}{}^-(x - \xi, y) d\xi + \sum_{n=-\infty}^{-1} \int_{1+\mu n y_A}^{1+B+\mu n y_A} B_n(\xi) Q_{0,2-n}{}^+(x - \xi, y) d\xi \qquad 0 < y < y_A \quad (20)$$

The first term of the right-hand side of equation (20) accounts for the disturbances generated at the lower blades and wakes, while the second term represents the reflections of these disturbances by the first blade.

Finally, the pressure component $P_5(x, y)$ is due to disturbances originating at the blade and wakes above the first blade which propagate downward and are not reflected upward by lower blades and, in addition, the reflections of upward propagating disturbances by the blades above the first blade, which are not intercepted by the blades below. For the example flow geometry, the expression for $P_5(x, y)$ has the form

$$P_5(x, y) = -\sum_{n=2}^{\infty} \int_{1+nx_A+D}^\infty B_n(\xi) Q_{0,n}{}^+(x - \xi, y) d\xi - \sum_{n=2}^{\infty} \int_{nx_A}^{1+nx_A+D} B_n(\xi) Q_{0,n+2}{}^+(x - \xi, y) d\xi + \sum_{n=1}^{\infty} \int_{nx_A+C}^{1+nx_A+B} B_n(\xi) Q_{0,n+2}{}^+(x - \xi, y) d\xi + \sum_{n=1}^{\infty} \int_{nx_A+B}^{nx_A+C} B_n(\xi) Q_{0,n+4}{}^+(x - \xi, y) d\xi + \sum_{N=2}^{\infty} \sum_{n=-\infty}^{N-2} \int_{1+(N-1)B+\mu n y_A}^{1+NB+\mu n y_A} B_n(\xi) Q_{0,2N-n}{}^+(x - \xi, y) d\xi \qquad 0 < y < y_A \quad (21)$$

This pressure component accounts for disturbances which have no influence on the zeroth and first blade pressure distributions. The first summation term in equation (21) represents disturbances originating at the upper blades and their wakes and the remaining terms account for the reflections by the upper blades of disturbance waves which impinge on them from below.

The complete solution for the unsteady, modified relative pressure in the reference passage region is obtained by a summation of the foregoing component terms, i.e.,

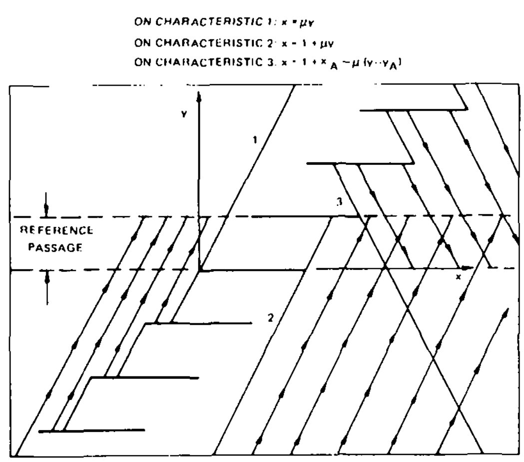

Fig. 3 Schematic representation of disturbance propagation toward the reference passage

$$P(x, y) = \sum_{i=1}^{5} P_i(x, y) \qquad 0 < y < y_A \tag{22}$$

This result satisfies the governing differential equation (1), the downstream propagation and far-field requirements, and the reflection and transmission conditions at blade and wake surfaces. The blade-to-blade periodicity and continuity conditions on the pressure upstream of the blade row, equation (4) for $x < 0$, and on the normal pressure gradient, $\partial P/\partial y$, for the entire reference passage, equation (5), are met if the pressure disturbance functions are required to satisfy the relation

$$B_n(x + nx_A)e^{-in\Omega} = B_0(x) = B(x) \qquad n = 0, \pm 1, \pm 2, \ldots \tag{23}$$

Thus, for the example flow geometry, $D < 1 < x_A + 3\mu y_A$, it follows after some algebra that

$$\begin{aligned}
P(x, y) = &- \int_0^B B(\xi) \sum_{n=-\infty}^{-1} e^{in\Omega} Q_{n,n}^-(x - \xi, y) d\xi \\
&- \int_0^\infty B(\xi) Q_{0,0}^-(x - \xi, y) d\xi \\
&- \sum_{n=-\infty}^{-1} \int_{1-nB}^\infty B(\xi) e^{in\Omega} Q_{n,n}^-(x - \xi, y) d\xi \\
&- \int_B^C B(\xi) Q_{0,-2}^-(x - \xi, y) d\xi + \int_B^1 B(\xi) Q_{0,2}^+(x - \xi, y) d\xi \\
&+ \int_C^1 B(\xi) \sum_{n=1}^{\infty} e^{in\Omega} Q_{n,n+2}^+(x - \xi, y) d\xi \\
&+ \int_B^C B(\xi) \sum_{n=0}^{\infty} e^{in\Omega} Q_{n,n+4}^+(x - \xi, y) d\xi \\
&- \int_0^\infty B(\xi) e^{i\Omega} Q_{1,1}^+(x - \xi, y) d\xi \\
&- \int_{1-D}^\infty B(\xi) \sum_{n=2}^{\infty} e^{in\Omega} Q_{n,n}^+(x - \xi, y) d\xi \\
&- \int_0^{1-D} B(\xi) \sum_{n=1}^{\infty} e^{in\Omega} Q_{n,n+2}^+(x - \xi, y) d\xi \\
&+ \int_0^{1-D} e^{i\Omega} B(\xi) Q_{1,-1}^-(x - \xi, y) d\xi \\
&+ \sum_{N=1}^{\infty} \sum_{n=-\infty}^{N-1} \int_{1+(N-n-1)B}^{1+(N-n)B} B(\xi) e^{in\Omega} Q_{n,2N-n}^+(x - \xi, y) d\xi
\end{aligned}$$
$$0 < y < y_A \tag{24}$$

The pressure solution, equation (24), has the same form as the previous velocity potential solution. In fact, the velocity potential result is recovered if $P(x, y)$, $B(x)$, and $Q_{m,n}^{\pm}(x, y)$ are replaced by $\mu\psi(x, y)$, $A(x)$, and $\pm I_{m,n}^{\pm}(x, y)$, respectively. For flows in which the trailing edge Mach waves pass behind the blades below ($D > 1$), the explicit solution for the modified relative pressure is obtained by eliminating the fourth, seventh, tenth, and eleventh integral terms from the right-hand side of equation (24) and setting the lower limits on the sixth and ninth integrals equal to B and 0, respectively.

Pressure Disturbance Function Relations

It remains to evaluate the pressure disturbance function, $B(x)$, based on a known pressure distribution on the upper surface of the reference blade, and the condition of pressure continuity across the reference wake. A relation for the pressure on the upper surface of the reference blade in terms of the pressure disturbance function, $B(x)$, is obtained by setting $y = 0$ in equation (24) and noting from equation (16) that $Q_{m,n}^+(x, 0) = -Q_{m,n}^-(x, 0)$. It follows that

$$\begin{aligned}
P(x, 0^+) = &\int_0^B B(\xi) K(x - \xi) d\xi + B(x) \\
&+ 2 \int_B^C B(\xi) Q_{0,-2}^-(x - \xi, 0) d\xi \\
&+ 2 \int_0^{1-D} B(\xi) e^{i\Omega} Q_{1,-1}^-(x - \xi, 0) d\xi \qquad 0 \le x \le 1
\end{aligned} \tag{25}$$

For $0 \le x < B$ the disturbance function, $B(x)$, is determined as the solution of a Fredholm integral equation which has the same kernel function as the flow tangency integral equation (11). Further, for $B \le x < 1$, the pressure disturbance function is determined from equation (25) by straightforward integrations in analogy with the previous result for the velocity disturbance function. Values of the disturbance function, $B(x)$, on the reference blade provide sufficient information for determining the pressure in the reference passage up to the trailing-edge Mach wave of the zeroth blade, $x = 1 + \mu y$, and hence the pressure on the lower surface of a given blade from its leading edge to the point of impingement of the trailing edge Mach wave from the adjacent blade below. To complete the solution for the blade pressure distribution, the reference wake disturbance function distribution must be determined from the condition of pressure continuity across blade wakes, i.e., equation (4) for $x > 1$.

The modified pressure difference across the reference blade and wake is given by

$$\begin{aligned}
\Delta P(x) &= P(x, 0^-) - P(x, 0^+) \\
&= P(x + x_A, y_A^-)e^{-i\Omega} - P(x, 0^+) \qquad x \ge 0
\end{aligned} \tag{26}$$

and it follows from equations (24) and (16) that

$$\Delta P(x) = -2B(x) + \mathcal{F}[B(x), x] + \mathcal{G}[B(x), x] \qquad x \ge 0 \tag{27}$$

where

$$\begin{aligned}
\mathcal{F}[B(x), x] = &-2 \int_B^1 B(\xi) e^{-i\Omega} Q_{-1,-1}^-(x - \xi, 0) d\xi \\
&- 2 \int_B^C B(\xi)[e^{-i\Omega} Q_{-1,-3}^-(x - \xi, 0) - Q_{0,-2}^-(x - \xi, 0)] d\xi \\
&- 2 \int_0^{1-D} B(\xi)[e^{i\Omega} Q_{1,-1}^-(x - \xi, 0) - Q_{0,-2}^-(x - \xi, 0)] d\xi
\end{aligned} \tag{28}$$

and

$$\mathcal{G}[B(x), x] = -2 \sum_{n=-\infty}^{-1} e^{in\Omega} \int_{1-(n+1)B}^{1-nB} B(\xi) Q_{n,n}^-(x - \xi, 0) d\xi \tag{29}$$

The first term on the right-hand side of equation (27) represents the self-induced reference blade loading. The functional $\mathcal{F}[B(x), x]$ represents the loading on the reference blade due to disturbances originating at and reflected from the adjacent blades above and below ($n = \pm 1$). This term depends on the reference blade disturbance function distribution (i.e., $B(x)$ for $0 \le x \le 1$). Finally, the functional $\mathcal{G}[B(x), x]$ defines the reference blade loading due to the lower blades ($n \le -2$) and wakes ($n \le -1$). This functional depends on the reference wake disturbance function distribution (i.e., $B(x)$ for $x > 1$).

Since the pressure must be continuous across blade wakes, it follows from equation (27) that

$$2B(x) - \mathcal{G}[B(x), x] = \mathcal{F}[B(x), x] \tag{30}$$

This relation indicates that nonzero values of the pressure disturbance function on the reference wake are entirely due to disturbances produced by the neighboring blades and wakes of the cascade. Thus, the wake distribution due to upstream self-induced disturbances is zero in agreement with the isolated airfoil result [18]. Equation (30) is an integral equation for $B(x)$ in which the adjacent blade interaction term, $\mathcal{F}[B(x), x]$, can be regarded as a known function. The wake functional, $\mathcal{G}[B(x), x]$, introduces the complication that the value of the disturbance function at a given wake location, say x_0, depends on its values downstream of this location, $x > x_0$. To determine a solution to equation (30) the infinite series wake functional must be truncated, i.e., it is assumed that

$$\begin{aligned}
\mathcal{G}[B(x), x] &\approx \mathcal{G}_N[B(x), x] \\
&= -2 \sum_{n=-N}^{-1} e^{in\Omega} \int_{1-(n+1)B}^{1-nB} B(\xi) Q_{n,n}^-(x - \xi) d\xi
\end{aligned} \tag{31}$$

where N must be chosen large enough so that an accurate estimate of $[B(x), x]$ is obtained. This truncation is also applied in equation

(27) to evaluate the reference blade pressure difference distribution. Numerical results based on the truncation procedure confirm that $|B(x)|$ approaches zero with increasing distance along the wake.

Equations (11), (13), (25), and (30) are the disturbance-function relations required by the present solution procedure for the prediction of reference blade pressure difference distributions and aerodynamic coefficients. These equations have been presented for a flow configuration in which the trailing edge Mach wave from a given blade impinges once on the adjacent blade below, $D < 1 < x_A + 3\mu y_A$. Expressions for the simpler flow configuration, in which trailing edge Mach waves pass behind the lower blades, $D > 1$, are obtained by dropping the last two integral expressions on the right-hand sides of equations (13), (25), and the definition of $\mathcal{F}[B(x), x]$, equation (28). The solution procedure can be summarized as follows. First, the velocity disturbance function distribution, $A(x)$ on the reference blade, $0 \leq x \leq 1$, is determined in terms of a prescribed blade motion, $V(x, 0)$, from the flow tangency integral relation, equation (11). Then the pressure distribution, $P(x, 0^+)$, on the upper surface of the reference blade is obtained from equation (13), and the reference blade pressure disturbance function distribution, $B(x)$, $0 \leq x \leq 1$, is evaluated from the integral relation, equation (25). Finally, the pressure disturbance function distribution along the reference wake is determined from the wake integral relation, equation (30), using the truncation scheme given by equation (31). The results described in Part 2 of this paper have been obtained by numerically estimating the sum of the infinite series kernel function, $K(x)$, which appears in equations (11) and (25), and the infinite series, equation (14), which appears in the expression for upper surface pressure, equation (13). In addition, the three integral equations, i.e., equations (11) and (25) for $0 \leq x < B$, and equation (30), have been solved by finite difference procedures which reduce each of these equations to a system of linear algebraic equations. The finite difference approximations to equations (11) and (25) yield identical coefficient matrices.

The expression for the pressure difference across the reference blade and wake in terms of the velocity disturbance function, $A(x)$ [5], is recovered from equation (27) by replacing $B(\xi)$ and $Q_{m,n}{}^-(x - \xi, 0)$ by $2\mu^{-1}A(\xi)$ and $K_{m,n}(x - \xi)$, respectively. The significant difference between the wake integral relation for $A(x)$ and equation (30) is that in the former case, the value of the velocity disturbance function at a given wake location, say x_0, depends on reference blade and wake disturbances originating upstream of the location x_0. For example, if $\mathcal{F}[B(x), x]$ and $\mathcal{G}[B(x), x]$ are neglected, equation (30) reduces to $B(x) = 0$. The corresponding result for the velocity disturbance function is

$$A(x) - \int_0^x A(\xi)\{kJ_1[k(x - \xi)] - i\omega\mu^{-2}J_0[k(x - \xi]\}d\xi = 0 \qquad x > 1 \quad (32)$$

The solution to equation (32) represents the wake upwash produced by an isolated airfoil with $V(x, 0) = A(x)$ (cf., equation (15)). This solution indicates that simple truncation, e.g., that described by equation (31), cannot be applied to the infinite series appearing in the wake integral equation for the velocity disturbance function. Thus aerodynamic response predictions determined from the velocity potential solution alone [5] were based on an assumed periodic behavior of the partial sums of the wake series. The numerical results described in [5] are restricted to subresonant blade motions. A subsequent analysis conducted by the present investigator revealed that the truncation scheme based on the foregoing periodicity assumption could not be applied throughout the superresonant region. This factor coupled with the relative simplicity of equations (27) and (30) has led to the present approach in which the velocity potential formulation is used in conjunction with the pressure formulation to obtain response predictions.

Concluding Remarks

A method based on the successive solution of two boundary value problems has been developed to predict the unsteady pressure field produced by the small amplitude, harmonic motions of a cascade placed in a supersonic stream with subsonic velocity component normal to the cascade leading edge locus. The first boundary-value formulation treats the velocity potential and the second treats the pressure as the basic dependent flow variable. Explicit solutions for these dependent variables are determined in terms of blade and wake velocity and pressure disturbance-function distributions, respectively. Integral equations based on blade and wake boundary conditions are then used to evaluate the disturbance functions.

For design applications the basic information required from an unsteady aerodynamic solution is the blade pressure distributions and the aerodynamic response coefficients due to a prescribed blade motion or upwash distribution. Pressures acting on the upper surface of a given blade of the cascade (say the reference blade, $y = 0$, $0 \leq x \leq 1$) are readily evaluated from the velocity-potential solution. Such pressures depend only on values of the velocity-disturbance function over the extent of the reference blade and these are conveniently determined from a flow tangency boundary condition. However, the flow adjacent to the lower surface of the reference blade downstream of the point $x = 1 - B$ depends on the reference wake velocity-disturbance function distribution. Since values of the latter do not attenuate far downstream of the cascade, it has proven difficult to obtain numerical solutions to the wake integral equation of the velocity potential formulation, especially for superresonant blade motions, and hence to complete the solution for the lower surface pressure distribution.

Therefore, in the present approach, the velocity-potential solution is only used to provide the pressure distribution acting on the upper surface of the reference blade. Lower surface pressures follow from the explicit representation of the unsteady pressure field in terms of a reference blade and wake pressure disturbance function distribution. Values of the pressure disturbance function on the reference blade are governed by an integral relation which is similar in form to the flow tangency relation applied in the velocity potential formulation. On the reference wake the pressure disturbance function rapidly approaches zero with increasing distance downstream of the cascade. This feature permits a simple truncation procedure to be effectively employed for the evaluation of the wake pressure disturbance distribution and, in turn, lower surface reference blade pressures. With the successive solution procedure, predictions of blade pressure difference distributions and aerodynamic coefficients are readily obtained for both subresonant and superresonant blade motions.

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