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A Condensed and Sparse QP Formulation for Predictive Control

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Abstract— **The computational burden that model predictive control (MPC) imposes depends to a large extent on the way the optimal control problem is formulated as an optimization problem. In this paper, we present a new formulation that results in a compact and sparse optimization problem to be solved at each sampling interval. The approach is based on a change of variables that leads to a block banded Hessian when the horizon length is bigger than the controllability index of the plant. In this case the problem can be solved with an interior-point method in time linear in the horizon length. Existing dense approaches grow cubically with the horizon length, whereas existing sparse approaches grow at a significantly greater rate than with the method presented here.**

I. INTRODUCTION

Applications with fast dynamics impose high sampling frequency requirements on their control loops. In linear MPC, the optimal control input at every sampling instant is determined through the solution of a convex optimization problem with a quadratic cost and linear constraints. The very high computational demands stand as a barrier that has prevented the widespread use of MPC in applications with fast dynamics that could otherwise benefit from MPC's natural ability to deal with physical constraints. One method for reducing the computation time is to compute a piece-wise linear feedback control law offline as an explicit function of the current state of the plant [2]. Online implementation is reduced to a table look-up procedure and the evaluation of a linear function. The main drawback of explicit MPC is that the number of regions in the look-up table grows exponentially with the size of the control problem and so do the computational and memory requirements. Even for relatively small problems, solving the optimization problem online can be the faster alternative [6], [19].

The method employed when formulating the MPC problem as a quadratic program (QP) has a big impact on the problem size and structure and the resulting computational and memory requirements, as well as on the numerical conditioning. The standard approach makes use of the plant dynamics to eliminate the states from the decision variables by expressing them as an explicit function of the current state and future control inputs [12]. This *condensed* online formulation leads to compact and dense QPs. In this case, the complexity scales cubically in the horizon length when using an interior-point method. For MPC problems that require long horizon lengths, the *non-condensed* formulation, which keeps the states as decision variables and considers the system dynamics implicitly by enforcing equality constraints [14], [20], can result in significant speed-ups. With this approach the problem becomes larger but the structure can be exploited to find a solution in time linear in the horizon length. The different ways of formulating the optimization problem have an analogy in numerical integration methods, where the explicit Runge-Kutta method obtains slope estimates as an explicit function of previously computed values, whereas in the implicit method estimates are expressed as an implicit function of all other estimates. As with MPC formulations, the Runge-Kutta implicit method has better numerical properties since estimation errors are not propagated [5]. A similar analogy could be made between single and multiple shooting approaches to boundary value problems [3], [13] and the condensed and non-condensed approaches, respectively.

The non-condensed method is often also referred to as the sparse method due to the abundant structure in the resulting optimization problems. In this paper, we will show that this label does not provide the complete picture and that it is indeed possible to have a sparse condensed formulation that can be solved in time linear in the horizon length. In addition, we will show that this method is at least as fast as the standard condensed formulation and it is faster than the non-condensed formulation for a wide variety of control problems where the number of states is larger than the number of inputs. Our approach is based on the use of linear feedback policies to simulate a change of variables that results in a block banded Hessian in cases where the horizon length is larger than the controllability index of the plant. The use of feedback policies for pre-stabilization has been previously studied as an aid for proving stability [15] and as a way of improving the conditioning of the resulting optimization problem [16]. However, we find it surprising that it has not yet been applied to introduce and exploit structure in the problem, as we will do here, considering the important practical implications.

This paper is organized as follows: Section II reviews the constrained LQR problem; the condensed and non-condensed formulations are reviewed in Section III and their computational complexity is analyzed in the context of primal-dual interior-point methods. It is important to mention that the results stated in this paper will have a similar impact on active-set methods. In Section IV we present our sparse condensed approach and compare it with existing QP formulations. Numerical results are presented in Section V to verify the feasibility of the proposed approach and the paper is concluded in Section VI.

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II. CONSTRAINED LQR

Consider a discrete-time linear time-invariant model of the plant

$$x_{k+1} = Ax_k + Bu_k, \quad (1)$$
$$y_k = Cx_k, \quad (2)$$

where $x_k \in \mathbf{R}^n$ is the state vector at sample instant k, $u_k \in \mathbf{R}^m$ is the input vector and $y_k \in \mathbf{R}^p$ is the output vector. At every sampling instant, given an estimate or measurement of the current state of the plant $\widehat{x}$, the finite-horizon constrained LQR problem is to minimize

$$\frac{1}{2}x_N^T \widetilde{Q} x_N + \sum_{k=0}^{N-1} (\frac{1}{2}x_k^T Q_d x_k + \frac{1}{2}u_k^T R_d u_k + x_k^T S_d u_k) \quad (3)$$

subject to

$$x_0 = \widehat{x} \quad (4a)$$
$$x_{k+1} = Ax_k + Bu_k \qquad \text{for } k = 0, 1, 2, ..., N-1 \quad (4b)$$
$$Jx_k + Eu_k \leq d \qquad \text{for } k = 0, 1, 2, ..., N-1 \quad (4c)$$

where $Q_d \in \mathbf{R}^{n\times n}$ is symmetric positive semi-definite (SPSD), $R_d \in \mathbf{R}^{m\times m}$ is symmetric positive definite (SPD) to guarantee uniqueness of the solution, $S_d \in \mathbf{R}^{n\times m}$ is such that (3) is convex, $J \in \mathbf{R}^{l\times n}$, $E \in \mathbf{R}^{l\times m}$, $d \in \mathbf{R}^l$ and l is the number of constraints. We assume a linear control law, which is the solution to the unconstrained problem, from $k = N$ to infinity, and choose N sufficiently large such that x_N lies in an invariant set with respect to the linear control law [17]. $\widetilde{Q} \in \mathbf{R}^{n\times n}$ is the solution to the appropriate Ricatti equation.

III. REVIEW OF EXISTING QP FORMULATIONS

The optimal control problem (3)–(4) can be written as a general QP of the following form:

$$\min_\theta \frac{1}{2}\theta^T H \theta + h^T \theta \quad (5)$$

subject to

$$F\theta = f, \quad (6a)$$
$$G\theta \leq g \quad (6b)$$

where the size and structure of the matrices and vectors depends on the employed formulation.

Primal-dual interior-point methods can be used to solve for the optimal θ. Algorithm 1 is a variant of the infeasible primal-dual method [20], where ν and λ are Lagrange multipliers for the equality and inequality constraints respectively, s is a vector of slack variables, σ is a small constant between zero and one,

$$W_k := \Lambda_k S_k^{-1},$$

Λ_k and S_k are diagonal matrices containing the elements of λ_k and s_k, respectively, and

$$\mu_k := \frac{\lambda_k^T s_k}{Nl}$$

Algorithm 1 Primal-Dual Interior-Point Algorithm

Choose initial point $(\theta_0, \nu_0, \lambda_0, s_0)$ with $[\lambda_0^T, s_0^T]^T > 0$
for $k = 0$ to $P - 1$ **do**

1. $\mathcal{A}_k := \begin{bmatrix} H + G^T W_k G & F^T \\ F & 0 \end{bmatrix}$
2. $b_k := \begin{bmatrix} -h - G^T(\lambda_k - W_k g + \sigma\mu_k s_k^{-1}) \\ -F\theta_k + f \end{bmatrix}$
3. Solve $\mathcal{A}_k z_k = b_k$ for $z_k =: \begin{bmatrix} \theta_k + \Delta\theta_k \\ \Delta\nu_k \end{bmatrix}$
4. $\Delta\lambda_k := W_k(G(\theta_k + \Delta\theta_k) - g) + \sigma\mu_k s_k^{-1}$
5. $\Delta s_k := -s_k - (G(\theta_k + \Delta\theta_k) - g)$
6. Find $\alpha_k := \max_{(0,1]} \alpha : \begin{bmatrix} \lambda_k + \alpha\Delta\lambda_k \\ s_k + \alpha\Delta s_k \end{bmatrix} > 0.$
7. $(\theta_{k+1}, \nu_{k+1}, \lambda_{k+1}, s_{k+1}) :=$
   $(\theta_k, \nu_k, \lambda_k, s_k) + \alpha_k(\Delta\theta_k, \Delta\nu_k, \Delta\lambda_k, \Delta s_k)$

end for

is a measure of sub-optimality that approaches zero at the optimum. Real-time requirements will impose a hard bound on the number of interior-point iterations, hence P is assumed fixed a priori.

At each interior-point iteration, computing the matrix triple product $G^T W_k G$ and solving the system of linear equations $\mathcal{A}_k z_k = b_k$ account for most of the computation, hence the overall complexity can be expressed considering the cost of these operations only.

A. Condensed approach

The state variables can be eliminated from the decision variables of the optimization problem by expressing them as a function of the current state and input sequence:

$$\mathbf{x} = \mathbf{A}\widehat{x} + \mathbf{B}\mathbf{u} \quad (7)$$

where

$$\mathbf{x} := [x_0^T\, x_1^T\, x_2^T\, \ldots\, x_{N-1}^T\, x_N^T]^T,$$
$$\mathbf{u} := [u_0^T\, u_1^T\, u_2^T\, \ldots\, u_{N-2}^T\, x_{N-1}^T]^T,$$

$$\mathbf{A} := \begin{bmatrix} I_n \\ A \\ A^2 \\ \vdots \\ A^{N-1} \\ A^N \end{bmatrix}, \mathbf{B} := \begin{bmatrix} 0 & & & & \\ B & 0 & & & \\ AB & B & \ddots & & \\ \vdots & & \ddots & & \\ A^{N-2}B & & & B & 0 \\ A^{N-1}B & A^{N-2}B & \cdots & AB & B \end{bmatrix}.$$

In this case, $\theta := \mathbf{u}$, $F := 0$, $f := 0$ and we have an inequality constrained QP with dense matrices

$$H := \mathbf{B}^T\mathbf{Q}\mathbf{B} + \mathbf{R} + \mathbf{B}^T\mathbf{S} + \mathbf{S}^T\mathbf{B},$$
$$h := \widehat{x}^T\mathbf{A}^T(\mathbf{Q}\mathbf{B} + \mathbf{S}),$$
$$G := \mathbf{J}\mathbf{B} + \mathbf{E},$$
$$g := \mathbf{d} - \mathbf{J}\mathbf{A}\widehat{x},$$

where

$$\mathbf{Q} := \begin{bmatrix} I_N \otimes Q_d & 0 \\ 0 & \widetilde{Q} \end{bmatrix}, \mathbf{R} := I_N \otimes R_d, \mathbf{S} := \begin{bmatrix} I_N \otimes S_d \\ 0 \end{bmatrix},$$

$$\mathbf{J} := I_N \otimes J,\ \mathbf{E} := I_N \otimes E,\ \mathbf{d} := 1_N \otimes d,$$

and $\otimes$ denotes the Kronecker product.

By simple inspection of the prediction matrices $\mathbf{A}$ and $\mathbf{B}$, it becomes apparent that ill-conditioning can arise for large values of N, especially when A has eigenvalues outside the unit circle.

With this formulation, G is a lower block triangular matrix. The cost of computing $G^T W_k G$ (Line 2 in Algorithm 1) can be split into $\frac{1}{2}N^2lm$ operations for the row update $W_k G$ and $\frac{1}{2}N^3lm^2$ operations for the matrix-matrix multiplication when exploiting the symmetry of the result. In terms of the system of linear equations (Line 3 in Algorithm 1), $\mathcal{A}_k \in \mathbf{R}^{Nm \times Nm}$ is a SPD dense matrix, hence the problem can be solved using an unstructured Cholesky factorization in $\frac{1}{3}N^3m^3 + 2N^2m^2$ operations [4, App. C].

It is also worth considering the memory requirements of each formulation, since it is an important aspect for embedded implementations [11]. With respect to Algorithm 1, most memory would be used for storing matrices H, G and $\mathcal{A}_k$. With the condensed approach, the cost of storing these matrices is approximately $\frac{1}{2}N^2(2m^2 + lm)$ elements.

B. Non-condensed approach

The future states can be kept as decision variables and the system dynamics can be incorporated into the problem by enforcing equality constraints [14], [20]. In this case $\theta := [\mathbf{x}^T\ \mathbf{u}^T]^T$, $h := 0$, and the remaining matrices have sparse structures that describe the optimal control problem (3)–(4) exactly.

Assuming dense constraint matrices J and E, the cost of computing $G^T W_k G$ is broken down into $Nl(n+m)$ for the row update and $Nl(n+m)^2$ for the highly structured matrix-matrix multiplication. For solving $\mathcal{A}_k z_k = b_k$, the coefficient matrix $\mathcal{A}_k \in \mathbf{R}^{N(2n+m)\times N(2n+m)}$ is an indefinite symmetric matrix that can be made banded through appropriate row re-ordering (or interleaving of variables $\Delta\theta$ and $\Delta\nu$). The resulting banded matrix has a half-band of size $2n+m$. Such a linear system can be solved using a banded symmetric LDL^T factorization in $N(2n+m)^3 + 4N(2n+m)^2 + N(2n+m)$ floating point operations [4, App. C], or through a block factorization method based on a sequence of Cholesky factorizations in $\mathcal{O}(N(n+m)^3)$ operations [14], [20].

In terms of memory requirements, storage of matrices G, H and $\mathcal{A}_k$ requires $Nl(n+m)$, $\frac{1}{2}N(n+m)^2$ and $N(2n+m)^2$ elements, respectively. The linear growth of computational and memory requirements with respect to the horizon length, in contrast to the cubic and quadratic growth exhibited by the condensed formulation, suggests that the non-condensed approach could be favored by applications that require long horizons.

IV. SPARSE CONDENSED APPROACH

In this section we present a novel way to formulate the optimal control problem (3)–(4) as a structured optimization problem.

The reason why the non-condensed approach leads to a cost that is linear in the horizon length is due to the size of matrices H, G and $\mathcal{A}_k$ growing linearly with N, but the size of their non-zero bands is independent of N. In this section we will show that a feedback law of the form

$$u_k = Kx_k + z_k \quad \text{for } k = 0, 1, 2, ..., N-1 \tag{8}$$

can simulate a change of variables [15], [16] such that H, G and $\mathcal{A}_k$ become banded matrices with the size of the non-zero band independent of N. As a result, an interior-point method solving the new optimization problem will have a cost linear in the horizon length. It is important to clarify that this K does not have to be the same as the feedback gain being assumed from $k = N$ to infinity [17], hence stability and feasibility properties are independent of the choice of K. In this context, the effect of the change of variables is not plant pre-stabilization but a mathematical trick to introduce structure into the problem. The gain K is never implemented in practice.

With respect to the optimal control problem (3)–(4), the effect on the change of variables (8) is as follows:

$$\begin{aligned} Q_K &:= Q_d + K^T R_d K + S_d K + K^T S_d^T \\ S_K &:= S_d + K^T R_d \\ J_K &:= J + EK \\ A_K &:= A + BK \end{aligned}$$

The new transformed system is given by

$$x_{k+1} = (A+BK)x_k + Bz_k = A_K x_k + Bz_k \tag{9}$$

and the states can now be expressed as a function of the new control variables z_k

$$\mathbf{x} = \mathbf{A_K}\widehat{x} + \mathbf{B_K}\mathbf{z}$$

where

$$\mathbf{z} := [z_0^T\ z_1^T\ z_2^T\ \dots\ z_{N-2}^T\ z_{N-1}^T]^T,$$

$$\mathbf{A_K} = \begin{bmatrix} I_n \\ A_K \\ \vdots \\ A_K^{r-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{B_K} = \begin{bmatrix} 0 & & & & \\ B & 0 & & & \\ \vdots & & \ddots & & \\ A_K^{r-1}B & & & & \\ 0 & \ddots & & & \\ \vdots & \ddots & \ddots & & \\ 0 & \cdots & 0 & A_K^{r-1}B & \cdots & B \end{bmatrix}.$$

If the pair (A, B) is reachable it is possible to design a state feedback law that yields a closed-loop system with arbitrary eigenvalues [1]. By inspection of the new prediction matrices, it becomes clear that if K is chosen such that $A + BK$ is nilpotent i.e. $(A+BK)^i = 0$ for all $i \geq r$, the prediction matrix $\mathbf{B_K}$ becomes banded lower triangular if $N > r+1$. The parameter r is the controllability index of the pair (A, B) and it lies in the range

$$\left\lceil \frac{n}{m} \right\rceil \leq r \leq n$$

when B has full column rank. In practice, the value of r is equal to the lower bound in most cases.

This simple observation has very significant practical implications. When the horizon length is large enough, the matrices become banded and the structured optimization problem can be solved in time linear in the horizon length with an interior-point method. This is also the main advantage of the non-condensed approach. However, the sparse condense approach is superior in terms of computational and memory requirements for most control problems, and starts to outperform the dense condensed approach at smaller values of horizon length.

The problem of obtaining a suitable matrix K such that $A+BK$ is nilpotent is analogous to finding the deadbeat gain in the context of static state feedback. The computation of the deadbeat feedback gain in a numerically reliable way is not a trivial task and the problem has been addressed by several authors [8], [9], [18]. All proposed methods start by transforming the original system into the controllability staircase form [7], which unlike the controller canonical form, can be obtained through a numerically reliable process. The algorithm proposed by Van Dooren [8] optimizes for the minimum norm K. Instead of approaching the problem of assigning an eigenvalue of multiplicity n at zero, which is known to be a numerically sensitive problem, the algorithm uses the fact that a strictly triangular matrix is always nilpotent. Norm-preserving unitary transformations are applied to the original system to arrive at the desired nilpotent form, yielding a numerically stable algorithm.

The resulting optimization problem (5)–(6) has $\theta := \mathbf{z}$, $F := 0$, $f := 0$,

$$\begin{aligned}
H :=& \mathbf{B_K}^T\mathbf{Q}\mathbf{B_K} + (\mathbf{K}\mathbf{B_K} + I)^T(\mathbf{R}(\mathbf{K}\mathbf{B_K} + I) + \mathbf{S}^T\mathbf{B_K}) + \\
& \mathbf{B_K}^T\mathbf{S}(\mathbf{K}\mathbf{B_K} + I), \\
h :=& \widehat{x}^T\mathbf{A_K}^T(\mathbf{Q}\mathbf{B_K} + \mathbf{S}(\mathbf{K}\mathbf{B_K} + I) + \\
& \mathbf{K}^T(\mathbf{R}(\mathbf{K}\mathbf{B_K} + I) + \mathbf{S}\mathbf{B_K})), \\
G :=& (\mathbf{J} + \mathbf{E}\mathbf{K})\mathbf{B_K} + \mathbf{E}, \\
g :=& \mathbf{d} - (\mathbf{J} + \mathbf{E}\mathbf{K})\mathbf{A_K}\widehat{x},
\end{aligned}$$

where

$$\mathbf{K} := \left[\begin{array}{cc} I_N \otimes K & 0 \end{array} \right].$$

H is a block banded symmetric positive definite matrix of size $Nm \times Nm$ with half-band equal to $r+1$ blocks of size $m \times m$. G is a block banded lower triangular matrix with a half-band of $r+1$ blocks of size $l \times m$, whereas $\mathcal{A}_k$ has the same dimensions as H. It is now clear that the optimization problem can be solved using an interior-point method in time linear with respect to N. The solution of the new optimization problem applied to (8) yields exactly the same control sequence as with the other formulations for any matrix K. It is important to note that if $K = 0$ we arrive at the original condensed formulation.

The cost of computing $G^T W_k G$ is now approximated by $\frac{1}{2}Nm(r+1)l$ for the row update plus $\frac{1}{2}Nm^2(r+1)^2l$ for the matrix multiplication. The coefficient matrix $\mathcal{A}_k \in \mathbf{R}^{Nm \times Nm}$ is now an SPD banded matrix, hence the linear system can be solved using a banded Cholesky routine with a cost of $Nm^3(r+1)^2 + 4Nm^2(r+1)$ operations [4].

TABLE I

COMPARISON OF THE COMPUTATIONAL COMPLEXITY IMPOSED BY THE DIFFERENT QP FORMULATIONS.

	Computation
Condensed	$\mathcal{O}(N^3m^2(l+m))$
Non-condensed	$\mathcal{O}(N(m+n)^2(l+m+n))$
Sparse condensed	$\mathcal{O}(Nm^2r^2(l+m))$

TABLE II

COMPARISON OF THE MEMORY REQUIREMENTS IMPOSING BY THE DIFFERENT QP FORMULATIONS.

	Memory
Condensed	$\mathcal{O}(N^2m(l+m))$
Non-condensed	$\mathcal{O}(N(m+n)(l+m+n))$
Sparse condensed	$\mathcal{O}(Nrm(l+m))$

The memory requirements for storing the main matrices are now approximately $\frac{1}{2}N(r+1)(2m^2+lm)$.

Being able to directly apply methods for positive definite matrices instead of methods for indefinite matrices is another benefit over the non-condensed approach. For instance, Cholesky factorization is more numerically stable than LDL^T, it requires slightly less computation, and the possibility of choosing an arbitrary permutation matrix allows for a simpler pivoting procedure and the possibility of making use of the block structure inside the non-zero band to reduce computation and memory requirements further. Another advantage is that input rate constraints can be added to the optimal control problem (3)–(4) without affecting the structure of the matrices in the optimization problem (5)–(6), whereas with the non-condensed approach the inclusion of rate constraints increases the bandsize of G and consequently $\mathcal{A}_k$.

Tables I and II compare the upper bound computational complexity and memory requirements for the three different QP formulations that have been discussed in this paper. The expressions for the sparse condensed approach assume that $N > r+1$, otherwise the matrices are dense. Hence, the sparse condensed approach is always at least as good as the standard condensed approach in terms of computational complexity and memory requirements. Taking a conservative assumption for the largest possible nilpotency index $r = n$, the expressions suggest that if the number of states is larger than the number of inputs the formulation presented in this paper will provide an improvement over the non-condensed approach, both in terms of computation and memory usage. Both these approaches will outperform the standard condensed approach for large N. These predictions are confirmed by Figure 1.

In Section III we discussed how the condensed approach can lead to ill-conditioned predictions for large N as a consequence of the state-transition matrix being raised to high powers. This issue has been addressed in the literature

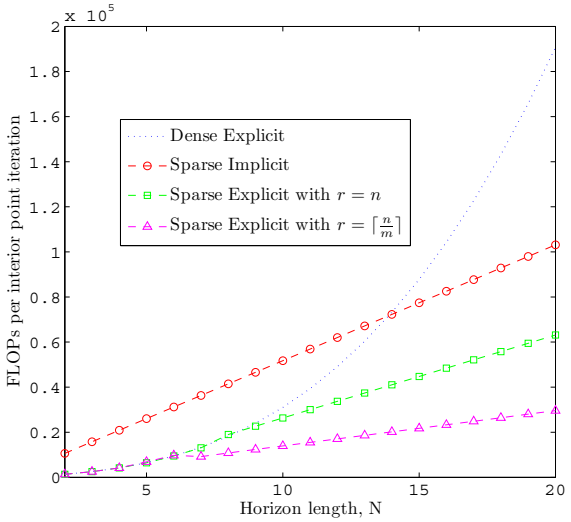

Fig. 1. Accurate count of the number of floating point operations per interior-point iteration for the different QP formulations discussed in this paper. The size of the control problem is $m = 2$, $n = 6$, $l = 6$. The upper and lower bounds for r are shown.

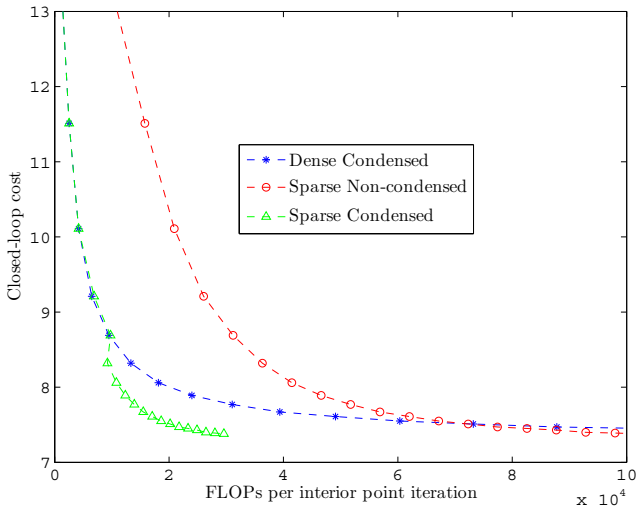

Fig. 2. Trade-off between closed-loop control cost and computational cost for all different QP formulations.

by suggesting a pre-stabilizing feedback law [16]. This approach can improve the conditioning of $\mathbf{A_K}$ and $\mathbf{B_K}$, but it still requires raising A to powers up to N, which causes model mismatch and finite representation errors to accumulate. The sparse condensed approach presented here limits the power to which A is raised to a maximum of $r-1$, hence reducing this error accumulation.

A. Limitations of the sparse condensed approach

It is well-known that control and signal processing problems can be ill-conditioned when there is a large mismatch between the requested sampling frequency and the dynamics of the continuous-time time system [10]. The method presented in this paper is no exception to the rule, however, the conditioning is acceptable for most control problems, especially when targeting systems with fast dynamics.

The matrix K is the deadbeat feedback gain that can control any state to the origin in r steps. For systems with fast dynamics it is easier to steer the state quickly, hence relatively small values of K are necessary and the conditioning of the problem is acceptable. It is precisely applications with fast dynamics that can benefit most from methods for solving optimization problems faster that can turn the possibility of employing MPC into a feasible option.

A well-known drawback of deadbeat control is that if the sampling period is small with respect to the system's dynamics, a very large K could be necessary, since more energy would be needed to steer the state to zero in less time. In the context of this paper, a large value of K can result in an ill-conditioned optimization problem. Our numerical simulations have indeed confirmed that for systems with slow unstable dynamics, when sampling in the millisecond range using the sparse condensed approach, the QP problems become badly conditioned and the trajectory of the system is very similar to the open-loop trajectory. However, for plants with stable dynamics, the control quality is good at most sampling frequency regimes, hence the problem could be solved by using a pre-stabilising gain.

B. Linear time-varying systems

In the proposed method the matrix K is computed offline for a given A and B, hence this approach might not be suitable for time-varying MPC applications where the model changes are not known a priori. However, the computation of the deadbeat gain K does not depend on the horizon length and it is significantly simpler than solving the QPs so ocassional online recomputation of K could still be feasible for some applications, especially since the QP matrices also need to be recomputed when using condensing.

For time-varying systems where the model changes are known ahead of time, such as periodic systems, a deadbeat K needs to be computed for every distinct model. If there are model changes during the prediction horizon, the effectiveness of the proposed approach in introducing structure will depend on the sequence of model changes.

V. NUMERICAL RESULTS

In order to investigate the numerical properties of the proposed approach we use an example oscillatory system composed of six masses connected by springs and dampers. The masses, spring constants and damping coefficients are 0.1, 150 and 0.01, respectively. The control objective is to keep all the masses at their rest position. All masses can be actuated and the constraints are given by

$$\begin{aligned} -0.5 \leq \quad u_k \quad &\leq 0.5, \\ -4 \leq \quad y_k \quad &\leq 4, \end{aligned}$$

hence the control problem has $m = 6$, $n = 12$, $p = 6$ and $l = 24$. Matrices Q_d and R_d are obtained from the continuous-time matrices Q_c and R_c assuming a zero order hold. These are chosen such that the inputs and outputs are

weighed equally. The horizon length is 0.3 seconds and the number of steps is given by $N = \lceil \frac{0.3}{\Delta} \rceil$ where Δ is the sampling period. The plant is controllable with controllability index $r = 2$ and has a maximum frequency pole at 12Hz.

All simulations start with all masses at their rest position, except mass 6 which is displaced to the constraint. This starting condition guarantees that input constraints will become active during the simulations, which last for 6 seconds. Sampling faster leads to better quality control for all formulations as the controller is able to respond faster to disturbances. However, this also means that the number of steps in the horizon increases, hence the amount of computation required also increases. Figure 2 shows the trade-off between closed-loop control quality and computational requirements for all QP formulations described in this paper. The plot is obtained by obtaining the closed-loop cost and computational demands for a range of sampling frequencies. For a given control quality the proposed approach requires less computation, and for a fixed computational power the proposed approach achieves a better control quality because it allows faster sampling.

VI. CONCLUSION

In this paper, we have presented a novel way to formulate a constrained optimal control problem as a structured optimization problem that can be solved in time linear in the horizon length with an interior-point method. The structure is introduced through a suitable change of variables that results in banded prediction matrices. The proposed method has been compared against the current standard approaches and it has been shown to offer reduced computational and memory requirements for most control problems. As a result, employing the proposed approach could allow one to push the boundaries of MPC to applications where the computational burden has so far been too great, or it could allow current MPC applications to run on cheaper commodity hardware.

The limitations of the approach have also been identified. Existing algorithms for computing the deadbeat feedback gain K attempt to find the minimum nilpotency index r, because the goal of the feedback is to provide a closed-loop system that steers the state to zero in the least number of steps. In the context of this paper, deadbeat control is used as a mathematical trick to introduce structure into the optimization problem, and a smaller value of r means that the non-zero bands of the matrices will be smaller. However, in order to improve the numerical conditioning of the problem, it may be preferable to increase the nilpotency index beyond the controllability index of the plant, especially since any r smaller than $N-1$ provides an improvement over the standard condensed approach. A methodology that allows trading computational time and memory requirements for numerical conditioning of the resulting optimization problem could be a target for future research.

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