Structure of diffeomorphism-invariant Lagrangians on the product bundle of metrics and linear connections

J. Muñoz Masqué* and M. Eugenia Rosado María†

*Instituto de Física Aplicada, CSIC
C/ Serrano 144, 28006-Madrid, Spain email=jaime@iec.csic.es,
†Departamento de Matemática Aplicada
Escuela Técnica Superior de Arquitectura, UPM
Avda. Juan de Herrera 4, 28040-Madrid, Spain email=eugenia.rosado@upm.es

Abstract. Let $p_C \colon C = CN \to N$ be the bundle of linear connections on a smooth manifold N and let $p_M \colon M \to N$ be the bundle of pseudo-Riemannian metrics of a given signature (n^+, n^-), $n^+ + n^- = n = \dim N$ on N. The structure of the first-order Lagrangians defined on the bundle $M \times_N C \to N$ that are invariant under the natural action of the diffeomorphisms of N, is determined.

Keywords: Lagrangian density, Jet bundles, Bundle of metrics, Bundle of linear connections, Diffeomorphism invariance, Infinitesimal contact transformation .
PACS: 02.40.Hw, 02.40.Ma, 04.20.Fy, 04.50.+h.

PRELIMINARIES

Jet bundles

Let $p \colon E \to N$ be an arbitrary fibred manifold, i.e., p is a surjective submersion; we set $\dim N = n$ and $\dim E = m + n$. An automorphism of p is a pair of diffeomorphisms $\phi \colon N \to N$, $\Phi \colon E \to E$ such that, $p \circ \Phi = \phi \circ p$. The group of all such automorphisms is denoted by $\mathrm{Aut}(p)$, and its 'Lie algebra' is the space $\mathrm{aut}(p) \subset \mathfrak{X}(E)$ of p-projectable vector fields on E. Equivalently, a vector field belongs to $\mathrm{aut}(p)$ if and only if each transformation Φ_t of its flow belongs to $\mathrm{Aut}(p)$. A natural group homomorphism $\mathrm{Aut}(p) \to \mathrm{Diff}N$, $\Phi \mapsto \phi$ exists, the kernel of which is the subgroup of vertical automorphisms of the fibred manifold, denoted by $\mathrm{Aut}^v(p)$.

Latin (resp. Greek) indices run from 1 to n (resp. m). A system of coordinates (x^i, y^α) on an open subset $V \subseteq E$ is said to be a 'fibred coordinate system' for the submersion p if (x^i) is a coordinate system for N on $U = p(V)$.

Let $p^1 \colon J^1E \to N$ be the 1-jet bundle of local sections of p, with natural projections $p^{1,0} \colon J^1E \to E$, $p^{1,0}(j^1_x s) = s(x)$, $j^1_x s$ denoting the 1-jet at x of a section s of p defined on a neighbourhood of $x \in N$. A fibred coordinate system (x^i, y^α) on V induces a coordinate system $(x^i, y^\alpha, y^\alpha_{,j})$ on $(p^{1,0})^{-1}(V) = J^1V$ as follows: $y^\alpha_{,j}(j^1_x s) = (\partial(y^\alpha \circ s)/\partial x^j)(x)$. Every morphism $\Phi \colon E \to E'$ whose associated map $\phi \colon N \to N'$ is a diffeomorphism, induces a map

$$\Phi^{(1)} \colon J^1E \to J^1E', \quad \Phi^{(1)}(j^1_x s) = j^1_{\phi(x)}(\Phi \circ s \circ \phi^{-1}). \tag{1}$$

If Φ_t is the flow of a vector field $X \in \mathrm{aut}(p)$, then $\Phi_t^{(1)}$ is the flow of a vector field $X^{(1)} \in \mathfrak{X}(J^1E)$, called the infinitesimal contact transformation of first order associated to the vector field X. The mapping $\mathrm{aut}(p) \ni X \mapsto X^{(1)} \in \mathfrak{X}(J^1E)$ is an injection of Lie algebras, namely, one has $(\lambda X + \mu Y)^{(1)} = \lambda X^{(1)} + \mu Y^{(1)}$, and $[X,Y]^{(1)} = [X^{(1)}, Y^{(1)}]$ for all $\lambda, \mu \in \mathbb{R}$, $X, Y \in \mathrm{aut}(p)$. In particular,

$$
\begin{aligned}
X &= u^i \frac{\partial}{\partial x^i} + v^\alpha \frac{\partial}{\partial y^\alpha}, \quad u^i \in C^\infty(N), v^\alpha \in C^\infty(E), \\
X^{(1)} &= u^i \frac{\partial}{\partial x^i} + v^\alpha \frac{\partial}{\partial y^\alpha} + v_i^\alpha \frac{\partial}{\partial y_i^\alpha}, \quad v_i^\alpha = \frac{\partial v^\alpha}{\partial x^i} + y_i^\beta \frac{\partial v^\alpha}{\partial y^\beta} - y_k^\alpha \frac{\partial u^k}{\partial x^i}.
\end{aligned}
$$

The bundle $J^1(M(N) \times_N C(N))$

Let N be an n-dimensional orientable and oriented connected smooth manifold. Let $p_M \colon M = M(N) \to N$ (resp. $p_F \colon F(N) \to N$, resp. $p_C \colon C = C(N) \to N$) be the bundle of pseudo-Riemannian metrics of a given signature (n^+, n^-), $n^+ + n^- = n$ (resp. the bundle of linear frames, resp. linear connections) on N, see [3, 10]. Every coordinate system (x^i) on an open domain $U \subseteq N$ induces the following coordinate systems:

1. (x^i, y_{jk}) on $(p_M)^{-1}(U)$, where the functions $y_{jk} = y_{kj}$ are defined by,

$$
g_x = \sum_{i \le j} y_{ij}(g_x)(dx^i)_x \otimes (dx^j)_x, \ \forall g_x \in (p_M)^{-1}(U).
$$

2. (x^i, x^i_j) on $(p_F)^{-1}(U)$, where the functions x^i_j are defined by,

$$
u = \left((\partial/\partial x^1)_x, \dots, (\partial/\partial x^n)_x \right) \cdot \left(x^i_j(u) \right), \ x = p_F(u), \forall u \in (p_F)^{-1}(U).
$$

3. (x^i, A^j_{kl}) on $(p_C)^{-1}(U)$, where the functions A^j_{kl} are defined as follows. We first recall some basic facts. Connections on $F(N)$ (i.e., linear connections of N) are the splittings of the Atiyah sequence,

$$
0 \to \mathrm{ad}F(N) \to T_{Gl(n,\mathbb{R})}F(N) \xrightarrow{(p_F)_*} T(N) \to 0,
$$

(cf. [1, 4, 7, 9]) where $\mathrm{ad}F(N) = T^*(N) \otimes T(N)$ is the adjoint bundle, $T_{Gl(n,\mathbb{R})}(F(N)) = T(F(N))/Gl(n,\mathbb{R})$, see [3, 6], and $\mathrm{gau}F(N) = \Gamma(N, \mathrm{ad}F(N))$ is the gauge algebra of $F(N)$.

We think of $\mathrm{gau}F(N)$ as the 'Lie algebra' of the gauge group $\mathrm{Gau}F(N)$. Moreover, $p_C \colon C \to N$ is an affine bundle modelled over the vector bundle $\otimes^2 T^*(N) \otimes T(N)$. The section of p_C induced tautologically by the linear connection Γ is denoted by $s_\Gamma \colon N \to C$. Every $B \in \mathfrak{gl}(n,\mathbb{R})$ defines a one-parameter group $\varphi_t^B \colon U \times Gl(n,\mathbb{R}) \to U \times Gl(n,\mathbb{R})$ of gauge transformations by setting (cf. [3]), $\varphi_t^B(x,\Lambda) = (x, \exp(tB) \cdot \Lambda)$. Let us denote by $\bar{B} \in \mathrm{gau}(p_F)^{-1}(U)$ the corresponding infinitesimal generator. If (E^i_j) is the standard basis of $\mathfrak{gl}(n,\mathbb{R})$, then $\bar{E}^i_j = \sum_{h=1}^n x^j_h \partial/\partial x^i_h$, is a basis of $\mathrm{gau}(p_F)^{-1}(U)$. Let $\tilde{E}^i_j = \bar{E}^i_j \mathrm{mod} G$ be the class of $\bar{E}^i_j$ on $\mathrm{ad}F(N)$. Unique smooth functions A^i_{jk} on $(p_C)^{-1}(U)$ exist such that,

$$s_\Gamma\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial x^j} - (A^i_{jk}\circ\Gamma)\tilde{E}^i_j$$

for every s_Γ and $A^i_{jk}(\Gamma_x) = \Gamma^i_{jk}(x)$, where Γ^i_{jk} are the Christoffel symbols of the linear connection Γ in the coordinate system (x^i), see [8, III, Poposition 7.4].

Natural lifts

Let $f_M\colon M \to M$, cf. [10] (resp. $\tilde{f}\colon F(N) \to F(N)$, cf. [8, p. 226]) be the natural lift of $f \in \mathrm{Diff}N$ to the bundle of metrics (resp. linear frame bundle); namely $f_M(g_x) = (f^{-1})^* g_x$ (resp. $\tilde{f}(X_1,\dots,X_n) = (f_*X_1,\dots,f_*X_n)$, where $(X_1,\dots,X_n) \in F_x(N)$); hence $p_M \circ f_M = f \circ p_M$ (resp. $p_F \circ \tilde{f} = f \circ p_F$), and $f_M\colon M \to M$ (resp. $\tilde{f}\colon F(N) \to F(N)$) have a natural extension to 1-jet bundles $f_M^{(1)}\colon J^1(M) \to J^1(M)$ (resp. $\tilde{f}^{(1)}\colon J^1(FN) \to J^1(FN)$) as defined in the formula (1), i.e.,

$$f_M^{(1)}\left(j^1_x g\right) = j^1_{f(x)}(f_M \circ g \circ f^{-1}) \quad (\text{resp. } \tilde{f}^{(1)}\left(j^1_x s\right) = j^1_{f(x)}(\tilde{f}\circ s\circ f^{-1})).$$

As $\tilde{f}$ is an automorphism of the principal $Gl(n,\mathbb{R})$-bundle $F(N)$, it acts on linear connections by pulling back connection forms, i.e., $\Gamma' = \tilde{f}(\Gamma)$ where $\omega_{\Gamma'} = (\tilde{f}^{-1})^*\omega_\Gamma$ (see[8, II, Propisition 6.2-(b)], [3, 3.3]). Hence, a unique diffeomorphism $\tilde{f}_C\colon C \to C$ exists such that, for every linear connection Γ, 1) $p_C \circ \tilde{f}_C = f \circ p_C$ and 2) $\tilde{f}_C \circ s_\Gamma = s_{\tilde{f}(\Gamma)}$.

If f_t is the flow of a vector field $X \in \mathfrak{X}(N)$, then the infinitesimal generator of $(f_t)_M$ (resp. $\tilde{f}_t$, resp. $(\tilde{f}_t)_C$) in $\mathrm{Diff}M$ (resp. $\mathrm{Diff}F(N)$, resp. $\mathrm{Diff}C$) is denoted by X_M (resp. $\tilde{X}$, resp. $\tilde{X}_C$) and the following Lie-algebra homomorphisms are obtained:

$$\begin{cases} \mathfrak{X}(N) \to \mathfrak{X}(M), & X \mapsto X_M, \\ \mathfrak{X}(N) \to \mathfrak{X}(F(N)), & X \mapsto \tilde{X}, \\ \mathfrak{X}(N) \to \mathfrak{X}(C), & X \mapsto \tilde{X}_C. \end{cases}$$

If $X = u^i\partial/\partial x^i \in \mathfrak{X}(N)$ is the local expression for X, then

1. From ([10, eqs. (2)-(4)]) we know that the natural lift of X to M is given by,

$$X_M = u^i\frac{\partial}{\partial x^i} - \sum_{i\le j}\left(\frac{\partial u^h}{\partial x^i}y_{hj} + \frac{\partial u^h}{\partial x^j}y_{ih}\right)\frac{\partial}{\partial y_{ij}} \in \mathfrak{X}(M).$$

and its 1-jet prolongation,

$$\begin{aligned} X_M^{(1)} &= u^i \frac{\partial}{\partial x^i} - \sum_{i \le j} \left(\frac{\partial u^h}{\partial x^i} y_{hj} + \frac{\partial u^h}{\partial x^j} y_{hi} \right) \frac{\partial}{\partial y_{ij}} \\ &\qquad - \sum_{i \le j} \left(\frac{\partial^2 u^h}{\partial x^i \partial x^k} y_{hj} + \frac{\partial^2 u^h}{\partial x^j \partial x^k} y_{hi} \right. \\ &\qquad \left. + \frac{\partial u^h}{\partial x^i} y_{hj,k} + \frac{\partial u^h}{\partial x^j} y_{hi,k} + \frac{\partial u^h}{\partial x^k} y_{ij,h} \right) \frac{\partial}{\partial y_{ij,k}}. \end{aligned}$$

2. From [5, Proposition 3](also see [8, VI, Proposition 21.1]) we know that the natural lift of X to $F(N)$ is given by,

$$\tilde{X} = u^i \frac{\partial}{\partial x^i} + \frac{\partial u^i}{\partial x^l} x_j^l \frac{\partial}{\partial x_j^i},$$

and its 1-jet prolongation,

$$\tilde{X}^{(1)} = u^i \frac{\partial}{\partial x^i} + \frac{\partial u^i}{\partial x^l} x_j^l \frac{\partial}{\partial x_j^i} + \left(\frac{\partial u^i}{\partial x^r} x_{j,k}^r - \frac{\partial u^r}{\partial x^k} x_{j,r}^i + \frac{\partial^2 u^i}{\partial x^k \partial x^r} x_j^r \right) \frac{\partial}{\partial x_{j,k}^i}.$$

3. Finally,

$$\tilde{X}_C = u^i \frac{\partial}{\partial x^i} - \left(\frac{\partial^2 u^i}{\partial x^k \partial x^j} - \frac{\partial u^i}{\partial x^r} A_{jk}^r + \frac{\partial u^s}{\partial x^k} A_{js}^i + \frac{\partial u^l}{\partial x^j} A_{lk}^i \right) \frac{\partial}{\partial A_{jk}^i},$$

$$\begin{aligned} \tilde{X}_C^{(1)} &= u^i \frac{\partial}{\partial x^i} + w_{jk}^i \frac{\partial}{\partial A_{jk}^i} + w_{jkh}^i \frac{\partial}{\partial A_{jk,h}^i}, \\ w_{jk}^i &= -\frac{\partial^2 u^i}{\partial x^k \partial x^j} + \frac{\partial u^i}{\partial x^r} A_{jk}^r - \frac{\partial u^s}{\partial x^k} A_{js}^i - \frac{\partial u^r}{\partial x^j} A_{rk}^i, \qquad (2) \\ w_{jkh}^i &= -\frac{\partial^3 u^i}{\partial x^h \partial x^k \partial x^j} + \frac{\partial^2 u^i}{\partial x^h \partial x^r} A_{jk}^r - \frac{\partial^2 u^s}{\partial x^h \partial x^k} A_{js}^i - \frac{\partial^2 u^r}{\partial x^h \partial x^j} A_{rk}^i \\ &\quad + \frac{\partial u^i}{\partial x^r} A_{jk,h}^r - \frac{\partial u^s}{\partial x^k} A_{js,h}^i - \frac{\partial u^l}{\partial x^j} A_{lk,h}^i - \frac{\partial u^t}{\partial x^h} A_{jk,t}^i. \qquad (3) \end{aligned}$$

Let $p\colon M \times_N C \to N$ be the natural projection. We denote by $\bar{f} = (f_M, f_C)$ (resp. $\bar{X} = (X_M, \tilde{X}_C) \in \mathfrak{X}(M \times_N C)$) the natural lift of f (resp. X) to $M \times_N C$. The prolongation to the bundle $J^1(M \times_N C)$ of $\bar{X}$ is as follows:

$$\bar{X}^{(1)} = \left(X_M^{(1)}, \tilde{X}_C^{(1)} \right) = u^i \frac{\partial}{\partial x^i} + \sum_{i \le j} v_{ij} \frac{\partial}{\partial y_{ij}} + \sum_{i \le j} v_{ijk} \frac{\partial}{\partial y_{ij,k}} + w_{jk}^i \frac{\partial}{\partial A_{jk}^i} + w_{jkh}^i \frac{\partial}{\partial A_{jk,h}^i},$$

where

$$\begin{aligned} v_{ij} &= -\frac{\partial u^h}{\partial x^i} y_{hj} - \frac{\partial u^h}{\partial x^j} y_{hi}, \\ v_{ijk} &= -\frac{\partial^2 u^h}{\partial x^i \partial x^k} y_{hj} - \frac{\partial^2 u^h}{\partial x^j \partial x^k} y_{ih} - \frac{\partial u^h}{\partial x^i} y_{hj,k} - \frac{\partial u^h}{\partial x^j} y_{ih,k} - \frac{\partial u^h}{\partial x^k} y_{ij,h}, \end{aligned}$$

and w^i_{jk}, w^i_{jkh} are given in the formulas (2), (3), respectively.

Diff$N$- and $\mathfrak{X}(N)$-invariance

A differential form $\omega_r \in \Omega^r(J^1(M\times_N C))$, $r\in\mathbb{N}$, is said to be Diff$N$-invariant—or invariant under diffeomorphisms—(resp. $\mathfrak{X}(N)$-invariant) if the following equation holds: $(\bar{f}^{(1)})^*\omega_r = \omega_r$, $\forall f \in \mathrm{Diff}N$ (resp. $L_{\bar{X}^{(1)}}\omega_r = 0$, $\forall X\in\mathfrak{X}(N)$). Obviously, "Diff$N$-invariance" implies "$\mathfrak{X}(N)$-invariance" and the converse is almost true. Because of this, below we consider $\mathfrak{X}(N)$-invariance only.

A linear frame $(X_1,\dots,X_n)\in F_x(N)$ is said to be orthonormal with respect to $g_x \in M_x(N)$ (or simply g_x-orthonormal) if $g_x(X_i,X_j)=0$ for $1\le i<j\le n$, $g(X_i,X_i)=1$ for $1\le i\le n^+$, $g(X_i,X_i)=-1$ for $n^+ +1\le i\le n$.

As N is an oriented manifold, there exists a unique p-horizontal n-form $\mathbf{v}$ on $M\times_N C$ such that, $\mathbf{v}_{(g_x,\Gamma_x)}(X_1,\dots,X_n)=1$, for every g_x-orthonormal basis $(X_1,\dots,X_n)$ belonging to the orientation of N. Locally $\mathbf{v}=\rho v_n$, where $\rho=\sqrt{(-1)^{n^-}\det(y_{ij})}$ and $v_n = dx^1\wedge\cdots\wedge dx^n$. As proved in [10, Proposition 7], the form $\mathbf{v}$ is Diff$N$-invariant and hence $\mathfrak{X}(N)$-invariant. A Lagrangian density Λ on $J^1(M\times_N C)$ can be globally written as $\Lambda = \mathscr{L}\mathbf{v}$ for a unique function $\mathscr{L}\in C^\infty(J^1(M\times_N C))$ and Λ is $\mathfrak{X}(N)$-invariant if and only if the function $\mathscr{L}$ is $\mathfrak{X}(N)$-invariant; that is $\bar{X}^{(1)}(\mathscr{L})=0$, $\forall X\in\mathfrak{X}(N)$. Therefore, the invariance of Lagrangian densities is reduced to that of scalar functions.

As the values for u^h, $\partial u^h/\partial x^i$, $\partial^2 u^h/\partial x^i\partial x^j$ $(i\le j)$, and $\partial^3 u^h/\partial x^i\partial x^j\partial x^k$ $(i\le j\le k)$ at a point $x\in N$ can be taken arbitrarily, we deduce that the equation $\bar{X}^{(1)}(\mathscr{L})=0$, $\forall X\in\mathfrak{X}(N)$ is equivalent to the following system of partial differential equations

$$\begin{array}{ll} 0=\frac{\partial}{\partial x^i}(\mathscr{L}), & \forall i,\\ 0=X^i_h(\mathscr{L}), & \forall h,i,\\ 0=X^{ik}_h(\mathscr{L}), & \forall h, i\le k,\\ 0=X^{jkh}_i(\mathscr{L}), & \forall i, j\le k\le h, \end{array} \tag{4}$$

where

$$\begin{aligned} X^i_h \;=\;& -y_{hi}\frac{\partial}{\partial y_{ii}} - y_{hj}\frac{\partial}{\partial y_{ij}} - y_{ih,k}\frac{\partial}{\partial y_{ii,k}} - y_{hj,k}\frac{\partial}{\partial y_{ij,k}} - \sum_{s\le j} y_{sj,h}\frac{\partial}{\partial y_{sj,i}} + A^i_{jk}\frac{\partial}{\partial A^h_{jk}}\\ & -A^r_{jh}\frac{\partial}{\partial A^r_{ji}} - A^r_{hk}\frac{\partial}{\partial A^r_{ik}} + A^i_{jk,s}\frac{\partial}{\partial A^h_{jk,s}} - A^s_{jh,r}\frac{\partial}{\partial A^s_{ji,r}} - A^s_{hk,r}\frac{\partial}{\partial A^s_{ik,r}} - A^r_{jk,h}\frac{\partial}{\partial A^r_{jk,i}},\\ X^{ik}_h \;=\;& -y_{ih}\frac{\partial}{\partial y_{ii,k}} - y_{kh}\frac{\partial}{\partial y_{kk,i}} - y_{hj}\frac{\partial}{\partial y_{ij,k}} - y_{hj}\frac{\partial}{\partial y_{kj,i}} - \frac{\partial}{\partial A^h_{ik}} - \frac{\partial}{\partial A^h_{ki}}\\ & +A^k_{js}\frac{\partial}{\partial A^h_{js,i}} - A^s_{jh}\frac{\partial}{\partial A^s_{jk,i}} - A^s_{hr}\frac{\partial}{\partial A^s_{kr,i}} + A^i_{js}\frac{\partial}{\partial A^h_{js,k}} - A^s_{jh}\frac{\partial}{\partial A^s_{ji,k}} - A^s_{hr}\frac{\partial}{\partial A^s_{ir,k}},\\ X^{jkh}_i \;=\;& \frac{\partial}{\partial A^i_{jk,h}} + \frac{\partial}{\partial A^i_{jh,k}} + \frac{\partial}{\partial A^i_{hk,j}} + \frac{\partial}{\partial A^i_{hj,k}} + \frac{\partial}{\partial A^i_{kj,h}} + \frac{\partial}{\partial A^i_{kh,j}}, \end{aligned}$$

From the above expressions it is easy to prove that the number of equations in system (4) is

$$n+n^2+n\binom{n+1}{2}+n\binom{n+2}{3}=n\binom{n+3}{3}.$$

As a simple computation shows, we have

Proposition 1 *The vector fields* $\partial/\partial x^i, X_h^i, X_h^{ik}, X_i^{jkh}$ *are linearly independent and span an involutive distribution on* $J^1(M\times_N C)$ *of rank* $n\binom{n+3}{3}$. *Hence, the number of functionally invariant Lagrangians on* $J^1(M\times_N C)$ *is* $\frac{1}{6}\left(5n^4+3n^3-5n^2+3n\right)$.

THE FACTORIZATION RESULT FOR THE INVARIANTS

Let B_N be the sub-vector bundle of all the tensors $t\in\wedge^2T^*N\otimes T^*N\otimes TN$ such that $\mathfrak{S}_{X,Y,Z}t(X,Y,Z)=0,\quad\forall X,Y,Z\in T_xN,\ \forall x\in N$, and let consider the following mapping

$$\begin{aligned}&\Upsilon\colon J^1(M\times_N C)\to M\times_N\left(\left(\wedge^2T^*N\otimes TN\right)\oplus\left(\wedge^2T^*N\otimes T^*N\otimes TN\right)\oplus B_N\right),\\&\Upsilon\left(j_x^1(g,s_\Gamma)\right)=\left(g_x,\left(\nabla^\Gamma-\nabla^g\right)_x,\left(R^\Gamma\right)_x,\Upsilon^3\left(j_x^1(g,s_\Gamma)\right)\right),\end{aligned}$$

where $\Upsilon^3\left(j_x^1(g,s_\Gamma)\right)(X,Y,Z)=\left(\nabla_Z^\Gamma T^\Gamma\right)(X,Y)+T^\Gamma(T^\Gamma(X,Y),Z)-R^\Gamma(X,Y)(Z)$.

Theorem 2 *Every* $\mathfrak{X}(N)$*-invariant Lagrangian function* $\mathscr{L}$ *on* $J^1(M\times_N C)$ *factors through* Υ *as follows:* $\mathscr{L}=\bar{\mathscr{L}}\circ\Upsilon$, *where*

$$\bar{\mathscr{L}}\colon M\times_N\left(\left(\otimes^2T^*N\otimes TN\right)\oplus\left(\wedge^2T^*N\otimes T^*N\otimes TN\right)\oplus B_N\right)\to\mathbb{R}$$

is a smooth function that is, in turn, invariant under the natural action of $\mathfrak{X}(N)$ *on such tensorial bundle, namely,* $\hat{X}(\bar{\mathscr{L}})=0$, *for all* $X\in\mathfrak{X}(N)$, *where* $\hat{X}$ *denotes the natural lift of* $X\in\mathfrak{X}(N)$ *to* $M\times_N\left(\left(\otimes^2T^*N\otimes TN\right)\oplus\left(\wedge^2T^*N\otimes T^*N\otimes TN\right)\oplus B_N\right)$.

ACKNOWLEDGMENTS

Supported by Ministerio of Ciencia y Tecnología of Spain, under grant #MTM2008–01386.

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