Adaptive Sinusoidal Disturbance Rejection in Linear Discrete-Time Systems—Part I: Theory

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An adaptive regulation approach against disturbances consisting of linear combinations of sinusoids with unknown and/or time varying amplitudes, frequencies, and phases for SISO LTI discrete-time systems is considered. The new regulation approach proposed is based on constructing a set of stabilizing controllers using the Youla parametrization of stabilizing controllers and adjusting the Youla parameter to achieve asymptotic disturbance rejection. Three adaptive regulator design algorithms are presented and their convergence properties analyzed. Conditions under which the on-line algorithms yield an asymptotic controller that achieves regulation are presented. Conditions both for the case where the disturbance input properties are constant but unknown and for the case where they are unknown and time varying are given. In the case of error feedback, the on-line controller construction amounts to an adaptive implementation of the Internal Model Principle. The performance of the adaptation algorithms is illustrated through a simulation example. A companion paper [4] describes the implementation and evaluation of the algorithms for the problem of noise cancellation in an acoustic duct.

1 Introduction and Motivation

Many engineering systems are subject to periodic disturbances which could adversely affect performance. Examples of such systems include flexible structures subject to periodic excitations [34], motion control systems with position dependent disturbances [21] or with friction [31], disk drive systems with repeatable runout error [25], manufacturing processes such milling operations [16, 26], and steel casting processes [19]. Since mechanical systems behave like low pass filters, the effects of only few of the periodic disturbance harmonics are visible at the system output. Moreover, a digital controller is typically required for the implementation of a controller to reject the periodic disturbance input. Since the digital control system includes an anti-aliasing filter, the effects of only the first few harmonics of the disturbance input are fed back to the controller. This makes it practical to consider disturbance models which include only the first few harmonics of the periodic disturbance signal. The disturbance would then be represented as a linear combination of sinusoids where the frequencies of the sinusoids are integer multiples of the disturbance frequency. It turns out that in some situations, such as in vibration control problems, the disturbance period may not be known a priori and may be slowly time-varying, which prompts the consideration of disturbance representations with time-varying frequencies. Finally, in order to make the disturbance rejection problem even more general, arbitrary (i.e., not rationally related) frequencies are considered and the disturbance would then be represented as

$$w(k) = \sum_{n=1}^{k_0} c_n(k) \cos(\omega_n(k)k + \phi_n(k)), \tag{1}$$

where $c_n(k)$, $\omega_n(k)$, and $\phi_n(k)$ represent, respectively, the amplitudes, frequencies, and phase of the sinusoids, all assumed unknown and time-varying. Given a linear time-invariant plant subject to a disturbance of the form (1), it is desired to design a controller that yields internal stability and asymptotic disturbance rejection which is robust in the face of variations in the disturbance amplitudes, frequencies, and phases. The disturbance model in (1) represents a class of *almost periodic signals* [9].

When the disturbance is as in (1) and is partially known, certain approaches for regulation such as Adaptive Feedforward Control (AFC) algorithms [20, 5], LQG methods for controller design [7], and disturbance accommodating control approach [15] have been proposed in the literature. When the disturbance model parameters in (1) are completely unknown, adaptive techniques can be used to achieve regulation. Adaptive regulation consists mainly of estimating the disturbance model on-line and using the estimated model in the controller design. Different approaches were proposed in the literature, and include adaptive versions of the Internal Model Principle [10, 11], an adaptive version of the External Model Principle [8, 35], augmenting a stabilizing controller with an adaptation mechanism to improve the overall system tracking and disturbance rejection performance [33], and using a special adaptive disturbance observer [22].

In this paper, we consider the specific problem of adaptively rejecting sinusoidal discrete disturbance inputs in (1) with unknown and time-varying frequencies $\omega_n(k)$, amplitudes $c_n(k)$, and phases $\phi_n(k)$, $1 \leq n \leq k_0$. More precisely, it is assumed that the frequencies, amplitudes, and phases are piecewise constant functions of time. The regulation problem is solved within a parametrized set of stabilizing controllers constructed using the Youla parametrization [36]. The purpose of the adaptation is to tune the Youla parameter in the stabilizing controller in order to asymptotically satisfy a set of interpolation conditions that are equivalent to disturbance rejection. For a given nominal plant model, we present three adaptive regulator design algorithms and their convergence properties. The first two algorithms represent recursive least squares (RLS) algorithms with time varying dead zone width. These algorithms are to be used in the case where the disturbance properties are unknown but constant. It is shown that asymptotic disturbance rejection can be achieved if an exponentially decaying bound on the disturbance response of the desired closed loop control system is known. In order to deal with disturbance inputs with time varying frequencies, a RLS algorithm with time-varying

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weighting is considered. In this case, the persistent excitation assumption is invoked to show asymptotic rejection of disturbance inputs with time varying properties. When the performance variable is same as the signal that is fed back to the controller, the on-line controller construction amounts to an adaptive implementation of the internal model principle.

The solution approach to the regulation problem presented in this paper differs from those presented in the literature [10, 11] in that the latter are based on explicitly estimating the disturbance model and then using the estimated model in the controller design. In contrast to the aforementioned, a direct adaptive controller design approach is presented in this paper which does not require the estimation of the disturbance model. For example, the adaptation algorithm does not have to deal, in the case of periodic signals of the form (1), with the explicit estimation of the period of the disturbance as is done in [14, 29, 30]. Moreover, the solution is proposed for a more general class of feedback systems than those treated in [10, 11]. More specifically, the regulation problem posed in this paper is for feedback systems where the performance variable is not necessarily the signal being fed back to the controller, whereas the opposite is true for the work in [10, 11]. It should be noted that the work presented in [10, 11] treats a more general problem where the controller is adapted with respect to both the disturbance and plant model uncertainties, but for a special case of the general feedback systems considered here. The adaptation approach used in this paper is the same as that used in [33, 27]. The purpose of the adaptation in [27] is to improve the performance of a nominal optimal disturbance rejection controller in the face of plant model uncertainties by adjusting the Youla parameter. In this work, the primary objective of the adaptation is to construct, on-line and for a given nominal plant, an asymptotic controller capable of performing asymptotic disturbance rejection.

The rest of the paper is organized as follows. Section 2 presents the framework within which the regulation problem is solved and gives conditions for regulation. Recursive algorithms for adaptive regulator design are presented and their properties analyzed in Section 3. The performance of the adaptation algorithms is illustrated through an example in Section 4.

2 Preliminaries

The solution approach is based on working within a parametrized set of stabilizing controllers where the Youla parameter is the design parameter. More details can be found in [3, 32, 13, 18].

Let R_p denote the set of proper real rational transfer matrices and RH_∞ the subset of asymptotically stable real rational transfer matrices. Consider a plant $G = \left[\begin{smallmatrix} G_{we} & G_{ue} \\ G_{wy} & G_{uy} \end{smallmatrix}\right]$ with inputs $\left[\begin{smallmatrix} w \\ u \end{smallmatrix}\right]$ and outputs $\left[\begin{smallmatrix} e \\ y \end{smallmatrix}\right]$, where w, u, e, and y are scalar signals representing, respectively, the disturbance input, the control signal, an error or performance variable, and a measurement signal. Consider a coprime factorization of G_{uy} given by $G_{uy} = NM^{-1}$ where N and M are in RH_∞. Let U and V in RH_∞ be such that the following Bezout identity $MV - UN = 1$ is satisfied. A stabilizing controller K_0 for the system G_{uy} is then given by $K_0 = UV$. Using the base stabilizing controller K_0, the set of all stabilizing controllers in R_p for the system G_{uy} can be constructed using the Youla parametrization [36]. In fact, for any $Q \in RH_\infty$, the controller K given by $K = (U + MQ)(V + NQ)^{-1}$, is a stabilizing controller for the plant G_{uy}. Moreover, every rational stabilizing controller K has the form given above for some $Q \in RH_\infty$. Under some very mild conditions [13, 18], a stabilizing controller for G_{uy} is also a stabilizing controller for G.

Let K be a stabilizing controller constructed using the Youla parametrization of stabilizing controllers where the Youla parameter is denoted by $Q \in RH_\infty$. Then the closed-loop system can be reconfigured as shown in Fig. 1 where $\mathbf{T} = \left[\begin{smallmatrix} T_{we} & T_{se} \\ T_{wr} & 0 \end{smallmatrix}\right]$, and where $T_{we} = G_{we} + G_{ue}UMG_{wy}$, $T_{se} = G_{ue}M$, and $T_{wr} = MG_{wy}$ are all in RH_∞. The closed-loop system transfer function is given by,

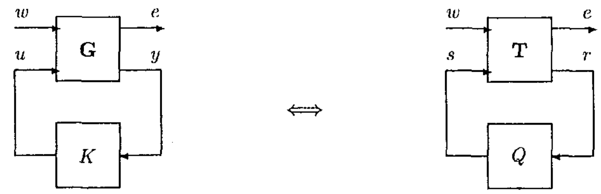

Fig. 1 Reparametrization and equivalent block diagrams of the closed-loop system

$$F_{T,Q}(z) = \frac{E(z)}{W(z)} = [T_{we}(z) + T_{se}(z)Q(z)T_{wr}(z)].$$

It was shown in [3] that if $Q(z)$ is chosen to be of the form of a Finite Impulse Response (FIR) filter, $Q(z) = \sum_{i=1}^{n_q} q_i z^{1-i}$, then the regulation requirement can be put in the form of interpolation conditions. Satisfying the latter is equivalent to solving a linear equation $\mathbf{A}\theta + B = 0$ in the parameter vector $\theta = [q_1, \ldots, q_{n_q}]^T$, where $\mathbf{A}$ and B are real matrices computed from the data and $\mathbf{A}$ is $n_p \times n_q$.

3 Adaptive Regulation

In adaptive regulation, it is desired to construct, on-line, an asymptotic controller that rejects disturbance inputs of the form (1) with unknown and/or time varying properties. The adaptation approach is based on tuning the Youla parameter Q to asymptotically achieve the desired regulator. Two types of controller parameter adaptation algorithms are considered. The first type of algorithms represent recursive least squares (RLS) algorithms with dead zone. Two algorithms of this type are presented. These algorithms are to be used in the case where the disturbance input properties are unknown but constant. In the case where the disturbance input properties are unknown and time varying, a second type of algorithms, representing the RLS algorithm with a time varying forgetting factor, is considered. The RLS algorithms with dead zone require only a decaying upper bound on the magnitude of the response of the nominal (desired) closed-loop system to show asymptotic regulation, whereas the RLS algorithm with time varying forgetting factor requires a persistent excitation assumption. It is shown that if the number of parameters is greater than the minimal number of parameters needed to achieve regulation, the RLS algorithms with dead zone yield zero steady state closed loop system response. The latter property implies that the adaptive system using the RLS algorithms with dead zone can tolerate over-parametrization, whereas this is not the case for the adaptive systems using the RLS algorithm with a time varying forgetting factor.

3.1 The Performance Variable. The purpose of this section is to derive an expression for the performance variable e suitable for use with adaptation algorithms, i.e., an expression where e can be considered an "affine" function of the parameter estimation error.

During adaptation, the subsystem represented by the operator Q is time varying. Therefore, in subsequent equations, transfer function notation is not used. Instead, systems will be considered operators on signals. Let B^{-l} denote the l time step delay operator. The performance variable $e(k)$ is given by

$$\begin{aligned} e(k) &= [T_{we}(B^{-1}) + T_{se}(B^{-1})Q_k T_{wr}(B^{-1})]w(k), \\ &= T_{we}(B^{-1})w(k) + T_{se}(B^{-1})Q_k r(k), \end{aligned} \tag{2}$$

where

$$r(k) = T_{wr}(B^{-1})w(k).$$

In the above expression for $e(k)$, the operators T_{ij}, $i, j = 1, 2$, are time-invariant whereas Q_k is a time-varying operator. Define a pseudo-performance variable

$$e_1(k) = [T_{we}(B^{-1}) + Q_k T_{se}(B^{-1}) T_{wr}(B^{-1})] w(k),$$
$$= T_{we}(B^{-1}) w(k) + Q_k T_{se}(B^{-1}) r(k).$$

Assuming Q_k is of the form

$$Q_k = \sum_{i=1}^{n_q} q_i(k-1) B^{1-i}, \tag{3}$$

the signal e_1 can be expressed as

$$e_1(k) = \underbrace{[T_{we}(B^{-1}) w(k)]}_{v_0(k)} + \sum_{i=1}^{n_q} q_i(k-1) B^{1-i} \underbrace{[T_{se}(B^{-1}) r(k)]}_{v_1(k)},$$
$$= v_0(k) - \phi^T(k) \theta(k-1),$$

where

$$\phi(k) = [-v_1(k) \; \dots \; -v_1(k-n_q+1)]^T, \tag{4}$$

$$\theta(k) = [q_1(k) \; \dots \; q_{n_q}(k)]^T. \tag{5}$$

Let θ_0 be a parameter vector satisfying the interpolation conditions. Let Q_0 be the Youla parameter that results from using θ_0 in (3) and $\{e_0(\cdot)\}$ the decaying disturbance response that results from the use of Q_0 in (2). Then

$$\begin{aligned} e_0(k) &= [T_{we}(B^{-1}) + T_{se}(B^{-1}) Q_0 T_{wr}(B^{-1})] w(k) \\ &= T_{we}(B^{-1}) w(k) + T_{se}(B^{-1}) Q_0 r(k) \\ &= T_{we}(B^{-1}) w(k) + Q_0 T_{se}(B^{-1}) r(k) \\ &= v_0(k) - \phi^T(k) \theta_0, \end{aligned} \tag{6}$$

where $\phi(k)$ is given by (4). Define the signals $e_2(k) = [T_{se}(B^{-1}) Q_k - Q_k T_{se}(B^{-1})] r(k)$ and $d(k) = e_0(k) + e_2(k)$. The performance variable is then

$$\begin{aligned} e(k) &= e(k) - e_1(k) + e_1(k) - e_0(k) + e_0(k) \\ &= [e_1(k) - e_0(k)] + [e_0(k) + (e(k) - e_1(k))] \\ &= [e_1(k) - e_0(k)] + [e_0(k) + ([T_{se}(B^{-1}) Q_k \\ &\quad - Q_k T_{se}(B^{-1})] r(k))] \\ &= [e_1(k) - e_0(k)] + [e_0(k) + e_2(k)] \\ &= [e_1(k) - e_0(k)] + d(k) \\ &= \phi^T(k) \tilde{\theta}(k-1) + d(k), \end{aligned} \tag{7}$$

where $\tilde{\theta}(k) = \theta_0 - \theta(k)$. An exponentially decaying upper bound $\bar{e}_0 = \bar{\alpha} \bar{\beta}^k$, $\bar{\alpha} > 0$, $0 < \bar{\beta} < 1$, on the response e_0 is used to define an upper bound $\bar{d}(k) = \bar{e}_0(k) + e_2(k)$ on the signal $d(k)$.

In order to be able to compute the regression vector ϕ without knowing the disturbance input w, we need to find an expression of r that is independent of w. We have that

$$\begin{aligned} y(k) &= G_{wy}(B^{-1}) w(k) + G_{uy}(B^{-1}) u(k) \\ &= G_{wy}(B^{-1}) w(k) + M^{-1}(B^{-1}) N(B^{-1}) u(k). \end{aligned} \tag{8}$$

We also have that $T_{wr}(B^{-1}) = M(B^{-1}) G_{wy}(B^{-1})$ [13, 18]. Since $r(k) = T_{wr}(B^{-1}) w(k) = M(B^{-1}) G_{wy}(B^{-1}) w(k)$, then from (8) we obtain

$$r(k) = M(B^{-1}) y(k) - N(B^{-1}) u(k). \tag{9}$$

It can be seen from the derivations given above that the regression vector does not require knowledge of the disturbance input values. In fact, according to (9) it is only necessary to know u and y in order to compute r, and consequently, compute v_1 which is the only variable present in the regression vector.

3.2 Case of Unknown But Time-Invariant Disturbance Properties. In this case, the disturbance input is of the form $w(k) = \sum_{n=1}^{k_0} c_n \cos(\omega_n k + \phi_n)$, where the number of sinusoids k_0, the amplitudes c_n, frequencies ω_n, and phases ϕ_n, $n = 1, \dots, k_0$, are unknown. It is assumed that a bound $\underline{k}_0$ on the number of sinusoids that can be assumed to exist in the expression above is known (i.e. in the design of the adaptation algorithm, the number of sinusoids in the expression above can be taken arbitrarily large as long as it is greater than or equal to k_0, the true number of sinusoids). Two RLS algorithms with dead-zone are presented and their performances analyzed. The following assumption is invoked since it will be used in subsequent theorems.

Assumption 1: The constants $\bar{\alpha}$ and $\bar{\beta}$ in the expression for $\bar{e}_0$ are assumed known a priori.

The adaptation algorithm is given by

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \lambda(k+1) \frac{P(k)\phi(k+1)}{1 + \lambda(k+1)\phi^T(k+1)P(k)\phi(k+1)} e(k+1), \tag{10}$$

$$P(k+1) = P(k) - \lambda(k+1) \frac{P(k)\phi^T(k+1)\phi(k+1)P(k)}{1 + \lambda(k+1)\phi^T(k+1)P(k)\phi(k+1)}, \tag{11}$$

with $\hat{\theta}(0) = \hat{\theta}_0$, $P(0) = P_0 > 0$, and where

$$\lambda(k) = \begin{cases} 1 & \text{if} \quad \left| \dfrac{e(k)}{1 + \phi^T(k)P(k-1)\phi(k)} \right| \geq |\bar{d}(k)|, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

The convergence properties of the algorithm are given in the following theorem.

Theorem 1 [2]: If Assumption 1 is satisfied and $n_q \geq 2k_0$, then the algorithm given by (10), (11), and (12) yields

a/ The adaptive system is Bounded-Input Bounded-Output stable.
b/ $\lim_{k\to\infty} \|\hat{\theta}(k) - \hat{\theta}(k-l)\| = 0$, $\forall 0 < l < \infty$.
c/ $\lim_{k\to\infty} d(k) = 0$.
d/ $\lim_{k\to\infty} [e^2(k)/1 + \phi^T(k)P(k-1)\phi(k)] = 0$.
e/ $\lim_{k\to\infty} e(k) = 0$.

The adaptation algorithm presented above is based on using the error signal e in (10). A second algorithm, which uses a modified error $\tilde{e} = e - e_2$, can also be derived and is somewhat easier to analyze. Note that at time k, the signal $e_2(k)$ is a function of $\theta(k-1)$ and consequently can be computed and used in the expression for $\tilde{e}$. Consider the algorithm given below

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \lambda(k+1) \frac{P(k)\phi(k+1)}{1 + \lambda(k+1)\phi^T(k+1)P(k)\phi(k+1)} \tilde{e}(k+1), \tag{13}$$

$$P(k+1) = P(k) - \lambda(k+1) \frac{P(k)\phi(k+1)\phi^T(k+1)P(k)}{1 + \lambda(k+1)\phi^T(k+1)P(k)\phi(k+1)}. \tag{14}$$

with $\hat{\theta}(0) = \hat{\theta}_0$ and $P(0) = P_0 > 0$ and where

$$\lambda(k) = \begin{cases} 1 & \text{if} \quad \dfrac{\tilde{e}^2(k)}{1 + \phi^T(k)P(k-1)\phi(k)} \geq \tilde{e}_0^2(k), \\ 0 & \text{otherwise.} \end{cases} \quad (15)$$

Theorem 2 [2]: If Assumption 1 is satisfied and $n_q \geq 2k_0$, then the algorithm given by (13), (14), and (15) yields

a/ The adaptive system is Bounded-Input Bounded-Output stable.
b/ $\lim_{k\to\infty} [\tilde{e}^2(k)/1 + \phi^T(k)P(k-1)\phi(k)] = 0$.
c/ $\lim_{k\to\infty} \|\hat{\theta}(k) - \hat{\theta}(k-l)\| = 0, \forall 0 < l < \infty$.
d/ $\lim_{k\to\infty} e(k) = 0$.

Remark: There is no indication in the above theorems as to whether the parameters converge or not, and if they do converge, where they converge to.

Remark: The adaptation algorithms presented above work well only in the case where the nominal parameter vector θ_0 is constant. When θ_0 changes with time, the performance of the adaptation algorithms deteriorates due to the fact that the magnitude of the adaptation gain, given by $[P(k-1)\phi(k)/1 + \lambda(k)\phi^T(k)P(k-1)\phi(k)]$ in (10) and (13), decreases with time after the algorithm is started. Therefore, the estimation algorithms are not capable of effectively tracking changes in the nominal parameter vector. A third adaptation algorithm, capable of tracking a piece-wise constant nominal parameter vector θ_0 is given in the next section.

3.3 Case of Unknown and Time-Varying Disturbance Properties. In this section, it is assumed that k_0 in (1) is known and that the coefficients c_n, frequencies ω_n, and phases ϕ_n, $n = 1, \ldots, k_0$, in (1) are unknown and possibly time-varying. In order to be able to reject such disturbance inputs, it is necessary to use an adaptation algorithm capable of tracking time varying parameters. The recursive least squares algorithm with time varying forgetting factor is considered for this purpose. Some of the properties of such algorithm are discussed in [17, 6] and the references therein. The algorithm is given by

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \lambda(k+1)P(k+1)\phi(k+1)\tilde{e}(k+1), \quad (16)$$

$$P^{-1}(k+1) = \lambda(k+1)[P^{-1}(k) + \phi(k+1)\phi^T(k+1)], \quad (17)$$

with $\hat{\theta}(0) = \hat{\theta}_0$, $P(0) = P_0 > 0$, and where $\lambda(k)$ is the time varying forgetting factor satisfying $0 < \lambda_{\min} \leq \lambda(k) \leq \lambda_{\max} < 1$. Using the Matrix Inversion Lemma [1], the above equations can be rewritten as

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \frac{P(k)\phi(k+1)}{1 + \phi^T(k+1)P(k)\phi(k+1)}\tilde{e}(k+1), \quad (18)$$

$$P(k+1) = \frac{1}{\lambda(k+1)} \times \left[P(k) - \frac{P(k)\phi(k+1)\phi^T(k+1)P(k)}{1 + \phi^T(k+1)P(k)\phi(k+1)}\right]. \quad (19)$$

Note that the parameter adaptation algorithm is driven by the modified error $\tilde{e}$ and not the actual error (or performance variable) e. In the following, it is assumed that the disturbance parameters in (1) are piecewise constant functions of time and that changes in the parameters are sufficiently spaced in time to allow parameter convergence.

Assumption 2: At any time k, there exists a unique parameter vector $\theta_0(k)$ satisfying the regulation conditions corresponding to the disturbance input properties at time k.

We can then define the parameter error at time k, $\tilde{\theta}(k) = \theta_0(k) - \hat{\theta}(k)$, and the change $\Delta\theta_0(k)$ in the parameter vector satisfying the interpolation conditions $\Delta\theta_0(k) = \theta_0(k) - \theta_0(k-1)$. Using Eq. (16) and the expression for $\tilde{e}$, we have

$$\begin{aligned}\theta_0(k+1) - \hat{\theta}(k+1) &= [\theta_0(k+1) - \theta_0(k)] + [\theta_0(k) - \hat{\theta}(k)] \\ &\quad - \lambda(k+1)P(k+1)\phi(k+1)[\phi^T(k)\tilde{\theta}(k-1) + e_0(k)].\end{aligned}$$

which can be rewritten as

$$\begin{aligned}\tilde{\theta}(k+1) &= [I - \lambda(k+1)P(k+1)\phi(k+1)\phi^T(k+1)]\tilde{\theta}(k) \\ &\quad + \Delta\theta_0(k) - \lambda(k+1)P(k+1)\phi(k+1)e_0(k+1).\end{aligned}$$

Using Eq. (17), we have

$$\begin{aligned}\tilde{\theta}(k+1) &= [\lambda(k+1)P(k+1)P^{-1}(k)]\tilde{\theta}(k) + \Delta\theta_0(k) \\ &\quad - \lambda(k+1)P(k+1)\phi(k+1)e_0(k+1).\end{aligned}$$

Using the approach in [17] for the convergence analysis of the adaptation algorithm, the effects of the initial conditions, changes in the parameter vector θ_0, and the signal e_0 on the parameter estimation error $\tilde{\theta}(k)$ can be studied separately using the following equations

$$\tilde{\theta}_1(k+1) = [\lambda(k+1)P(k+1)P^{-1}(k)]\tilde{\theta}_1(k), \quad \tilde{\theta}_1(0) = \tilde{\theta}(0), \quad (20)$$

$$\tilde{\theta}_2(k+1) = [\lambda(k+1)P(k+1)P^{-1}(k)]\tilde{\theta}_2(k) + \Delta\theta_0(k), \quad \tilde{\theta}_2(0) = 0, \quad (21)$$

$$\begin{aligned}\tilde{\theta}_3(k+1) &= [\lambda(k+1)P(k+1)P^{-1}(k)]\tilde{\theta}_3(k) \\ &\quad - \lambda(k+1)P(k+1)\phi(k+1)e_0(k+1), \quad \tilde{\theta}_3(0) = 0. \quad (22)\end{aligned}$$

where $\tilde{\theta}_1(\cdot)$ represents the response to the initial condition $\tilde{\theta}(0)$, $\tilde{\theta}_2(\cdot)$ the response to the change in the nominal parameter vector θ_0, and $\tilde{\theta}_3(\cdot)$ the response to the forcing signal $e_0(\cdot)$. Note that the separation of the effects of the initial conditions, changes in the nominal parameter vector, and the perturbation term e_0 was possible due to the fact that $[\lambda(k+1)P(k+1)P^{-1}(k)]$ is independent of $\hat{\theta}(\cdot)$ as v_1, and consequently $\phi(\cdot)$, are independent of $\hat{\theta}(\cdot)$. Consider now the assumption below to be used in the next theorem.

Assumption 3: The signal v_1 is persistently exciting of order n_q.

The above assumption is easily satisfied if $T_{se}T_{wr}$ does not have zeros at the poles of $W(z)$. The properties of the algorithm are given by the following theorem.

Theorem 3 [2]: If Assumptions 2 and 3 are satisfied, then the algorithm given by (18) and (19) yields

a/ $\lim_{k\to\infty} \tilde{\theta}_i(k) = 0$, $i = 1, 2, 3$.
b/ The adaptive system is Bounded-Input Bounded-Output stable.

4 Examples

Consider the SISO plant given by the following state space representation

$$x(k+1) = .8x(k) + u(k) + .5w(k), \quad x(0) = 0,$$

$$y(k) = x(k).$$

The disturbance input is given by $w(k) = \sin(\omega_1 kT_s) + \sin(\omega_2 kT_s)$ where ω_1 and ω_2 are the frequencies of the continuous time sinusoids and $T_s = 1$ sec is the sampling period.

In the example given above, the plant P is such that $G_{we} = G_{wy}$ and $G_{ue} = G_{uy}$. Hence the disturbance response e is the same as the plant output y. The stabilizing controller $K_0(z) = -(0.06/z - 0.1)$ is considered. In order to construct a set of stabilizing controllers, the Youla parameter Q is chosen to be of the form $Q(z) = q_1 + q_2 z^{-1} + q_3 z^{-2} + q_4 z^{-3}$. Notice that the number of parameters n_q is the same as the number of poles n_p of $W(z)$. The resulting parametrized controller transfer function is

$$K(z) = \frac{q_1 z^4 + (-.06 + q_2 - .8q_1)z^3 + (q_3 - .8q_2)z^2 + (q_4 - .8q_3)z - .8q_1}{z^4 + (q_1 - .1)z^3 + q_2 z^2 + q_3 z + q_4}, \qquad (23)$$

and the transfer function $F_{T,Q}(z) = [E(z)/W(z)$ relating the disturbance input w to the disturbance response e is given by

$$F_{T,Q}(z) = \frac{z^4 + (q_1 - .1)z^3 + q_2 z^2 + q_3 z + q_4}{z^3(z^2 - .9z + .14)}. \qquad (24)$$

For the disturbance input considered above, the disturbance response is given by

$$E(z) = \left[\frac{z^4 + (q_1 - .1)z^3 + q_2 z^2 + q_3 z + q_4}{z^3(z^2 - .9z + .14)}\right] \times \left[\frac{\sin(\omega_1 T_s)z}{z^2 - 2\cos(\omega_1 T_s)z + 1} + \frac{\sin(\omega_2 T_s)z}{z^2 - 2\cos(\omega_2 T_s)z + 1}\right].$$

In order to have $E(z) \in RH_\infty$, we must have $q_1 = 0.1 - 2(\cos(\omega_1 T_s) + \cos(\omega_2 T_s))$, $q_2 = 2(1 + 2\cos(\omega_1 T_s)\cos(\omega_2 T_s))$, $q_3 = -2(\cos(\omega_1 T_s) + \cos(\omega_2 T_s))$, and $q_4 = 1$. It is important to notice that if, for given ω_1 and ω_2, the values of q_1, q_2, q_3, and q_4 given above are used in the controller K in (23), the controller would contain a model of the disturbance input (1) (the poles of $W(z)$ are also poles of $K(z)$). Hence, if during adaptation, the adjusted Q parameters converge to the nominal parameters given above, then the controller design would represent an adaptive implementation of the Internal Model Principle. This is due to the fact that, for the plant given above, we have $e = y$. The frequencies of the disturbance are $\omega_1(k) = 3$ rad/sec and $\omega_2(k) = 1$ rad/sec for $0 \le k < 500$ and then change to $\omega_1(k) = 2.5$ rad/sec and $\omega_2(k) = 1.5$ rad/sec for $500 \le k$. Therefore, the parameter vector $\theta = [q_1, q_2, q_3, q_4]^T$ that should be used to achieve regulation is as follows: for $0 \le k < 500$, $\theta = [.9994, -.1396, .8994, 1]^T$ and for $500 \le k$, $\theta = [1.5608, 1.7733, 1.4608, 1]^T$. The performance of the two adaptation algorithms is discussed below.

4.1 The RLS Algorithms With Dead Zone. In order to use the RLS algorithms with dead zone, it is necessary to determine $\bar{\beta}$ and $\bar{\alpha}$ in the expression for $\bar{e}$. A conservative value for $\bar{\beta}$ can be given by examining the poles of $F_{T,Q}(z)$ in (24) which are located at $z = 0$, .2, and .7. The slowest pole of $F_{T,Q}(z)$ is located at $z = .7$. Therefore, we can take $\bar{\beta} = .95 > .7$. The value of $\bar{\alpha}$ is set equal to 1. The performance of the adaptation algorithms is illustrated in Figs. 2 and 3. The initial conditions are $\hat{\theta}(0) = [0, 0, 0, 0]^T$ and $P(0) = 100I$ where I is a 4×4 identity matrix. For $0 \le k < 500$, the closed-loop system was able to slowly reject the disturbance input. The estimated parameters converged to the nominal parameters.

The simulation results presented above correspond to the case where the minimal number of parameters (4 in this case) needed to achieve regulation are used. To illustrate the fact that regulation takes place even when the number of parameters in Q is larger than 4, we consider adjusting the parameters of a tenth order Youla parameter Q, i.e., there are ten parameters to be adjusted. The parameter vector $\hat{\theta}$ is initialized at the origin. The performances of the two algorithms are shown in Figs. 4 and 5. In both cases, the error goes to zero asymptotically and the parameter estimates converge to

$$[0.3092, -0.2254, 0.5020, -0.0972, 0.0509, -0.2855, -0.1507, 0.1471, 0.0525, 0.3944]^T.$$

4.2 The RLS Algorithm With a Forgetting Factor. The forgetting factor in this algorithm is a constant $\lambda = .95$. The initial conditions of the algorithm are $\hat{\theta}(0) = [0, 0, 0, 0]^T$ and $P(0) = 100I$ where I is a 4×4 identity matrix. The performance of the closed-loop control system is shown in Fig. 6. It can be seen that the adaptive control system was capable of rejecting the disturbance input even when the frequency of the disturbance input changes at time 500 s. The estimated parameters converged to the

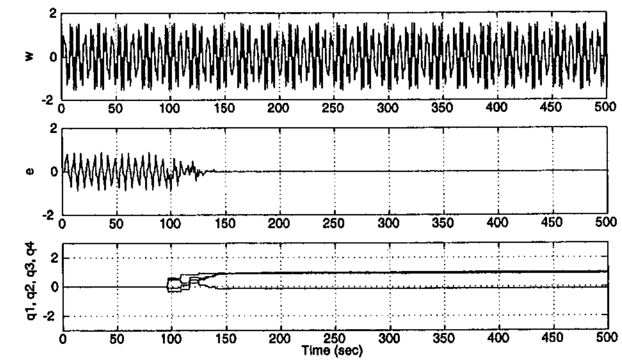

Fig. 2 Response of the adaptive control system using the RLS algorithm with dead zone and driven by the performance variable *e*. Top: disturbance input *w(k)*. Middle: response of the adaptive control system to the disturbance input *w(k)*. Bottom: parameters of the controller parametrizing mapping *Q* (sampling period = 1 s).

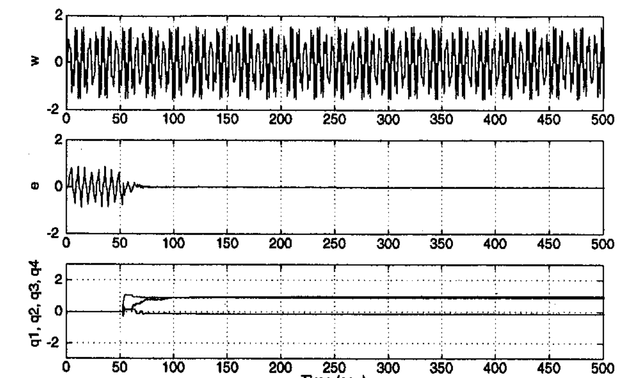

Fig. 3 Response of the adaptive control system using the RLS algorithm with dead zone and driven by the modified performance variable *e*. Top: disturbance input *w(k)*. Middle: response of the adaptive control system to the disturbance input *w(k)*. Bottom: parameters of the controller parametrizing mapping *Q* (sampling period = 1 s).

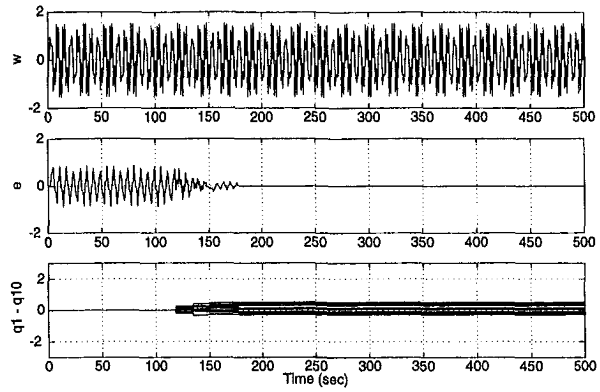

Fig. 4 Response of the adaptive control system using the RLS algorithm with dead zone and driven by the performance variable *e*, for the case of an overparametrized *Q*. Top: disturbance input *w*(*k*). Middle: response of the adaptive control system to the disturbance input *w*(*k*). Bottom: parameters of the controller parametrizing mapping *Q* (sampling period = 1 s).

nominal parameters. Hence, for both values of the disturbance frequency, the adaptive control algorithm was able to construct an internal model of the disturbance input in the controller.

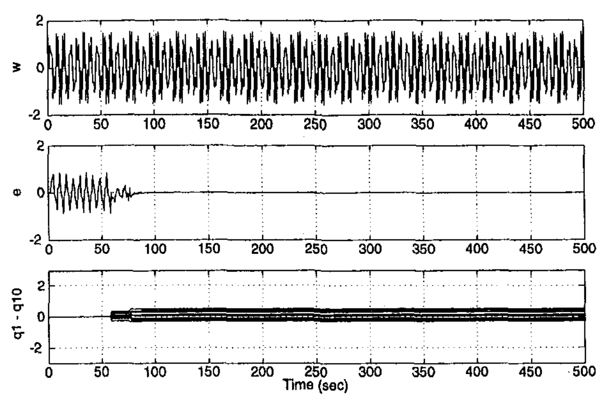

Fig. 5 Response of the adaptive control system using the RLS algorithm with dead zone and driven by the modified performance variable *e*, for the case of an overparameterized *Q*. Top: disturbance input *w*(*k*). Middle: response of the adaptive control system to the disturbance input *w*(*k*). Bottom: parameters of the controller parametrizing mapping *Q* (Sampling period = 1 s).

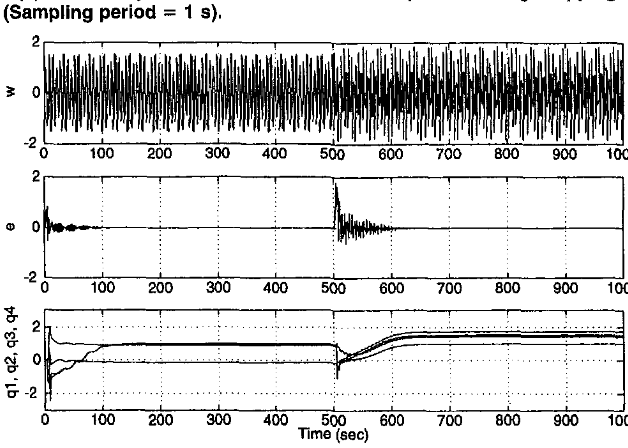

Fig. 6 Response of the adaptive control system using the RLS algorithm with forgetting factor. Top: disturbance input *w*(*k*). Middle: response of the adaptive control system to the disturbance input *w*(*k*). Bottom: parameters of the controller parametrizing mapping *Q* (sampling period = 1 s).

5 Summary and Conclusions

The problem of adaptively rejecting sinusoidal disturbance inputs with unknown and/or time varying characteristics was considered for SISO plants. The adaptation approach is based on searching, on-line, within a parametrized set of stabilizing controllers, for the controller that achieves asymptotic disturbance rejection. Adaptive disturbance rejection algorithms which are robust with respect to changes in the disturbance input modes are presented and their properties analyzed. When the performance variable is the same as the measurement signal, the adaptation results in an on-line implementation of the Internal Model Principle. Experimental implementation and evaluation of the proposed controllers, for noise cancellation in an acoustic duct, is presented in a companion paper [4].

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References

1 K. J. Astrom and B. Wittenmark, *Adaptive Control*, Addison-Wesley, 1995.
2 F. Ben Amara, "Adaptive Sinusoidal Disturbance Rejection in Linear Systems With Application To Noise Cancellation," Ph.D. thesis, The University of Michigan, 1996.
3 F. Ben Amara, P. T. Kabamba, and A. G. Ulsoy, "Adaptive Band-Limited Disturbance Rejection in Linear Discrete-Time Systems," *Mathematical Problems in Engineering*, Vol. 1, No. 2, pp. 139–177, 1995.
4 F. Ben Amara, P. T. Kabamba, and A. G. Ulsoy, "Adaptive Algorithms for Sinusoidal Disturbance Rejection—Part II: Experiments," ASME Journal of Dynamic Systems, Measurement and Control, published in this issue pp. 655–659.
5 M. Bodson, "Adaptive Algorithms for the Rejection of Sinusoidal Disturbances with Unknown Frequency," *Proceeding of the 13th IFAC Triennial World Congress*, San Francisco, CA, Vol. K, pp. 229–234, 1996.
6 R. M. Canetti and M. D. Espana, "Convergence Analysis of the Least-Squares Identification Algorithm with a Variable Forgetting Factor for Time-Varying Linear Systems," *Automatica*, Vol. 25, pp. 609–612, 1989.
7 G. Celantano and R. Setola, "A Technique for Narrow-Band Persistent-Disturbance Attenuation," *Proceedings of the 1996 IFAC World Congress*, San Francisco, CA, 1996.
8 C. K. Chak and G. Feng, "Adaptive Control with External Model for Periodic Disturbance Rejection," *International Journal of Systems Sciences*, pp. 1965–1976, 1994.
9 C. Corduneanu, *Almost Periodic Functions*, Wiley, NY, 1961.
10 H. Elliott and G. C. Goodwin, "Adaptive Implementation of the Internal Model Principle," *Proceedings of the 23rd Conference on Decision and Control*, pp. 1292–1297, 1984.
11 G. Feng, "Robust Adaptive Rejection of Unknown Deterministic Disturbances," *Proceedings of the American Control Conference*, pp. 1624–1625, 1994.
12 G. Feng and M. Palaniswami, "A Stable Adaptive Implementation of the Internal Model Principle," *IEEE Transactions on Automatic Control*, Vol. 37, pp. 1120–1225, 1992.
13 B. A. Francis, *A Course in H^∞ Control Theory*, Springer-Verlag, New York, 1987.
14 G. Hillerstrom and J. Sternby, "Rejection of Periodic Disturbances with Unknown Period—A Frequency Domain Approach," *Proceedings of the American Control Conference*, pp. 1626–1631, 1994.
15 C. D. Johnson, "A Discrete-Time Disturbance-Accomodating Control Theory for Digital Control of Dynamic System," *Control of Dynamic Systems*, Vol. 18, pp. 223–315, 1982.
16 S. Y. Liang and S. A. Perry, "In-Process Compensation for Milling Cutter Runout Via Chip Load Manipulation," ASME *Journal of Engineering for Industry*, Vol. 116, 1994.
17 R. Lozano, "Identification of Time-Varying Linear Models," *Proceedings of the IEEE Conference on Decision and Control*, pp. 604–606, 1983.
18 J. M. Maciejowski, *Multivariable Feedback Design*, Addison-Wesley, 1989.
19 T. J. Manayathara, T.-C. Tsao, J. Bentsman, and D. Ross, "Rejection of Unknown Periodic Load Disturbances in Continuous Steel Casting Process Using Learning Repetitive Control Approach," *IEEE Transactions on Control Systems Technology*, pp. 259–265, 1996.
20 W. Messner and M. Bodson, "Design of Adaptive Feedforward Algorithms Using Internal Model Equivalence," *International Journal of Adaptive Control and Signal Processing*, pp. 199–212, 1995.
21 M. Nakano, J.-H. She, Y. Mastuo, and T. Hino, "Elimination of Position-Dependent Disturbances in Constant-Speed-Rotation Control Systems," *Control Engineering Practice*, Vol. 4, pp. 1241–1248, 1996.
22 V. O. Nikiforov, "Adaptive Servocompensation of Input Disturbances," *Proceedings of the 13th IFAC Triennial World Congress*, San Francisco, CA, Vol. K, pp. 175–180, 1996.
23 M. Palaniswami, "Adaptive Internal Model for Disturbance Rejection and Control," *IEE Proceedings-D*, Vol. 140, pp. 51–59, 1993.
24 H. L. Royden, *Real Analysis*, Macmillan, 1988.

25 A. Sacks, M. Bodson, and P. Khosla, "Experimental Results of Adaptive Periodic Disturbance Cancellation in a High Performance Magnetic Disk Drive," ASME Journal of Dynamic Systems, Measurement, and Control, Vol. 118, pp. 416–424, 1996.

26 A. J. Stevens and S. Y. Steven, "Runout Rejection in End Milling Through Two-Dimensional Repetitive Force Control," *Mechatronics*, Vol. 5, pp. 1–13, 1995.

27 T. T. Tay and J. B. Moore, "Enhancement of Fixed Controllers Via Adaptive—*Q* Disturbance Estimate Feedback," *Automatica*, Vol. 27, pp. 39–53, 1991.

28 M. Tomizuka, T. C. Tsao, and K. K. Chew, "Discrete Time Domain Analysis and Synthesis of Repetitive Controllers," ASME Journal of Dynamic Systems, Measurement, and Control, Vol. 111, pp. 353–358, 1989.

29 T. C. Tsao and M. Nemani, "Asymptotic Rejection of Periodic Disturbances With Uncertain Period," *Proceedings of the American Control Conference*, pp. 2696–2699, 1992.

30 T. C. Tsao and Y. X. Qian, "An Adaptive Repetitive Control Scheme for Tracking Periodic Signals with Unknown Period," *Proceedings of the American Control Conference*, pp. 1726–1730, 1993.

31 E. D. Tung, G. Anwar, and M. Tomizuka, "Low Velocity Friction Compensation and Feedforward Solution Based on Repetitive Control," ASME Journal of Dynamic Systems, Measurement, and Control, Vol. 115, pp. 279–284, 1993.

32 M. Vidyasagar, *Control System Synthesis: A Factorization Approach*, M.I.T. Press, MA, 1985.

33 Z. Wang, I. M. Y. Mareels, and J. B. Moore, "Adaptive Disturbance Rejection," *Proceedings of the IEEE Conference on Decision and Control*, pp. 2836–2841, 1991.

34 B. Wie and M. Gonzalez, "Control Synthesis for Flexible Space Structures Excited by Persistent Disturbances," *Journal of Guidance, Control, and Dynamics*, 1992.

35 W. C. Yang and M. Tomizuka, "Disturbance Rejection Through an External Model for Non-Minimum Phase Systems," ASME Journal of Dynamic Systems, Measurement, and Control, Vol. 116, pp. 39–44, 1994.

36 D. C. Youla, H. A. Jabr, and J. J. Bongiorno Jr., "Modern Wiener-Hopf Design of Optimal Controllers—Part II: The Multivariable Case," *IEEE Transactions on Automatic Control*, Vol. 21, pp. 319–338, 1976.