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Research Article

Generalized Derivations of Prime Rings

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Let R be an associative prime ring, U a Lie ideal such that $u^2 \in U$ for all $u \in U$. An additive function $F : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. In this paper, we prove that $d = 0$ or $U \subseteq Z(R)$ if any one of the following conditions holds: (1) $d(x) \circ F(y) = 0$, (2) $[d(x), F(y) = 0]$, (3) either $d(x) \circ F(y) = x \circ y$ or $d(x) \circ F(y) + x \circ y = 0$, (4) either $d(x) \circ F(y) = [x, y]$ or $d(x) \circ F(y) + [x, y] = 0$, (5) either $d(x) \circ F(y) - xy \in Z(R)$ or $d(x) \circ F(y) + xy \in Z(R)$, (6) either $[d(x), F(y)] = [x, y]$ or $[d(x), F(y)] + [x, y] = 0$, (7) either $[d(x), F(y)] = x \circ y$ or $[d(x), F(y)] + x \circ y = 0$ for all $x, y \in U$.

1. Introduction

Let R be an associative ring with center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and denote by $x \circ y$ the anticommutator $xy + yx$. Given two subsets A and B of R, $[A, B]$ will denote the additive subgroup of R generated by all elements of the form $[a, b]$, where $a \in A$, $b \in B$. For a nonempty subset S of R, we put $C_R(S) = \{x \in R \mid [x, s] = 0 \text{ for all } s \in S\}$. Recall that R is *prime* if $aRb = 0$ implies $a = 0$ or $b = 0$. An additive map $d : R \to R$ is called a *derivation* if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive function $F : R \to R$ is called a *generalized* derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Many analysts have studied generalized derivation in the context of algebras on certain normed spaces (see [1] for reference). An additive subgroup U of R is said to be a *Lie ideal* of R if $[u, r] \in U$ for all $u \in U$ and $r \in R$.

In [2], Ashraf et al. investigates the commutativity of a prime ring R admitting a generalized derivation F with associated derivation d satisfying anyone of the following properties: $d(x) \circ F(y) = 0$, $[d(x),F(y)] = 0$, $d(x) \circ F(y) = x \circ y$, $d(x) \circ F(y) + x \circ y = 0$, $d(x) \circ F(y) - xy \in Z(R)$, $d(x) \circ F(y) + xy \in Z(R)$, $[d(x),F(y)] = [x,y]$, $[d(x),F(y)] + [x,y] = 0$ for all $x,y \in I$, where I is a nonzero ideal of R.

In [3], Bergen et al. investigate the relationship between the derivations and Lie ideals of a prime ring, and obtain some useful results. In [4], P. H. Lee and T. K. Lee get six sufficient conditions of central Lie ideal, which extend some results of commutativity on a prime ring. Motivated by the above, in this paper, we extend M. Ashraf's results to a Lie ideal of a prime ring. Throughout this paper, R will be a prime ring and U will always denote a Lie ideal of R.

2. Preliminaries

We begin with the following known results which will be used extensively to prove our theorems.

Lemma 2.1. *If U is a Lie ideal of R such that $u^2 \in U$ for all $u \in U$, then $2uv \in U$ for all u, $v \in U$.*

Proof. For all $w,u,v \in U$,

$$uv + vu = (u+v)^2 - u^2 - v^2 \in U. \tag{2.1}$$

On the other hand,

$$uv - vu \in U. \tag{2.2}$$

Adding two expressions, we have $2uv \in U$ for all $u,v \in U$. □

Lemma 2.2 [3]. *If $U \not\subseteq Z(R)$ is a Lie ideal of R, then $C_R(U) = Z(R)$.*

Lemma 2.3 [3]. *If U is a Lie ideal of R, then $C_R([U,U]) = C_R(U)$.*

Lemma 2.4. *Set $V = \{u \in U \mid d(u) \in U\}$. If $U \not\subseteq Z(R)$, then $V \not\subseteq Z(R)$.*

Proof. Assume that $V \subseteq Z(R)$. Since $[U,U] \subseteq U$ and $d([U,U]) \subseteq U$, we have $[U,U] \subseteq V \subseteq Z(R)$. Hence $C_R([U,U]) = R$. From Lemma 2.2, $C_R(U) = Z(R)$. But by Lemma 2.3, $C_R([U,U]) = C_R(U)$. That is, $R = Z(R)$, a contradiction. □

Lemma 2.5 [3]. *If $U \not\subseteq Z(R)$ is a Lie ideal of R and if $aUb = 0$, then $a = 0$ or $b = 0$.*

Lemma 2.6. *A group can not be a union of two of its proper subgroups.*

Lemma 2.7 [4]. *Let $d \neq 0$ be a derivation of R such that $[u,d(u)] \in Z(R)$ for all $u \in U$. Then $U \subseteq Z(R)$.*

Lemma 2.8 [4]. *Let d and δ be nonzero derivations of R such that $d\delta(U) \subseteq Z(R)$. Then $U \subseteq Z(R)$.*

Lemma 2.9 [5]. *If U is a Lie ideal of a semiprime ring R and $[U,U] \subseteq Z(R)$, then $U \subseteq Z(R)$.*

Lemma 2.10 [3]. *If $d \neq 0$ is a derivation of R, and if U is a Lie ideal of R such that $d(U) \subseteq Z(R)$, then $U \subseteq Z(R)$.*

3. The proof of main theorems

Theorem 3.1. *Let R be a prime ring with char $R \neq 2$, U a Lie ideal such that $u^2 \in U$ for all $u \in U$, and F a generalized derivation associated with $d \neq 0$. If $d(x) \circ F(y) = 0$ for all $x, y \in U$, then $U \subseteq Z(R)s$.*

Proof. Assume that $U \not\subseteq Z(R)$, then $V \not\subseteq Z(R)$ by Lemma 2.4. Now we have $d(x) \circ F(y) = 0$ for all $x, y \in U$. Replacing y by $2yz$, by Lemma 2.1 and using char $R \neq 2$, we get $d(x) \circ F(yz) = 0$ for all $x, y, z \in U$ and we obtain $(d(x) \circ y)d(z) - y[d(x), d(z)] + (d(x) \circ F(y))z - F(y)[d(x), z] = 0$. Now using our hypotheses, the above relation yields $(d(x) \circ y)d(z) - y[d(x), d(z)] - F(y)[d(x), z] = 0$. For any $x \in V$, replace z by $d(x)$ to get

$$(d(x) \circ y)d^2(x) - y[d(x), d^2(x)] = 0. \tag{3.1}$$

Now, replace y by $2zy$ in (3.1) to get $(d(x) \circ (zy))d^2(x) - zy[d(x), d^2(x)] = 0$. This implies that

$$z(d(x) \circ y)d^2(x) + [d(x), z]yd^2(x) - zy[d(x), d^2(x)] = 0. \tag{3.2}$$

Combining (3.1) with (3.2), we get $[d(x), z]yd^2(x) = 0$ for $x \in V$ and $y, z \in U$. In particular, $[d(x), x]yd^2(x) = 0$, and hence $[d(x), x]Ud^2(x) = 0$ for $x \in V$. Then, either $[d(x), x] = 0$ or $d^2(x) = 0$ by Lemma 2.5. Now let $V_1 = \{x \in V \mid [d(x), x] = 0\}$ and $V_2 = \{x \in V \mid d^2(x) = 0\}$. Then V_1, V_2 are both additive subgroups of V and $V_1 \cup V_2 = V$. Thus, either $V = V_1$ or $V = V_2$ by Lemma 2.6. If $V = V_1$, then Lemma 2.7 gives $V \subseteq Z(R)$, a contradiction. On the other hand, if $V = V_2$ then $V \subseteq Z(R)$ by Lemma 2.8, again a contradiction. □

Theorem 3.2. *Let R be a prime ring with char $R \neq 2$, U a Lie ideal such that $u^2 \in U$ for all $u \in U$, and F a generalized derivation associated with $d \neq 0$. If $[d(x), F(y)] = 0$ for all $x, y \in U$, then $U \subseteq Z(R)$.*

Proof. Assume that $U \not\subseteq Z(R)$, then $V \not\subseteq Z(R)$ by Lemma 2.4. By hypotheses, we have

$$[d(x), F(y)] = 0 \tag{3.3}$$

for all $x, y \in U$. Replacing y by $2yz$ in (3.3) and using char $R \neq 2$, we get

$$F(y)[d(x), z] + y[d(x), d(z)] + [d(x), y]d(z) = 0 \tag{3.4}$$

for all $x, y, z \in U$. For any $x \in V$, replacing z by $2zd(x)$ in (3.4) and using (3.4), we get

$$yz[d(x), d^2(x)] + y[d(x), z]d^2(x) + [d(x), y]zd^2 = 0. \tag{3.5}$$

Again, replacing y by $2ty$ in (3.5) and using (3.5), we get $[d(x),t]yzd^2(x) = 0$ for all $x \in V$ and $y,z,t \in U$. In particular, $[d(x),x]yzd^2(x) = 0$. That is, $[d(x),x]UUd^2(x) = 0$. Then, either $[d(x),x] = 0$ or $Ud^2(x) = 0$ (i.e., $d^2(x) = 0$). Now, let $V_1 = \{x \in V \mid [d(x),x] = 0\}$ and $V_2 = \{x \in V \mid Ud^2(x) = 0\}$. Then V_1, V_2 are both additive subgroups of V and $V_1 \cup V_2 = V$. Thus, either $V = V_1$ or $V = V_2$ by Lemma 2.6. If $V = V_1$, then Lemma 2.7 gives $V \subseteq Z(R)$, a contradiction. On the other hand, if $V = V_2$ then $Ud^2(x) = 0$, and hence $d^2(x)Ud^2(x) = 0$ for all $x \in V$. Thus, Lemma 2.5 yields that $d^2(x) = 0$ for all $x \in V$. Now, we have obtained $V \subseteq Z(R)$ by Lemma 2.8, again a contradiction. □

Theorem 3.3. *Let R be a prime ring with char $R \neq 2$, U a Lie ideal such that $u^2 \in U$ for all $u \in U$, and F a generalized derivation associated with $d \neq 0$. If $d(x) \circ F(y) = x \circ y$ for all $x,y \in U$, then $U \subseteq Z(R)$.*

Proof. We are given $d(x) \circ F(y) = x \circ y$ for all $x,y \in U$. If $F = 0$, then $x \circ y = 0$ for all $x,y \in U$. Replacing y by $2yz$ and using char $R \neq 2$, we get $y[x,z] = 0$ for all $x,y,z \in U$. In particular, $[x,z]y[x,z] = 0$ (i.e., $[x,z]U[x,z] = 0$), and hence $[x,z] = 0$ (i.e., $[U,U] = 0$) by Lemma 2.5. Then Lemma 2.9 yields that the required result. Therefore, we assume that $F \neq 0$. For any $x,y \in U$, we have

$$d(x) \circ F(y) = x \circ y. \tag{3.6}$$

Replacing y by $2yz$, we get

$$(d(x) \circ y)d(z) - y[d(x),d(z)] + (d(x) \circ F(y))z - F(y)[d(x),z] = (x \circ y)z - y[x,z]. \tag{3.7}$$

Combining (3.6) with (3.7), we get

$$(d(x) \circ y)d(z) - y[d(x),d(z)] - F(y)[d(x),z] + y[x,z] = 0 \tag{3.8}$$

for all $x,y,z \in U$. For any $x \in V$, replacing z by $d(x)$ in (3.8), we get

$$(d(x) \circ y)d^2(x) - y[d(x),d^2(x)] + y[x,d(x)] = 0. \tag{3.9}$$

Now, replacing y by $2yz$ in (3.9), we get

$$(z(d(x) \circ y) + [d(x),z]y)d^2(x) - zy[d(x),d^2(x)] + zy[x,d(x)] = 0. \tag{3.10}$$

Combining (3.9) with (3.10), we get $[d(x),z]yd^2(x) = 0$ for all $x \in V$ and $y,z \in U$. In particular, $[d(x),x]yd^2(x) = 0$. So we get $[d(x),x]Ud^2(x) = 0$. The rest of the proof is the same with the end of Theorem 3.1, and we get the required result. □

Now, using the similar techniques, we also prove the following two theorems.

Theorem 3.4. *Let R be a prime ring with char $R \neq 2$, U a Lie ideal such that $u^2 \in U$ for all $u \in U$, and F a generalized derivation associated with $d \neq 0$. If $d(x) \circ F(y) + x \circ y = 0$ for all $x,y \in U$, then $U \subseteq Z(R)$.*

Theorem 3.5. *Let R be a prime ring with char $R \neq 2$, U a Lie ideal such that $u^2 \in U$ for all $u \in U$, and F a generalized derivation associated with $d \neq 0$. If either $[d(x),F(y)] = [x,y]$ or $[d(x),F(y)] + [x,y] = 0$ for all $x,y \in U$, then $U \subseteq Z(R)$.*

Theorem 3.6. *Let R be a prime ring with char $R \neq 2$, U a Lie ideal such that $u^2 \in U$ for all $u \in U$, and F a generalized derivation associated with $d \neq 0$. If $d(x) \circ F(y) - xy \in Z(R)$ for all $x,y \in U$, then $U \subseteq Z(R)$.*

Proof. Assume that $U \nsubseteq Z(R)$, then $V \nsubseteq Z(R)$ by Lemma 2.4. By hypotheses, we have $d(x) \circ F(y) - xy \in Z(R)$. If $F = 0$, then $xy \in Z(R)$ for all $x,y \in U$. In particular, $[xy,x] = 0$ and hence $x[y,x] = 0$. Replacing y by $2yz$ we get $xy[z,x] = 0$ (i.e., $xU[z,x] = 0$). Hence, by Lemma 2.5, either $x = 0$ or $[z,x] = 0$. But $x = 0$ also implies that $[z,x] = 0$; hence for any $x,z \in U$, we have $[z,x] = 0$. Then Lemma 2.9 gives us the required result. Now we assume that $F \neq 0$. For any $x,z \in U$, we have $d(x) \circ F(y) - xy \in Z(R)$. Replacing y by $2yz$ and using char $R \neq 2$, we get $d(x) \circ F(yz) - xyz \in Z(R)$ for any $x,y,z \in U$. That is, $(d(x)F(y) - xy)z + d(x)yd(z) \in Z(R)$. This implies that $[d(x)yd(z),z] = 0$. Then we have $d(x)[yd(z),z] + [d(x),z]yd(z) = 0$ for any $x,y,z \in U$. For any $x \in V$, replace y by $2d(x)y$ in the above relation to get $[d(x),z]d(x)yd(z) = 0$ (i.e., $[d(x),z]d(x)Ud(z) = 0$). Then we have $[d(x),z]d(x) = 0$ or $d(z) = 0$ by Lemma 2.5. Now, let $U_1 = \{z \in U \mid [d(x), z]d(x) = 0\}$ and $U_2 = \{z \in U \mid d(z) = 0\}$. Then U_1, U_2 are both additive subgroups of U and $U_1 \cup U_2 = U$. Thus, either $U = U_1$ or $U = U_2$ by Lemma 2.6. If $U = U_1$, replace z by $2zy$ to get $[d(x),z]yd(x) = 0$, and hence $[d(x),z] = 0$ (especially, $[d(x),x] = 0$) or $d(x) = 0$. For all $x \in U$ we have $[d(x),x] = 0$, thus $U \subseteq Z(R)$ by Lemma 2.7, a contradiction. On the other hand, if $U = U_2$ then $d(U) = 0$ and hence $U \subseteq Z(R)$ by Lemma 2.10, again a contradiction. □

The following is proved as in Theorem 3.6 with necessary variations.

Theorem 3.7. *Let R be a prime ring with char $R \neq 2$, U a Lie ideal such that $u^2 \in U$ for all $u \in U$, and F a generalized derivation associated with $d \neq 0$. If $d(x) \circ F(y) + xy \in Z(R)$ for all $x,y \in U$, then $U \subseteq Z(R)$.*

Theorem 3.8. *Let R be a prime ring with char $R \neq 2$, U a Lie ideal such that $u^2 \in U$ for all $u \in U$, and F a generalized derivation associated with $d \neq 0$. If $[d(x),F(y)] = [x,y]$ for all $x,y \in U$, then $U \subseteq Z(R)$.*

Proof. If $F = 0$ then $[x,y] = 0$. That is, $[U,U] = 0$, and hence $U \subseteq Z(R)$ by Lemma 2.9. Now we assume that $F \neq 0$. Then we have

$$[d(x),F(y)] = [x,y] \tag{3.11}$$

for all $x,y \in U$. Replacing y by $2yz$ in (3.11) and using (3.11), we have

$$F(y)[d(x),z] + y[d(x),d(z)] + [d(x),y]d(z) = y[x,z] \tag{3.12}$$

for all $x,y,z \in U$. Now, for any $x \in V$, replace z by $2zd(x)$ in (3.12) and use (3.12) to get

$$y[d(x),z]d^2(x) + yz[d(x),d^2(x)] + [d(x),y]zd^2(x) = yz[x,d(x)]. \tag{3.13}$$

Again, replace y by $2ty$ in (3.13) to get

$$tyz[d(x),d^2(x)]+ty[d(x),z]d^2(x)+t[d(x),y]zd^2(x)+[d(x),t]yzd^2(x)=tyz[x,d(x)]. \tag{3.14}$$

Combining (3.13) with (3.14), we have $[d(x),t]yzd^2(x)=0$. In particular, $[d(x),x]yzd^2(x)=0$ for all $x\in V$ and $y,z\in U$. Notice that the arguments in the end of the proof of Theorem 3.1 are still valid in the present situation, and hence we get the required results.

We also prove the following as in Theorem 3.8 with necessary variations. □

Theorem 3.9. *Let R be a prime ring with char $R\neq 2$, U a Lie ideal such that $u^2\in U$ for all $u\in U$, and F a generalized derivation associated with $d\neq 0$. If $[d(x),F(y)]+[x,y]=0$ for all $x,y\in U$, then $U\subseteq Z(R)$.*

Finally, using the similar techniques as in the above theorems, we prove the following.

Theorem 3.10. *Let R be a prime ring with char $R\neq 2$, U a Lie ideal such that $u^2\in U$ for all $u\in U$, and F a generalized derivation associated with $d\neq 0$. If either $[d(x),F(y)]=x\circ y$ or $[d(x),F(y)]+x\circ y=0$ for all $x,y\in U$, then $U\subseteq Z(R)$.*

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