ON THE CONSTRUCTION OF LOW-PASS FILTERS ON THE UNIT SPHERE

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ABSTRACT

This paper considers the problem of construction of low-pass filters on the unit sphere, which has wide ranging applications in the processing of signals on the unit sphere. We propose a design criterion for the construction of strictly bandlimited low-pass filters in the spectral domain with optimal concentration in the specified polar cap region in the spatial domain. Our approach uses the weighted sum of the first optimally concentrated eigenfunctions from appropriately formulated Slepian concentration problems on the sphere. Furthermore, in order to reduce the computational complexity of the proposed algorithm, we develop a closed-form expression to accurately model these eigenfunctions. We illustrate the construction of low-pass filters using the proposed approach and demonstrate the advantage of our method approach compared to a diffusion based approach in the literature in terms of control over both bandwidth in the spectral domain and concentration in the spatial domain.

Keywords: Filtering, unit sphere, bandlimited signals, Slepian concentration problem, convolution.

1. INTRODUCTION

Motivated by the growing number of applications in diverse fields such as geodesy and cosmology [1], 3D beamforming [2], image processing [3], computer graphics [4] and medical imaging [5], there has been an increasing interest in developing theories and techniques for processing of signals on the sphere. In this regard, the design and construction of low-pass filters on the sphere is an important problem which has direct applications in all the above fields where signals are defined in the unit sphere domain.

Recently, some work has been done to extend the well known filtering methods in the Euclidean domain to the spherical domain, but many key challenges still remain. In general, the closed geometry of the sphere makes it complex and non-trivial to emulate and extend familiar operations in Euclidean domain to the spherical domain. The problems of filtering and sampling on the sphere have been studied in [6] and [5], which provide the equivalents of Nyquist-Shannon sampling theorem and the generalised Papoulis sampling theorem on the sphere, respectively. A general framework, that unifies the different and often apparently conflicting notions of convolution on the sphere is proposed in [7]. The matched filter on sphere has been derived in [8] and is used for detection of objects embedded in stochastic backgrounds. The theoretical conditions on the invertibility of filter banks on the sphere are investigated in [5], but no design criterion is proposed. A low-pass filter based on diffusion approach is proposed in [3]. It is shown that this approach is equivalent to Gaussian smoothing on the sphere and is computationally efficient, but it does not provide explicit control over either the filter bandwidth or the concentration of filtering kernel in spatial domain. The work in [1] poses and solves the analogue of Slepian's concentration problem [9] on the sphere, that of optimally concentrating a signal in both spatial and spectral domains. The authors propose a numerical technique to determine a family of eigenfunctions which are optimally concentrated in spatial domain and strictly bandlimited in the spectral domain. The use of these eigenfunctions for localized spectral analysis on the unit sphere is investigated in [10]. It is suggested in [5, 10] that these eigenfunctions could be suitable candidates for filter design on the sphere. However, to the best of author's knowledge, no such technique exists in the literature.

In this work, we focus on the design of an azimuthally-symmetric low-pass filter. We propose a novel design criterion for low-pass filter construction in spectral domain such that the spatial response of the filter is optimally concentrated in a specified polar cap region. Our approach is based on Simon's bandlimited eigenfunctions [10] and the resultant filter confines the spatial response in a defined polar cap region at the cost of a small ripple in spectral domain. In addition, the proposed filter is perfectly band-limited in spectral domain with no side lobes. We develop a closed-form expression to accurately model the eigenfunctions being used in construction process, which greatly reduces the computational complexity of the proposed algorithm. We illustrate the construction of low-pass filters using the proposed approach and demonstrate the advantage of our technique compared with the diffusion based low-pass filtering in [3] in terms of control over filter response in both spectral and spatial domains.

This paper is organized as follows. The mathematical background is summarized in Section 2. The problem formulation is discussed in Section 3. The proposed low-pass filter design algorithm and closed-form formulation is proposed in Section 4. Results are discussed in Section 5. Finally, Section 6 concludes the paper.

2. MATHEMATICAL BACKGROUND

Let $g(\theta,\phi)$ be a square integrable function, defined on the unit sphere $\mathbb{S}^2 \triangleq \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| = 1\}$ in complex Hilbert space $L^2(\mathbb{S}^2)$, where **x** is the unit vector, $\theta \in [0,\pi]$ denotes the co-latitude measured with respect to positive z-axis, $\phi \in [0, 2\pi)$ denotes the longitude and is measured with respect to positive x-axis in the $x-y$ plane. Note that $\theta = 0$ corresponds to the north pole.

The spherical harmonics, $Y_\ell^m(\theta,\phi)$, for degree $\ell \geq 0$ and order $|m| \leq \ell$ are defined as [11]

$$Y_\ell^m(\theta,\phi) = N_\ell^m P_\ell^m(\cos\theta)e^{im\phi} \tag{1}$$

where i is the imaginary unit and N_ℓ^m is the normalization constant

$$N_\ell^m = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} \tag{2}$$

This work was supported by the Australian Research Council Grant DP1094350.

and P_ℓ^m are the Associated Legendre Polynomials defined as

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!}\sqrt{(1-x^2)^m}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell \tag{3}$$

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(x) \tag{4}$$

for $|x| \leq 1$ and $m \geq 0$. With above definitions, spherical harmonic functions form an orthonormal set of basis functions for $L^2(\mathbb{S}^2)$. Thus, any function $g(\theta,\phi)$ defined on unit sphere can be expanded in terms of spherical harmonics as

$$g(\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} g_\ell^m Y_\ell^m(\theta,\phi) \tag{5}$$

where g_ℓ^m is the spherical harmonics coefficient, which is obtained by projecting the function $g(\theta,\phi)$ onto $Y_\ell^m(\theta,\phi)$ as

$$g_\ell^m \triangleq \langle g, Y_\ell^m\rangle = \int_\Omega g(\theta,\phi)\overline{Y_\ell^m(\theta,\phi)}d\Omega \tag{6}$$

where Ω denotes integration over the whole unit sphere, $d\Omega = \sin(\theta)d\theta d\phi$ and $\overline{(\cdot)}$ denotes the complex conjugate operation. Note that if $g(\theta,\phi)$ is an azimuthally-symmetric function (independent of ϕ), then only order $m = 0$ harmonic coefficients are non-zero.

3. PROBLEM FORMULATION

A suitable approach to formulate filtering on the sphere is to emulate the filtering in Euclidean domain. Following the analogy, low-pass filtering on the unit sphere may be defined as spherical convolution. If $f(\theta,\phi)$ represents the signal to be filtered using filter kernel $h(\theta,\phi)$, the filtered output $g(\theta,\phi)$ is given by

$$g(\theta,\phi) = f(\theta,\phi) * h(\theta,\phi) \tag{7}$$

where '*' denotes the convolution operation. There are different notions of spherical convolution available in the literature [7]. Due to its importance in the sequel, we first summarize the definition of convolution used in this work, before defining the design criteria for the construction of low-pass filters on the sphere.

3.1. Convolution on Unit Sphere

In the Euclidean domain, convolution is a function of translations of one function relative to the other. On the sphere, rotations are the analog of translations in Euclidean domain. The convolution of spherical filter $h(\theta,\phi)$ and a spherical signal $f(\theta,\phi)$ is given by [6]

$$g(\alpha,\beta,\gamma) = \int_{\mathbb{S}^2}[D(\alpha,\beta,\gamma)h](\theta,\phi)f(\theta,\phi)d\Omega \tag{8}$$

where $g(\alpha,\beta,\gamma)$ is a function on $SO(3)$, $\alpha \in [0,2\pi)$, $\beta \in [0,\pi]$, $\gamma \in [0,2\pi)$ are the Euler angles which parameterize rotations on the sphere, $D(\alpha,\beta,\gamma)$ is the rotation operator which gives '$\alpha\beta\gamma$' rotation about 'zyz-axis'. The effect of rotation on spherical harmonics coefficients can be evaluated using Wigner-D functions [11]. For z-axis symmetric filter $h(\theta,\phi) = h(\theta)$, rotation by γ about z-axis causes no effect on $h(\theta,\phi)$. Thus with $\gamma = 0$, $g(\alpha,\beta,0) = g(\alpha,\beta)$ is a function on $\mathbb{S}^2$ parameterized by $\phi = \alpha$ and $\theta = \beta$.

If $h(\theta,\phi) = h(\theta)$, the convolution of two signals in (8) is equivalent to the following multiplication in spherical harmonics domain [7]

$$g_\ell^m = \sqrt{\frac{4\pi}{2\ell+1}}f_\ell^m h_\ell^0. \tag{9}$$

where $g_\ell^m = \langle g, Y_\ell^m\rangle$, $f_\ell^m = \langle f, Y_\ell^m\rangle$, $h_\ell^m = \langle h, Y_\ell^m\rangle$. Note that (9) is a frequency domain version of (8) and in general, not commutative. In addition, there is a scaling factor which decays with ℓ. Due to the $m = 0$ action on one of the arguments, we can see that the filter h scales spectral coefficients f_l^m for every degree of particular order by the same order spectral coefficient h_ℓ^0.

3.2. Design Criteria

An ideal low-pass filter in the time-frequency domain is a rectangular function in the frequency domain, i.e. the filter eliminates all frequencies above a desired cut-off frequency. Drawing inspiration from this analogy, we propose that the low-pass filter $h(\theta,\phi)$ should meet the following two conditions:

(A1) The low-pass filter is strictly limited in spectral domain with spectral bandwidth L_c, i.e.,

$$h_\ell^m = \begin{cases} 1 & ; 0 \leq \ell \leq L_c, \\ 0 & ; \text{otherwise.}\end{cases} \tag{10}$$

With this definition, the spectral response of filter $\mathbf{h} = [h_0^0\ h_1^0\ h_2^0 \cdots h_{L_c}^0]$ can be characterized as a row vector of size $L_c + 1$ with all ones.

(A2) The spatial response of the filter is optimally concentrated within the polar cap region characterized by angle Θ_c.

The notion of concentration in the spatial domain is explained in the next subsection.

3.3. Concentration Problem on Sphere

The bandlimited eigenfunctions in [10] are used as building blocks in this work. To maximize the spatial concentration of a signal $f(\theta,\phi)$ with maximum harmonic degree L within a region R on the sphere, one needs to maximize the spatial concentration ratio [10]

$$\lambda = \frac{\int_R f^H f d\Omega}{\int_\Omega f^H f d\Omega} \tag{11}$$

where $0 < \lambda < 1$ measures the spatial concentration. The ratio in (11) can be expressed in spectral domain using matrix form [10]

$$\lambda = \frac{\mathbf{f}^T\mathbf{D}\mathbf{f}}{\mathbf{f}^T\mathbf{f}} \tag{12}$$

Considering azimuthally symmetric region R (polar cap) as a special case, $\mathbf{D}$ is the $(L+1)\times(L+1)$ symmetric matrix where the entries are given by $D_{\ell\ell'} = \int_R Y_\ell^0(\theta,\phi)Y_{\ell'}^0(\theta,\phi)d\Omega$. The solution that maximizes (12) gives rise to the standard eigenvalue problem

$$\mathbf{D}\mathbf{f} = \lambda\mathbf{f} \tag{13}$$

which can be solved numerically and results in a constellation of real valued $(L+1)$ orthonormal eigenvectors, each of length $(L+1)$. The eigenvalue associated with each eigenvector is a measure of concentration of corresponding spectrally limited spatial eigenfunction in the desired region R. It has been shown in [1, 10] that most of these eigenvalues lie either near zero or unity. The largest of the $(L+1)$ eigenvalues is near unity and the smallest is slightly greater than zero. If the area of interest is the polar cap characterized by angle Θ, there will be $N_0 - 1$ significant optimally concentrated eigenfunctions with corresponding eigenvalues > 0.90, where N_0 is a space-bandwidth product given by [10]

$$N_0 = \frac{(L+1)\Theta}{\pi} \tag{14}$$

Note that the above relationship shows the inverse relationship between Θ in spatial domain and L in spectral domain.

The proposed algorithm for designing a low-pass filter by systematically choosing bandlimited spatially concentrated eigenfunctions is described in Section 4.

4. PROPOSED FILTER DESIGN ALGORITHM

Our objective is to design a filter with bandwidth L_c in spectral domain whose spatial response is optimally concentrated in polar cap angle Θ_c. As discussed in Section 3.3, the solution to the concentration problem on the sphere gives rise to $L+1$ eigenfunctions, N_0-1 of which are optimally concentrated in the chosen polar cap region of interest. Thus an intuitive approach to construct a low-pass filter is to use the weighted sum of these N_0-1 optimally concentrated eigenfunctions. However, if we choose $L = L_c$ in (14), the N_0-1 bandlimited eigenfunctions may not be sufficient to produce a filter that satisfies the strict condition (A1). On the other hand, if we choose $L > L_c$, the resultant filter exhibits some spectral leakage.

It must be noted that for a given bandwidth and polar cap region (L, Θ) pair, the first eigenfunction (with associated largest eigenvalue) is the *most optimally concentrated* in the polar cap region. The most concentrated eigenfunction exhibits the best localization in both spatial and spectral domains. Rather than using the weighted sum of N_0-1 optimally concentrated eigenfunctions arising from a given (L, Θ) pair, we propose to use weighted sum of the first perfectly concentrated eigenfunction from different (L, Θ) pairs to construct the filter. In this regard, we first modify (14) to ensure that the first N_0-1 eigenfunctions for a (L, Θ) pair have eigenvalues > 0.99 as

$$N_0 = \frac{(L-2)\Theta}{\pi} \tag{15}$$

The systematic method of choosing (L, Θ) pairs using (15), while meeting conditions (A1) and (A2) is described below:

Step 1: For a given filter bandwidth L_c and polar cap region Θ_c, calculate L_1 using (15) with $\Theta = \Theta_c$ and $N_0 = 2$. Choosing $N_0 = 2$ gives one eigenfunction with the *minimum possible* bandwidth L_1 which is optimally concentrated in polar cap region Θ_c.

Step 2: Starting from L_1, increment the value of L in (15) by 1 until $L = L_c$ and calculate the corresponding Θ using (15), to get (L, Θ) pairs of the form

$$(L, \Theta) = [(L_1, \Theta_1), (L_2, \Theta_2), \cdots, (L_c, \Theta_{L_c-L_1+1})] \tag{16}$$

Step 3: Solve the eigenvalue problem in (13) for each (L_k, Θ_k) pair, to get the corresponding first bandlimited eigenfunction $f_k(\theta, \phi)$ with bandwidth L_k and spatial response optimally concentrated in the polar cap Θ_k ($\Theta_1 = \Theta_c$ and $\Theta_k < \Theta_c$ for $k > 2$)

$$f_1(\theta, \phi), f_2(\theta, \phi), \cdots, f_{L_c-L_1+1}(\theta, \phi). \tag{17}$$

Define $\mathbf{F}$ to be the eigenfunction matrix of size $L_c \times (L_c - L_1 + 1)$

$$\mathbf{F} = [\mathbf{f}_1 \ \ \mathbf{f}_2 \cdots \mathbf{f}_{L_c-L_1+1}] \tag{18}$$

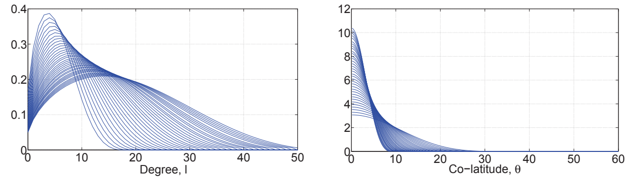
where

$$\mathbf{f}_k = [f^0_{k,0} \ \ f^0_{k,1} \ \ f^0_{k,2}, \cdots, f^0_{k,L_c+1}] \tag{19}$$

denotes the spectral response of $f_k(\theta, \phi)$.

Step 4: Let $\mathbf{w}$ denotes the row vector of weights assigned to $(L_c - L_1 + 1)$ eigenfunctions determined in Step 3. Define the error function between desired and constructed low-pass filter response

$$\mathbf{E}(\mathbf{w}) = \|\mathbf{F}\mathbf{w}^T - \mathbf{h}\| \tag{20}$$



(a) (b)

Fig. 1. Series of eigenfunctions in (17) for $L_c = 50$ and $\Theta_c = \pi/6$ in (a) spectral and (b) spatial domain.

where $\mathbf{h}$ is defined in condition (A1). The weights are calculated such that the error function above is minimized. Note that since the number of variables $L_c - L_1 + 1$ is less than the number of equations $L_c + 1$, we have an over determined system of linear equations which can be solved using standard norm ℓ_2-minimization technique [12].

Step 5: Generate the desired low-pass filter in spectral domain (10) as a weighted sum of eigenvectors as

$$\hat{\mathbf{h}} = \mathbf{F}\mathbf{w}^T \tag{21}$$

where $\hat{\mathbf{h}}$ is the constructed version of the desired response $\mathbf{h}$.

Note that since $L_c - L_1 + 1$ eigenfunctions are being used to construct the filter, this imposes a constraint on the choice of the filter design parameters (L_c, Θ_c) that $L_c - L_1 + 1 > 0$ or $L_c - (2\pi/\Theta_c) - 1 > 0$.

4.1. Closed-Form Formulation of Eigenfunctions

The proposed filter design algorithm requires the first eigenfunction for each (L, Θ) pair in Step 3. This means that the eigenvalue problem in (13) must be solved $L_c - L_1 + 1$ times. Using the least number of samples in the spatial domain [6], the computational complexity of calculating the $\mathbf{D}$ matrix alone for each (L, Θ) pair, which is the first step in solving the eigenvalue problem, is $O(L^3)$. Utilizing the efficient numerical approach for solving eigenvalue problems in [13], the complexity to obtain first eigenvector of the eigenvalue problem in (13) is $O(L)$. For a given filter cut-off frequency L_c, the number of (L, Θ) pairs is L_c and hence the overall computational complexity of calculating the eigenfunction matrix $\mathbf{F}$ in (18) is $O(L_c^5)$.

In order to reduce the computational complexity of the proposed algorithm, we introduce a closed-form expression to accurately model the eigenfunctions in Step 3. Consider the series of eigenfunctions in (17) which are plotted in Fig. 1 for $\Theta_c = \pi/6$ and $L_c = 50$ for illustration. We can see that the eigenfunctions for different (L, Θ) pairs exhibit certain symmetry in their shape and form. We propose to model the eigenfunctions in the spectral domain using a polynomial-exponential expression of the form

$$f^0_{k,\ell} = \begin{cases} A(\ell+1)^B \exp(C(\ell+1)^2) & ; \text{if } 0 \le \ell \le (k+L-1), \\ 0 & ; \text{else} \end{cases} \tag{22}$$

where the variables A, B and C are computed as a function of spatial parameter Θ_k using non-linear least square curve fitting approach and their optimized values used in this work are given below:

$$A = 0.317\Theta_k^{0.9271}, \quad B = 0.876\Theta_k^{0.2133}, \quad C = -0.0545\Theta_k^{1.8344} \tag{23}$$

Note that with the above parameter values, the mean square error between the eigenfunction generated using (22) and the corresponding eigenfunction generated using exact numerical solution is close to 10^{-4} for a wide range of polar caps of interest ($\pi/16 \le \Theta_k \le \pi/2$).

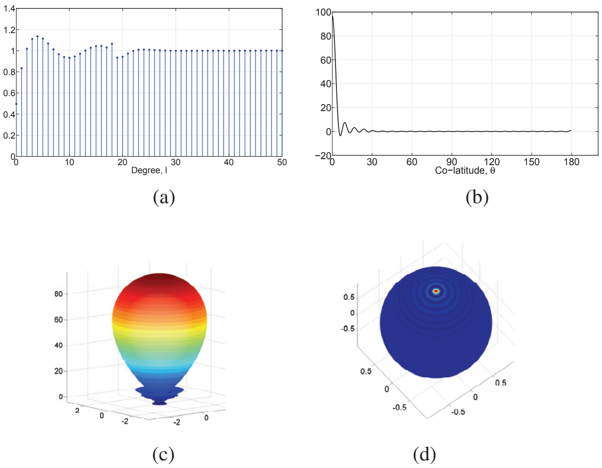


Fig. 2. Constructed low-pass filter using the proposed algorithm for $L_c = 50$ and $\Theta_c = \pi/6$ in (a) spectral domain, (b) spatial domain, (c) spatial domain on displaced unit sphere and (d) spatial domain on surface unit sphere.

The closed form in (22), along with the variable values in (23), generates an eigenvector with spectral bandwidth $L_k = k+L-1$ and optimal concentration in the polar cap region Θ_k which is required in Step 3. Thus using (22), the overall computational complexity of calculating eigenfunction matrix $\mathbf{F}$ in (18) is reduced to $O(L_c^2)$.

5. RESULTS

In this section, we demonstrate the low-pass filter design procedure formulated in the previous Section for $\Theta_c = \pi/6$ and $L_c = 50$. The weighted sum of the eigenfunctions in Fig. 1 is used to construct the filter response, shown in Fig. 2. It can be seen that the constructed filter meets both conditions (A1) and (A2). We can see that the proposed approach in Section 4 minimizes the concentration outside the desired polar cap region at the cost of a small ripple in the pass-band.

We also compare our proposed approach with a parametric form of low-pass filtering based on spherical diffusion in [3]. This Bulow's filter has the form $h_\ell^0 \triangleq \sqrt{(2\ell+1)/(4\pi)}\exp(-\ell(\ell+1)kt)$, where kt controls the spherical harmonic bandwidth. Fig. 3 shows the comparison of the proposed filter (in blue) and Bulow's filter (in red) for $L_c = 26$, and $\Theta_c = \pi/6$ in both spatial and spectral domains. The parameter kt is set to 0.015, such that the energy within the desired bandwidth L_c is equal for both filters. The response of both filters is also normalized to have unit energy for fair comparison. We can see that in terms of energy concentration, both filters meet condition (A2) and have above 99% energy concentration in the desired polar cap region $\Theta_c = \pi/6$. However, the spectral response of Bulow's filter depicts exponential decay and does not satisfactorily meet condition (A1). In addition, the relationship between kt and the filter parameters (L_c, Θ_c) is not clear. Our proposed filter, while having a small ripple in the spectral response, satisfies both design criteria and is directly parameterized by the spectral bandwidth L_c and width of polar cap region Θ_c. This makes the proposed approach a good alternative for spherical filter design.

6. CONCLUSIONS

In this paper, we have proposed a novel algorithm for construction of low-pass filters on the sphere with strictly limited bandwidth in the spectral domain and optimal concentration in the spatial domain. The proposed algorithm is based on a weighted sum of the first perfectly concentrated eigenfunction from appropriately formulated spatial concentration problems on the sphere. We have also presented a closed-form expression to accurately model these eigenfunctions in the spectral domain to reduce the computational complexity of our proposed algorithm. Finally, we have demonstrated the advantage of the proposed technique over an existing popular diffusion based method, in terms of providing precise and systematic control over both bandwidth in the spectral domain and concentration in the spatial domain.

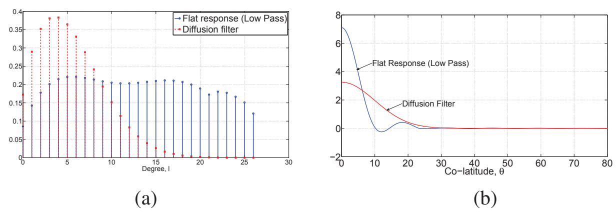


Fig. 3. Comparison of the proposed approach with Bulows' filter for $L_c = 50$ and $\Theta_c = \pi/6$ in (a) spectral and (b) spatial domain.

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