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The EHTA for Two Shallow Water Wave Equations

Ping He [1], Defei Zhang and Wei Li

Department of Mathematics, Honghe University

Mengzi, 661100, Yunnan, China

Abstract

In this paper, two shallow water wave equations are studied by using the extended homoclinic test approach (EHTA). The new exact solutions for the shallow water wave equations are obtained. Their dynamic properties of some exact solutions are discussed and their profiles of these solutions are given by using of software Maple.

Mathematics Subject Classification: 35B05, 35B10, 35B15

Keywords: Shallow water wave equation, soliton solution, dynamic properties

1 Introduction.

There is a wide variety of approaches to nonlinear problems to seek for exact solutions. A new technique called" extended homoclinic test technique" is proposed very recently [1], which is a fairly effective method to seek periodic solitary wave solutions of integrable equations[2,3]. By the method, Dai gets some new exact soliton solutions for nonlinear evolution equations[4,5].

In this paper, applying the extension of homoclinic test approach to two shallow water wave equations, we obtain more exact the blow-up soliton solutions, breather-type soliton solutions, kink-soliton solutions, anti-kink soliton solutions and two-soliton solutions.

[1] Corresponding author.

e-mail address: hepingky@163.com (P. He)

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Now we consider the model equations for shallow water waves

$$u_t - u_{xxt} - 3uu_t - 3u_x \int^x u_t dx + u_x = 0, \tag{1}$$

and

$$u_t + 6uu_x - 3uu_t - u_{xxt} - 3u_x \int^x u_t dx + u_{3x} = 0. \tag{2}$$

The Eq.(1) can be transformed to the bilinear form

$$D_x^3 D_t(f \cdot f) - D_x^2(f \cdot f) - D_x D_t(f \cdot f) = 0. \tag{3}$$

The Eq.(2) can be transformed to the bilinear form

$$D_x^3 D_t(f \cdot f) - D_x^4(f \cdot f) - D_x D_t(f \cdot f) = 0, \tag{4}$$

where the bilinear operator[6] is defined as:

$$D_t^m D_x^n f \cdot g = (\partial_t - \partial_{t'})^m (\partial_x - \partial_{x'})^n f(t,x) g(t',x')|_{t'=t,x'=x}.$$

Suppose the solutions of Eq.(1)and Eq.(2) as follows

$$u(x,t) = 2(lnf)_{xx}. \tag{5}$$

we suppose that

$$f = e^{\eta_1} + p_1 cos(\eta_2) + p_2 e^{-\eta_1}, \tag{6}$$

where $\eta_i = a_i x + b_i t$ $(i = 1, 2)$ and a_i, b_i, p_i, $(i = 1, 2)$ are parameters to be determined later. Using Eq. (6), we will investigate exact solutions of Eqs.(1) and (2). Some new exact solutions will be obtained and some interesting phenomena will be found.

2 The exact solutions of Eq.(1).

In Eq.(3), let the test function be Eq.(6). Substituting (6) into (3) yields

$$L_1 + \{L_2 e^{(a_1 x + b_1 t)} + L_3 e^{(-a_1 x - b_1 t)}\} cos(a_2 x + b_2 t) + \{L_4 e^{(a_1 x + b_1 t)} + L_5 e^{(-a_1 x - b_1 t)}\} sin(a_2 x + b_2 t) = 0, \tag{7}$$

where

$$L_2 = p_1 b_2 a_2^3 - 3a_1^2 p_1 b_2 a_2 - 3p_1 a_2^2 a_1 b_1 - a_1^2 p_1 + p_1 a_2^2 - a_1 b_1 p_1 + a_1^3 b_1 p_1 + p_1 b_2 a_2,$$
$$L_4 = -2a_1 p_1 a_2 + 3a_1^2 b_1 p_1 a_2 - 3p_1 b_2 a_2^2 a_1 + a_1^3 p_1 b_2 - p_1 a_2^3 b_1 - a_1 p_1 b_2 - p_1 a_2 b_1,$$

Equating all the coefficients of the above equations, we obtain

$$L_1 = 0,\ L_2 = 0,\ L_3 = 0, L_4 = 0, Ł_5 = 0. \tag{13}$$

Solving the above the set of algebraic equation with the aid of Maple, we obtain

Case (I).

$$p_1 = p_1, p_2 = -\frac{(3a_2^2 + 3 - a_1^2)p_1^2 a_2^2}{4(a_2^2 + 3 - 3a_1^2)a_1^2},$$

$$a_2 = a_2, b_1 = \frac{a_1(a_1^2 + a_2^2 - 1)}{(a_2^2 + 1 + 2a_1 + a_1^2)(a_2^2 + 1 - 2a_1 + a_1^2)}, b_2 = -\frac{a_2(a_2^2 + a_1^2 + 1)}{(a_2^2 + 1 + 2a_1 + a_1^2)(a_2^2 + 1 - 2a_1 + a_1^2)}.$$

Case (II).

$$p_1 = p_1, p_2 = p_2,$$
$$a_2 = -ia_1, b_1 = \frac{a_1}{4a_1^2 - 1}, b_2 = \frac{ia_1}{4a_1^2 - 1}.$$

Case (III).

$$p_1 = p_1, p_2 = \frac{p_1^2}{4},$$
$$a_2 = ia_1, b_1 = \frac{2a_1 + ib_2 - 4ia_1^2 b_2}{-1 + 4a_1^2}, b_2 = b_2.$$

Under the condition of case (I), We obtain a periodic breather-type solitary wave solution of Eq.(1) as follows:

$$u_1 = \frac{8p_1 a_1^2(-a_2^2 - 3 + 3a_1^2)\{(\gamma_2+\gamma_4)sin\beta cosh\alpha + (\gamma_2-\gamma_4)sin\beta sinh\alpha + (\gamma_3+\gamma_5)cos\beta cosh\alpha + (\gamma_3-\gamma_5)cos\beta sinh\alpha + \gamma_1 + \gamma_6\}}{[(\gamma_7+\gamma_8)cosh\alpha + (\gamma_7-\gamma_8)sinh\alpha + p_1\gamma_7 cos\beta]^2},$$

Where $\alpha = -\frac{a_1(xa_2^4 + 2xa_2^2 + 2xa_2^2a_1^2 + x - 2xa_1^2 + xa_1^4 + ta_1^2 + ta_2^2 - t)}{(a_2^2+1+2a_1+a_1^2)(a_2^2+1-2a_1+a_1^2)}$,

$\beta = \frac{a_2(-xa_2^4 - 2xa_2^2 - 2xa_2^2a_1^2 - x + 2xa_1^2 - xa_1^4 + ta_1^2 + ta_2^2 + t)}{(a_2^2+1+2a_1+a_1^2)(a_2^2+1-2a_1+a_1^2)}$,

$\gamma_1 = 12p_1 a_2^2 a_1^2 - 4p_1 a_2^2 a_1^4 + 12p_1 a_2^4 a_1^2$,

$\gamma_2 = 8a_1^3 a_2^3 + 24a_1^3 a_2 - 24a_1^5 a_2$,

$\gamma_3 = 12a_1^6 - 16a_1^4 a_2^2 - 12a_1^4 + 4a_1^2 a_2^4 + 12a_2^2 a_1^2$,

$\gamma_4 = 6p_1^2 a_2^5 a_1 + 6p_1^2 a_2^3 a_1 - 2p_1^2 a_2^3 a_1^3$,

$\gamma_5 = -3p_1^2 a_2^6 - 3p_1^2 a_2^4 + 3p_1^2 a_2^2 a_1^2 - p_1^2 a_2^2 a_1^4 + 4p_1^2 a_2^4 a_1^2$,

$\gamma_6 = -12p_1 a_2^2 a_1^4 + 4p_1 a_2^4 a_1^2 + 12p_1 a_2^2 a_1^2$,

$\gamma_7 = -4a_2^2a_1^2 - 12a_1^2 + 12a_1^4$,

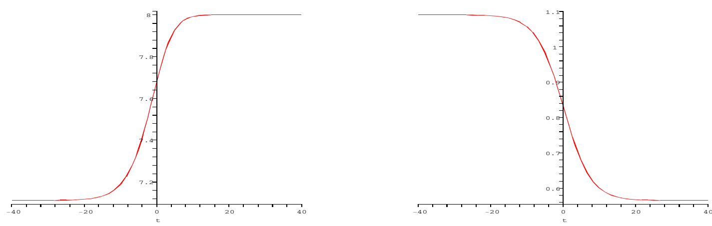
$\gamma_8 = 3p_1^2a_2^4 + 3p_1^2a_2^2 - p_1^2a_2^2a_1^2$.

Under the condition of case (II) , we then obtain a new cross-kink solitary solution of Eq.(1)

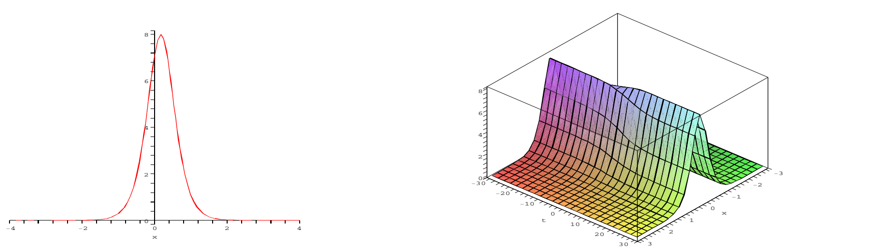
$$u_2 = \frac{2a_1^2\{2p_1(1-p_2)[sinh\beta cosh\alpha + cosh\beta sinh\alpha] + 2p_1(1+p_2)[sinh\beta sinh\alpha + cosh\beta cosh\alpha] + \gamma\}}{\{(1+p_2)cosh\alpha + (1-p_2)sinh\alpha + p_1cosh\beta\}^2},$$

Where $\alpha = \frac{a_1(-x+4xa_1^2+t)}{4a_1^2-1}, \beta = \frac{a_1(x-4xa_1^2+t)}{4a_1^2-1}, \gamma = 4p_2 + p_1^2$.

The Fig.1 show that the solution u_2 has different wave-forms. It depends on the x.When x increases from 0 to 1, the anti-kink-solitary wave is changed to a kink-solitary wave. However, the smooth solitary wave is obtained for given $t = -30$. The Fig.1 show the profiles of solution u_2 under the fixed parameters $a_1 = 2, p_1 = 2, p_2 = 2$.



$(a).x = 0$, anti-kink-solitary wave $(b).x = 1$, kink-solitary wave



$(c).t = -30$, solitary wave (d). solitary wave in three-dimensional space

Fig. 1. The profiles of solutions u_2.

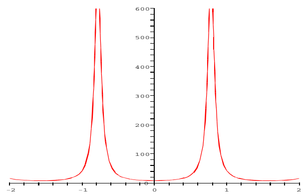
Under the condition of case (III) , we obtain a blow-up soliton solution of Eq.(1) as follows:

$$u_3 = \frac{16a_1^2p_1\{(4-p_1^2)(isin\beta cosh\alpha - cos\beta sinh\alpha) + (4+p_1^2)(cos\beta cosh\alpha - isin\beta sinh\alpha) + 4p_1\}}{(16+p_1^4)cosh2\alpha - (16-p_1^4)sinh2\alpha + (32p_1+8p_1^3)cos\beta cosh\alpha - (32p_1-8p_1^3)cos\beta sinh\alpha + 8p_1^2(1+2cos^2\beta)},$$

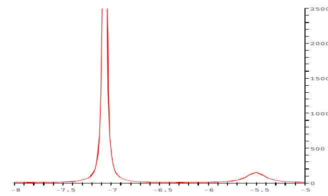
Where $\alpha = \frac{a_1x-4a_1^3x-2ta_1-tib_2+4tia_1^2b_2}{4a_1^2-1}, \beta = a_1ix + b_2t$.

It is very interesting that the solution u_3 has different wave-forms as one of parameters varied. When x increases from 0 to 0.48, two-blow-up solitary wave is changed to the blow-up-bell solitary wave. When x increases from 0.48 to 0.6, the blow-up-bell solitary wave is changed to the two-bell solitary wave.

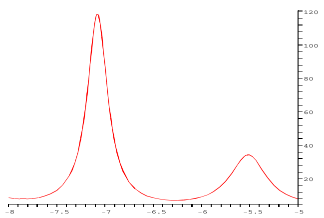
The profiles of its waveforms are shown in Fig.2(a)-(c), respectively. The Fig.2 show under the fixed parameters $a_1 = 2, p_1 = 2, b_2 = 2$. Fig.2(d) shows blow up phenomenon in three-dimensional space.



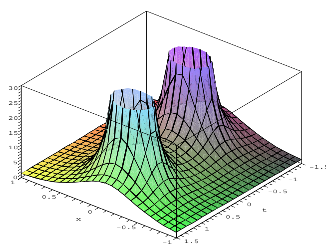
$(a).x = 0$, two-blow-up solitary wave



$(b).x = 0.48$, blow-up-bell solitary wave



$(c).x = 0.6$, two-bell solitary wave



(d). solitary wave in three-dimensional space

Fig. 2. The profiles of solutions u_3.

3 The exact solutions of Eq.(2).

In Eq.(4), let the test function be Eq.(6). Substituting (6) into (4) yields

$$A_1 + \{A_2 sin(a_2x + b_2t) + A_3 cos(a_2x + b_2t)\}e^{(a_1x+b_1t)} + \{A_4 sin(a_2x + b_2t) + A_5 cos(a_2x + b_2t)\}e^{(-a_1x-b_1t)} = 0, \quad (14)$$

where

(19)

$$A_2 = -3p_1b_2a_2^2a_1 + a_1^3p_1b_2 + 3a_1^2b_1p_1a_2 - p_1a_2^3b_1 + 4p_1a_2^3a_1 - 4a_1^3p_1a_2 - p_1a_2b_1 - a_1p_1b_2,$$

$$A_4 = -4p_1a_2^3p_2a_1 + 4p_2a_1^3p_1a_2 + 3p_1b_2a_2^2p_2a_1 - 3p_2a_1^2b_1p_1a_2 + p_1a_2p_2b_1 + p_2a_1p_1b_2 + p_1a_2^3p_2b_1 - p_2a_1^3p_1b_2,$$

Equating all the coefficients of the above equations, we obtain

$$A_1 = 0,\ A_2 = 0,\ A_3 = 0, A_4 = 0,\ A_5 = 0. \tag{20}$$

Solving the above the set of algebraic equation with the aid of Maple, we obtain

Case (I).

$$p_1 = 0, p_2 = p_2,$$
$$a_2 = a_2, b_1 = \frac{4a_1^3}{-1+4a_1^2}, b_2 = b_2.$$

Case (II).

$$p_1 = p_1, p2 = -\frac{(a_1^2 - 3a_2^2 - 3)p_1^2 a_2^2}{4a_1^2(3a_1^2 - 3 - a_2^2)},$$
$$a_2 = a_2, b_1 = \frac{a_1(a_1^4 + 2a_2^2a_1^2 - a_1^2 + a_2^4 + 3a_2^2)}{(a_1^2 - 2a_1 + 1 + a_2^2)(a_1^2 + 2a_1 + 1 + a_2^2)}, b_2 = \frac{a_2(a_1^4 - 3a_1^2 + a_2^2 + 2a_2^2a_1^2 + a_2^4)}{(a_1^2 - 2a_1 + 1 + a_2^2)(a_1^2 + 2a_1 + 1 + a_2^2)}.$$

Under the condition of case (I), we have a solution of Eq.(2)

$$u_1 = \frac{8p_2a_1^2}{[(1+p_2)cosh\theta + (1-p_2)sinh\theta]^2},$$

where $\theta = \frac{a_1(-x+4xa_1^2+4a_1^2t)}{4a_1^2-1}$, a_1, p_2 are free constants.

Under the condition of case (II), we have a two-solitary-wave solution of Eq.(2)

$$\mu_2 = \frac{8p_1a_1^2\{(\gamma_1+\gamma_3)cos\beta cosh\alpha + (\gamma_3-\gamma_1)cos\beta sinh\alpha + (\gamma_2+\gamma_4)sin\beta cosh\alpha + (\gamma_4-\gamma_2)sin\beta sinh\alpha + \gamma_5\}}{(\gamma_6+\gamma_8)cos\beta cosh\alpha + (\gamma_8-\gamma_6)cos\beta sinh\alpha + (\gamma_9+\gamma_7)cosh2\alpha + (\gamma_9-\gamma_7)sinh2\alpha + \gamma_{10}cos^2\beta + \gamma_{11}},$$

Where $\alpha = \frac{a_1(a_1^4x-2a_1^2x+2a_2^2a_1^2x+x+2a_2^2x+a_2^4x+a_1^4t+2a_2^2a_1^2t-a_1^2t+a_2^4t+3a_2^2t)}{(a_1^2-2a_1+1+a_2^2)(a_1^2+2a_1+1+a_2^2)}$,

$\beta = \frac{a_2(a_1^4x-2a_1^2x+2a_2^2a_1^2x+x+2a_2^2x+a_2^4x+a_1^4t-3a_1^2t+a_2^2t+2a_2^2a_1^2t+a_2^4t)}{(a_1^2-2a_1+1+a_2^2)(a_1^2+2a_1+1+a_2^2)}$,

$\gamma_1 = 3p_1^2a_2^8 + 12a_2^6p_1^2 + 13a_1^4p_1^2a_2^4 - 13a_1^2a_2^6p_1^2 - 3a_1^6p_1^2a_2^2 + 12a_1^4p_1^2a_2^2 - 9p_1^2a_2^2a_1^2 - 24a_1^2p_1^2a_2^4 + 9p_1^2a_2^4$,

$\gamma_2 = 18p_1^2a_2^3a_1 + 6p_1^2a_2^7a_1 - 20p_1^2a_2^5a_1^3 - 24p_1^2a_2^3a_1^3 + 24p_1^2a_2^5a_1 + 6p_1^2a_2^3a_1^5$,

$\gamma_3 = -36a_2^2a_1^2 - 24a_1^2a_2^4 - 4a_1^2a_2^6 + 28a_1^4a_2^4 - 60a_1^6a_2^2 + 96a_1^4a_2^2 + 36a_1^4 - 72a_1^6 + 36a_1^8$,

$\gamma_4 = -144a_1^5a_2 + 72a_1^3a_2 - 48a_1^5a_2^3 + 72a_1^7a_2 + 48a_1^3a_2^3 + 8a_1^3a_2^5$,

$\gamma_5 = -48p_1a_2^2a_1^6 + 120p_1a_2^2a_1^4 + 64p_1a_2^4a_1^4 - 72p_1a_2^4a_1^2 - 16p_1a_2^6a_1^2 - 72p_1a_1^2a_2^2$,

$\gamma_6 = 80p_1^3a_1^4a_2^4 - 24p_1^3a_1^2a_2^6 - 96p_1^3a_1^2a_2^4 - 24p_1^3a_2^2a_1^6 + 96p_1^3a_2^2a_1^4 - 72p_1^3a_2^2a_1^2$,

$\gamma_7 = -6p_1^4a_2^6a_1^2 - 6p_1^4a_2^4a_1^2 + p_1^4a_2^4a_1^4 + 9p_1^4a_2^4 + 9p_1^4a_2^8 + 18p_1^4a_2^6$,

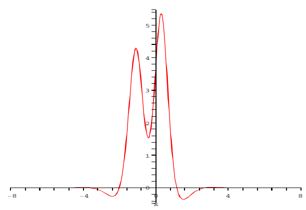
$\gamma_8 = 32p_1a_2^4a_1^4 - 192p_1a_2^2a_1^6 + 192p_1a_2^2a_1^4 + 288a_1^8p_1 - 576a_1^6p_1 + 288a_1^4p_1$,

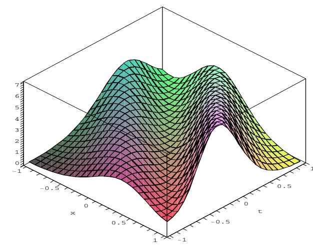
$\gamma_9 = 144a_1^8 - 288a_1^6 + 144a_1^4 + 96a_1^4a_2^2 - 96a_1^6a_2^2 + 16a_1^4a_2^4$,

$\gamma_{10} = 144p_1^2a_1^4 + 96a_1^4p_1^2a_2^2 + 144p_1^2a_1^8 - 288p_1^2a_1^6 + 16a_1^4p_1^2a_2^4 - 96a_1^6p_1^2a_2^2$,

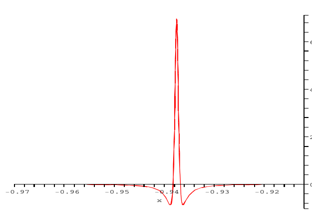
$\gamma_{11} = -24a_1^6p_1^2a_2^2 - 72p_1^2a_2^2a_1^2 - 96a_1^2p_1^2a_2^4 + 80a_1^4p_1^2a_2^4 - 24a_1^2a_2^6p_1^2 + 96a_1^4p_1^2a_2^2$.

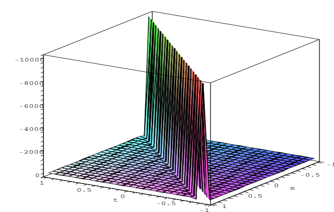
From Fig.3, it is easy to know that the Fig.3(a) and (c) show two kinds of periodic waves . It is clear that the profile of the solution u_2 is transformable. The Fig.3 (a) and (b)show under the fixed parameters $a_1 = 2, p_1 = 2, a_2 = 2$. Fig.3 (c) and (d)show under the fixed parameters $a_1 = 50, p_1 = 50, a_2 = 50$. The profile of the solution u_2 depends on the parameters a_1, a_2, p_1. However, the profiles of the solution u_2 are similar.





(a). two-solitary wave (b). solitary wave in three-dimensional space





(c). solitary wave (d). solitary wave in three-dimensional space

Fig.3. The profiles of solutions u_2.

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