Stochastic Analysis of Mobile Ad-hoc Networks

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Abstract

The MANET networks (Mobile Ad-hoc Networks) are known by their dynamicity of nodes and they are without pre-existing infrastructure. To study this kind of networks, we modelled them by a Random Geometric Graphs (RGG). We show that this type of graph is the best adapted to represent such networks, and this shows that the RGG are able to catch the dynamicity properties. We study then the evolution of the network by a continuous-time birth-death Markov process.

Keywords: MANET Networks, Mobility, Dynamicity, Random Geometric Graph, Birth-Death Markovian Process, Stochastic Analysis.

1. Introduction

A Mobile Ad-hoc Network (MANET) is composed by a set of stations. Those stations are self-organized and this with a decentralized manner. Thus, they form an autonomous dynamic network, witch is without pre-existing infrastructure. Each station communicates with each other one via a radio interface. So, only the items covered in the same transmission range are able to communicate directly and mutually. Otherwise, the communications between the remote components take place according to a multi-hop communication model; i.e. a message is forwarded station-to-station gradually until the destination is reached. In this case, it is not easy to find an efficient path between two remote elements. The mobility of stations and the lack of infrastructure lead to some worries about the network connectivity.

Traditionally, a computer network is modelled by a graph where the network components are represented by vertices and the edges represent the communication links between these components. Otherwise and for the reasons of network dynamism (such as appearance and/or disappearance of network nodes), it is necessary to represent this new generation of network by a graph model witch is able to catch some dynamicity properties.
The random graph model presented in [1], [2] constitutes a way to describe these networks. However, one of the problems posed is the generation of the related random graphs. In the literature, sparsely techniques are available to meet this purpose (see [3], [4]).
Moreover, in MANET, the communication between two stations is made only if the distance between them is less than a definite transmission range. This translate the presence of the proximity concept between the localization of these stations in the space. This allows us to give a geometric meaning to vertices and edges.

The random geometric graphs are proposed to meet the requirement distance. These graphs are introduced first in 1961 by Gilbert in [5]. Therefore, they served for modelling many natural and artificial phenomena, for example the communication between different distributed stations on a territory [6] and some fundamental algorithmic problems in networks (ad-hoc, wireless, sensors, etc.). See [7], [8], [9].

Although the static properties of these networks are well understood mathematically. The additional challenges, caused by the mobility of nodes, have received so far relatively little attentions from theorists.

In our work, we are interested to a random geometric graphs to model and we will study the dynamicity of MANET. So, The organization of this paper is as follows. In Section 2, we will present basic notions of random geometric graphs. Afterwards, in Section 3, we will analyse the dynamicity of nodes in the network by the birth and death Markov processes. A detailed study of this process in different regimes will be presented in this section. In the end, we will present in Section 4 a conclusion and we will discuss our future works.

2. Preliminary

2.1 Random geometric graph

In 1961, Edward Gilbert [5] defined the (*Random Plane Networks*) as an effective model to study the communication in the network, of distributed stations, in a wide area. This model has been used in many disciplines for example modeling wireless sensor networks [10], statistical physics and hypothesis testing [11], spread of disease in complex networks [12], social networks [13]. The reader can see[11] for more information.

Gilbert's model is known in graph theory as the **R**andom **G**eometric **G**raph (RGG). In this family graphs, the vertices are a random points distributed in a same metric space $\mathcal{S}$, generally a compact subset of a space $\mathbb{R}^d$ for $d \geq 1$.
An edge between two vertices will be created if the distance between these two vertices is less than a defined radius r. This radius represents, in our case, the transmission range. It specifies the distance over which nodes can send and receive informations.
Furthermore, the mobility of nodes in a network (physical move of nodes from one region to another) causes a variation in the network topology. Hence, network connectivity changes also, it may be that the network is no longer connected.
Similarly, when nodes appear or disappear the infrastructure will be also modified. So, at each appearance, the new links will be added, and any disappearance links will be deleted. With these changes over time, it is important to ensure ownership of the network connectivity.

To describe this dynamic network, we model it by a random geometric graph.

Mathew Penrose in [11] proposed a static model of random geometric graph $G(V_\lambda, r)$ where the points of the Poisson process, of intensity λ, represent the nodes. The set of these points is denoted V_λ. The edges are added between each pair of nodes whose the distance between them does not exceed r.

For our study, we use the model of mobile random geometric graphs introduced by Berg and White in [14]. This model is similar to Penrose's model, but the difference is that the nodes are able to move.

Let $\chi_0 = \{X_i\}_i$ the Poisson point process, with intensity λ, the points are the nodes. The displacement of a node X_i is made according to a Brownian motion $(\xi_i(t))_{t \geq 0}$ and this independently to other nodes. For any time t, the set $\chi_t = \{X_i + \xi_i(t)\}_i$ is a point process obtained after a displacement nodes of χ_0.

3. Dynamicity Analysis

3.1 Evolution of MANET

Among the main characteristics of a MANET, we find the instability of its topology. The causes are multiples, we cite:

- displacement nodes, in a given space, rendering their neighbourhoods unstable. We call this case: *node mobility*.
- disconnection of existing nodes (user disconnection, battery discharge, to be outside the transmission range, ...)
- arrival of new other nodes (connection of new users, charging of battery, placement of new sensors, ...). Idem, we call this case: *node dynamicity*.

The table I below summarizes the behaviour of nodes in mobile ad hoc networks.

t_i	$t_i + \Delta t$	**Interpretation**	**State**
●	●	the node has not moved	mobility
●	○ ↷ ●	it is moved	
●	●	it is remained in its state	dynamicity
×	●	it is appeared	
●	×	it is disappeared	

TABLE I
EVOLUTION OF THE MANETS NETWORKS

● describes a network node.
Δt is the time interval assumed to be very short, sufficient in order that a single node joins or leaves the network.

3.2 Analysis of the dynamicity

In this section, we model the dynamicity of a MANET by a process that evolves in the space and the time. The transitions between states are made through a process of a new node appearance or disappearance of an existing node. Moreover, the future state of the process depends only on the present state. This process can be seen as a *continuous-time birth-death Markov process*.

At each appearance (birth), the process increases by one and conversely each disappearance (death) decreases it by one. The times, in which they have appearances or disappearances, are random. The law that manages those times is characterized as bellow.

In the sequel, we use both the graph terminology or the network terminology to mean the same things.

Proposition 1: Let G be a graph and Δt a very short time. The probability that a new vertex appears in this graph is $\lambda\Delta t$, and the probability that an existing vertex disappears is $\mu\Delta t$.

Where λ and μ are two parameters strictly positives, independent either to the size of graph and either to the time. Those parameters are same for every vertex of the graph G.

We describe variations in the graph size by a random process $\{X_t : t \in \mathbb{R}^+\}$. We have the following proposition.

Proposition 2: Let X_t the size of the graph at a time t. For a sufficiently small time interval h, we have:

$$\begin{array}{lll} (i) & Pr(X_{t+h} - X_t = 0) & = 1 - X_{t+h}\ h\ (\lambda + \mu) + o(h), \\ (ii) & Pr(X_{t+h} - X_t = 1) & = \lambda\ h\ X_{t+h} + o(h), \\ (iii) & Pr(X_{t+h} - X_t = -1) & = \mu\ h\ X_{t+h} + o(h), \\ (iv) & Pr(|X_{t+h} - X_t| \geq 2) & = 0. \end{array}$$

The equation (i) expresses the fact that from the time t to the time $t+h$ the size of the graph is not changed; no appearance or disappearance hasn't occurred. The equation (ii) (respectively (iii)) reflects the appearance (resp. disappearance) of a vertex. The equation (iv) means that no more than one event is occurred in the interval $[t, t+h]$.

The associated expected size and its variance are:

$$\begin{aligned} \mathbb{E}(X_{t+h} - X_t | X_t = n) &= h(\alpha - \beta) + o(h) \\ \mathbb{V}(X_{t+h} - X_t | X_t = n) &= h(\alpha + \beta) - (h(\alpha - \beta))^2 + o(h) \end{aligned}$$

Proof:

$$\begin{aligned} \mathbb{E}(X_{t+h} - X_t | X_t = n) =& 1 \times ((n+1)\lambda\ h\ + o(h)) \\ & + 0 \times (1 - n\ h\ (\lambda + \mu) + o(h)) \\ & - 1 \times ((n-1)\mu\ h\ + o(h)) \\ =& (n+1)\lambda\ h - (n-1)\mu h + o(h) \\ =& h((n+1)\lambda - (n-1)\mu) + o(h). \end{aligned}$$

If we pose $(n+1)\lambda = \alpha$ et $(n-1)\mu = \beta$, then:

$$\mathbb{E}(X_{t+h} - X_t | X_t = n) = h(\alpha - \beta) + o(h).$$

$$\begin{aligned} \mathbb{V}(X_{t+h} - X_t | X_t = n) =& 1^2 \times ((n+1)\lambda\ h\ + o(h)) \\ & + 0^2 \times (1 - n\ h\ (\lambda + \mu) + o(h)) \\ & + (-1)^2 \times ((n-1)\mu\ h\ + o(h)) \\ & - (\mathbb{E}(X_{t+h} - X_t | X_t = n))^2 \\ =& h(\alpha + \beta) - (h(\alpha - \beta))^2 + o(h). \end{aligned}$$

■

Proposition 3: Let $p_n(t)$ the probability to reach a graph of size $X_t = n$ at time t. We have:

$$\begin{aligned} p_n(t+h) = & p_n(t) \times Pr(X_{t+h} - X_t = 0) \\ & + p_{n-1}(t) \times Pr(X_{t+h} - X_t = 1) \\ & + p_{n+1}(t) \times Pr(X_{t+h} - X_t = -1) \\ & + o(h). \end{aligned}$$

Using the notations of the proposition 2, we obtain:

$$\begin{aligned} p_n(t+h) = & p_n(t)(1 - n(\lambda + \mu)\ h) \\ & + p_{n-1}(t)(n-1)\lambda\ h \\ & + p_{n+1}(t)(n+1)\mu\ h + o(h). \end{aligned}$$

what gives:

$$\begin{aligned} p_n(t+h) - p_n(t) = & -p_n(t)n(\lambda + \mu)\ h \\ & + p_{n-1}(t)(n-1)\lambda\ h \\ & + p_{n+1}(t)(n+1)\mu\ h + o(h). \end{aligned}$$

The division of the two terms of this equation by h gives:

$$\begin{aligned}\frac{p_n(t+h)-p_n(t)}{h} = & -p_n(t)n(\lambda+\mu) \\ & + p_{n-1}(t)(n-1)\lambda \\ & + p_{n+1}(t)(n+1)\mu \\ & + \frac{o(h)}{h}.\end{aligned}$$

When h tends to 0, the previous equation becomes:

$$\begin{aligned}\lim_{h\to 0}\frac{p_n(t+h)-p_n(t)}{h} &= \frac{dp_n(t)}{dt} \\ &= p_n'(t) \\ &= -(\lambda+\mu)np_n(t) \\ &\quad +\lambda(n-1)p_{n-1}(t) \\ &\quad +\mu(n+1)p_{n+1}(t),\end{aligned}$$

with $\lim_{h\to 0}\frac{o(h)}{h}=0$.

The resolution of this differential equation is, in the general case particularly complex; i.e. for any initial size n_0. In our study, we fix n_0 to the value 1. Thus, we obtain as solution of this differential equation the solution below:

$$\begin{aligned}p_0(t) &= \mu g(t) \\ p_n(t) &= (1-\mu g(t))(1-\lambda g(t))(\lambda g(t))^{n-1},\end{aligned}$$

with:

$$g(t)=\frac{1-e^{(\lambda-\mu)^t}}{\mu-\lambda e^{(\lambda-\mu)^t}}.$$

3.3 Operational parameters of performance

In order to correctly describe the evolution of a MANET, it is necessary to specify the mechanisms of arrival and departure of vertices (nodes). Practically the question which arises this is: *which laws that obey the arrival process and exit process ?*

To describe this phenomenon, a first idea is to use the time interval between a successive arrivals and the time interval between a successive departures of vertices, or else the number of arrivals and departures in a given interval.

The arrivals (resp. the departures) of the vertices are done according to an arrival process (resp. of departure). The arrival process (resp. of departure) is described by the sequence of the successive arrival (resp. of departure) dates:

$$a_1, a_2, \ldots, a_n, a_{n+1}, \ldots \quad \text{and} \quad s_1, s_2, \ldots, s_n, s_{n+1}, \ldots$$

Where a_n and s_n are respectively the arrival and departure of n^{th} vertex.

The difference between two successive arrival dates is named *inter-arrival*. Similarly, The difference between two consecutive departure dates is named *inter-departure* .
We note generally by $\mathcal{A}_n$ the n^{th} inter-arrival (i.e. $\mathcal{A}_n = a_{n+1} - a_n$) and by $\mathcal{S}_n$ the n^{th} inter-departure (i.e. $\mathcal{S}_n = s_{n+1} - s_n$).

We describe the arrival process by the sequence of inter-arrivals:

$$\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n, \mathcal{A}_{n+1}, \ldots.$$

In the same way, the departure process is described by the sequence of inter-departures:

$$\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n, \mathcal{S}_{n+1}, \ldots.$$

With the arrival process, it is natural to associate a *counting function* that counts the number of arrivals. This function is defined by:

$$N_a(t) = \#\{n|a_n < t\} = \sum_{n\geq 1}\mathbf{1}_{a_n<t},$$

where $\#$ indicates the set *cardinal*.

Similarly, the counting function which returns the number of departures is defined by:

$$N_s(t) = \#\{n|s_n < t\} = \sum_{n\geq 1}\mathbf{1}_{s_n<t}.$$

The *counting function* which counts the number of vertices is defined by:

$$X_t = N(0) + (N_a(t) - N_s(t)).$$

3.4 Parameters of performance in transient regime

Consider the behaviour of the graph (network) in a given period-time, for example, between $t = 0$ and $t = \mathcal{T}$. Let X_t the total number of vertices in the graph at time t. To take an interest in the vertex behaviours during the time interval $[0, \mathcal{T}]$ come back to consider the *transient regime*.

Let the following operating parameters:

- A_i: arrival time of the i^{th} node in the network;
- S_i: departure time of the i^{th} node in the network;
- C_i: connection time of i^{th} node in the network:

$$C_i = A_i - S_i;$$

- $\mathcal{T}$: total time of the observation;
- $T(n, \mathcal{T})$: total time for which the network contains n nodes; we have:

$$\sum_{n \geq 0} T(n, \mathcal{T}) = \mathcal{T};$$

- $P(n, \mathcal{T}) = \dfrac{T(n, \mathcal{T})}{\mathcal{T}}$: proportion of time for which the network contains n nodes ;
- $\alpha(\mathcal{T})$: number of arrived nodes in the network during the period $[0, \mathcal{T}]$;
- $\delta(\mathcal{T})$: number of nodes that have left the network during the period $[0, \mathcal{T}]$.

From these quantities, we define the performance parameters in transient regime as follows:

- The *average debit of arrival* $d_a(\mathcal{T})$ is the average number of nodes entered in the network over time unit. During the observation period $[0, \mathcal{T}]$, we have:

$$d_a(\mathcal{T}) = \frac{\alpha(\mathcal{T})}{\mathcal{T}}.$$

- The *average debit of departure* $d_s(\mathcal{T})$ is the average number of nodes that have left the network over time unit. During the observation period $[0, \mathcal{T}]$, we have:

$$d_s(\mathcal{T}) = \frac{\delta(\mathcal{T})}{\mathcal{T}}.$$

- The *average number of present nodes* in the network $L(\mathcal{T})$ is the temporal average of X_t (or $X(t)$) during the observation period $[0, \mathcal{T}]$, so it is the area under the curve of X_t:

$$L(\mathcal{T}) = \frac{1}{\mathcal{T}} \sum_{n \geq 0} nT(n, \mathcal{T}) = \sum_{n \geq 0} nP(n, \mathcal{T}).$$

- The *average connection time* is the arithmetic average of the connection times of arrived nodes in the network during the time interval $[0, \mathcal{T}]$:

$$C(\mathcal{T}) = \frac{1}{\alpha(\mathcal{T})} \sum_{i=1}^{\alpha(\mathcal{T})} C_i.$$

3.5 Parameters of performance in permanent regime

All previous parameters define the network performance in transient regime (after a finite time $\mathcal{T}$). In *transient regime*, we are interested to the existence of limits and their convergence values when $\mathcal{T}$ tends to infinity, and this for all these parameters:

$$d_a = \lim_{\mathcal{T} \to +\infty} d_a(\mathcal{T}); \qquad d_s = \lim_{\mathcal{T} \to +\infty} d_s(\mathcal{T});$$
$$L = \lim_{\mathcal{T} \to +\infty} L(\mathcal{T}); \qquad C = \lim_{\mathcal{T} \to +\infty} C(\mathcal{T}).$$

Stability case:

A network is stable if and only if the average asymptotic debit of departure of nodes in the network is equal to the average debit of input nodes:

$$\lim_{\mathcal{T} \to +\infty} d_a(\mathcal{T}) = \lim_{\mathcal{T} \to +\infty} d_s(\mathcal{T}) = d.$$

According to the previous relations, this implies that $\alpha(\mathcal{T})$ (total number of node arrivals during the interval $[0, \mathcal{T}]$) does not grow more faster than the total number of nodes having left the network $\delta(\mathcal{T})$. So, when $\mathcal{T}$ tends to infinity we have:

$$\lim_{\mathcal{T} \to +\infty} \frac{\delta(\mathcal{T})}{\alpha(\mathcal{T})} = 1.$$

In term of complexity we note: $\delta(\mathcal{T}) = \Theta\alpha(\mathcal{T})$

Ergodicity:

The ergodicity is a very important notion in the stochastic processes field. On the one hand, the operational analysis focuses on a particular evolution of a network between two instants $t = 0$ and $t = \mathcal{T}$. We have seen that when $\mathcal{T}$ tends to infinity and also when we consider the limitations of all operational performance parameters, this reverts to focus on permanent regime of the network. In fact, it reverts to focus on permanent regime of a particular evolution of the network. It is then possible to study different network evolutions. In fact, it reverts to focus on permanent regime of a particular evolution of the network. It is then possible to study different network evolutions. Furthermore and on other hand, the stochastic analysis will associate at the network of random variables and stochastic processes:

- V_{A_i}: random variable measuring *the arrival time* of the i^{th} node in the network;
- V_{S_i}: random variable measuring *the departure time* of the i^{th} node;
- V_{C_i}: random variable measuring *the connection time* of the i^{th} node:
$$V_{C_i} = V_{A_i - V_{S_i}}.$$
- (α_t): processes measuring *the number of arrival nodes* in the network at time t.
- (δ_t): processes measuring *the number of nodes having left* the network at time t.
- (X_t): stochastic processes measuring *the number of nodes in the network* at time t.
For a given time t, we have:
$$X_t = \alpha_t - \delta_t.$$
- $p_n(t)$: probability for that the network contains n nodes at instant t: $p_n(t) = P([X_t = n])$.

We can then, as that was done within the framework of the operational analysis, calculate all the stochastic parameters of performances, in transient regime and in permanent regime. *The average number of nodes* present in the network at the moment t is calculated as follows:

$$L(t) = \sum_{n=0}^{+\infty} n p_n(t).$$

The notion of ergodicity enables us to define a class of network for which all the particular realizations of the network evolution are asymptotically and statistically identical.
A network is ergodic if and only if for any studied particular realization of the stochastic process we have:

$$\lim_{\mathcal{T}\to+\infty} \sum_{n=0}^{+\infty} n^k P(n, \mathcal{T}) = \lim_{t\to+\infty} \sum_{n=0}^{+\infty} n^k p_n(t),$$

That implies that all the operational parameters of performances in permanent regime are equal to the stochastic parameters of performances in permanent regime. We can therefore consider this property as the ergodicity definition.

Choosing as performance parameters the proportions of time spent by the network in the state where its size $X_t = n$ and the associated probabilities for that the network contains n nodes, we obtain in the case where the network is ergodic the following equality:

$$\lim_{\mathcal{T}\to+\infty} P(n, \mathcal{T}) = \lim_{t\to+\infty} p_n(t).$$

Moreover, the proportions of time spent in a state $X_t = n$ and the probabilities of being in this state can be confused in permanent regime.

Let us note however that it exists networks not ergodic. For example, An irreducible Markov chain constitutes a network not-ergodic (some realizations lead to a absorbent sub-chain while others lead to other realizations). A periodic chain is another example of a not-ergodic network. The stationary probabilities do not exist but we are able to determine the proportions of time spent in each state of the chain.

Let us note finally that a unstable network is not also a ergodic network, and this since the limit of the median number of nodes $L(\mathcal{T})$ does not exist when $\mathcal{T}$ tends to the infinity.

4. Conclusion

In this paper, we have studied the evolution of MANET networks. Those networks are different from static networks by their mobility and their dynamicity. We have represented them by random geometric graphs and we have modelled their evolution by a Markov birth and death process.

Our perspectives to this work are to finalize, in first, the mobility part of nodes to complete our study on the evolution of MANETs, and to integrate and study, in second, the routing functions that incorporate the management of the node localizations.

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