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Algorithms for finding distance-edge-colorings of graphs

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Abstract

For a bounded integer ℓ, we wish to color all edges of a graph G so that any two edges within distance ℓ have different colors. Such a coloring is called a distance-edge-coloring or an ℓ-edge-coloring of G. The distance-edge-coloring problem is to compute the minimum number of colors required for a distance-edge-coloring of a given graph G. A partial k-tree is a graph with tree-width bounded by a fixed constant k. We first present a polynomial-time exact algorithm to solve the problem for partial k-trees, and then give a polynomial-time 2-approximation algorithm for planar graphs.

Keywords: Algorithm; Approximation algorithm; Distance-edge-coloring; Partial k-trees; Planar graphs

1. Introduction

We denote by $G = (V, E)$ a graph with vertex set V and edge set E. An ordinary *edge-coloring of a graph G* is to color all edges of G so that any adjacent edges have different colors. For two vertices u and v, we denote by $\text{dist}(u, v)$ the *distance between u and v* in G, that is, the number of edges in a shortest path between u and v in G. For two edges $e = (u, v)$ and $e' = (u', v')$, the *distance between e and e'* in G is defined as follows:

$$\text{dist}(e, e') = \min\{\text{dist}(u, u'), \text{dist}(u, v'), \text{dist}(v, u'), \text{dist}(v, v')\}.$$

For a given bounded nonnegative integer ℓ, we wish to color all edges of G so that any two edges e and e' with $\text{dist}(e, e') \leqslant \ell$ have different colors. Such a coloring is called a *distance-edge-coloring* or an *ℓ-edge-coloring* of G. Thus a 0-edge-coloring is merely an ordinary edge-coloring, and a 1-edge-coloring is a "strong edge-coloring" [15, 16]. The *ℓ-chromatic index* $\chi'_\ell(G)$ of G is the minimum number of colors required for an ℓ-edge-coloring of G. The *distance-edge-coloring problem* or the *ℓ-edge-coloring problem* is to compute the ℓ-chromatic index $\chi'_\ell(G)$ of a given graph G. For example, the coloring of a graph in Fig. 1 is a 1-edge-coloring with six colors $c_1, c_2, \ldots, c_6$, and is of course a 0-edge-coloring, but is not a 2-edge-coloring. One can easily observe that $\chi'_1(G) = 6$ for the graph G in Fig. 1.

Since the edge-coloring problem is NP-hard [12], the ℓ-edge-coloring problem is NP-hard in general and hence it is very unlikely that the ℓ-edge-coloring problem can be efficiently solved for general graphs. A partial k-tree is a

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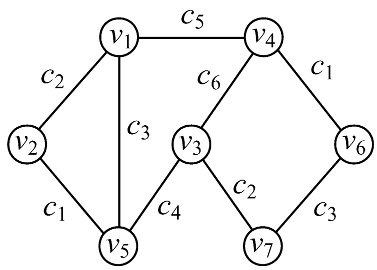

Fig. 1. A 1-edge-coloring of a partial 3-tree G with six colors.

graph with tree-width bounded by a fixed constant k. The class of partial k-trees is fairly large, and includes trees, outerplanar graphs, series-parallel graphs, etc. It is known that many combinatorial problems can be solved very efficiently for partial k-trees even if the problems are NP-hard for general graphs [2,3,8–10]. Such classes of problems have been characterized in terms of "forbidden subgraphs" or "extended monadic second-order logic" [2,3,8–10]. The ℓ-edge-coloring problem does not belong to such a class of the "maximum or minimum subgraph problems" [10]. The ℓ-edge-coloring problem is indeed one of the "edge-covering problems" which, as mentioned in [8], does not appear to be efficiently solved for partial k-trees. However, the following two results have been known. First, the ordinary edge-coloring problem can be solved in linear time for partial k-trees [18]. Second, the 1-edge-coloring problem can be solved in polynomial time for partial k-trees [16].

A vertex version of the distance-edge-coloring problem has been studied for partial k-trees and planar graphs. For a given bounded nonnegative integer ℓ, the *distance-vertex-coloring* or *ℓ-vertex-coloring* is to color all vertices of a graph G so that any two vertices u and v with $\mathrm{dist}(u, v) \leqslant \ell$ have different colors. The distance-vertex-coloring problem, which finds an ℓ-vertex-coloring of a given graph with the minimum number of colors, can be solved in polynomial time for partial k-trees [17]. There is a polynomial-time 2-approximation algorithm for the distance-vertex-coloring problem on planar graphs [1]. The distance-edge-coloring problem for a graph G can be reduced to an ordinary vertex-coloring problem for a new graph G' obtained from G by some operations. However, G' is not always a partial k-tree or a planar graph even if G is a partial k-tree or a planar graph.

In this paper we give two polynomial-time algorithms for the ℓ-edge-coloring problem. The first is a polynomial-time exact algorithm to solve the ℓ-edge-coloring problem for partial k-trees. The algorithm determines in time $\mathrm{O}(n(\alpha+1)^{2^{2(k+1)(\ell+1)+1}})$ whether a partial k-tree G has an ℓ-edge-coloring with a given number α of colors, where n is the number of vertices in G. Remember that we assume $k, \ell = \mathrm{O}(1)$. One may assume without loss of generality that α is smaller than the number of the edges in G; otherwise, G has a trivial ℓ-edge-coloring with α colors. Thus the ℓ-edge-coloring problem can be solved in polynomial time. The algorithm takes linear time if α is a fixed constant. The second algorithm is a polynomial-time 2-approximation algorithm for the ℓ-edge-coloring problem on planar graphs. An early version of the paper has been presented at [13].

The rest of the paper is organized as follows. Section 2 includes basic definitions and notations. Section 3 gives an exact algorithm to solve the ℓ-edge-coloring problem for partial k-trees. Section 4 gives a polynomial-time 2-approximation algorithm to solve the ℓ-edge-coloring problem for planar graphs. Finally, Section 5 is a conclusion.

2. Terminology and definitions

In this section we give some definitions. An edge joining vertices u and v is denoted by (u, v). We denote by n the number of vertices in G, and by m the number of edges in G. We assume that k is a bounded positive integer.

A *k-tree* is defined recursively as follows [5]:

(1) A complete graph with $k+1$ vertices is a k-tree.
(2) If G is a k-tree and k vertices induce a complete subgraph of G, then a graph obtained from G by adding a new vertex and joining it with each of the k vertices is a k-tree.

Every subgraph of a k-tree is called a *partial k-tree*. Thus a partial k-tree G is a simple graph, and $m < kn$.

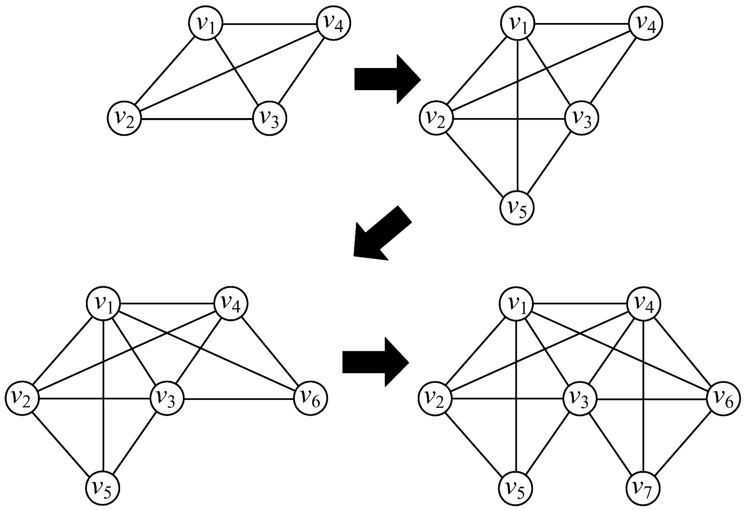

Fig. 2. A process of generating 3-trees.

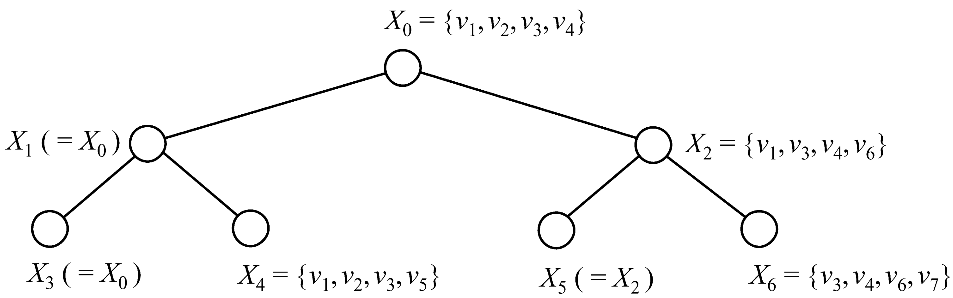

Fig. 3. Tree-decomposition of the partial 3-tree in Fig. 1.

Fig. 2 illustrates a process of generating 3-trees. The graph in Fig. 1 is indeed a partial 3-tree since it is a subgraph of the last 3-tree in Fig. 2.

A binary tree $T = (V_T, E_T)$ is called a *tree-decomposition of a partial k-tree* $G = (V, E)$ if T satisfies the following conditions (a)–(e):

(a) every node $X \in V_T$ of T is a subset of V, and $|X| \leqslant k + 1$;
(b) $\bigcup_{X \in V_T} X = V$;
(c) for each edge $e = (u, v)$ of G, T has a leaf $X \in V_T$ such that $u, v \in X$;
(d) if node X_q lies on the path in T from node X_p to node X_r, then $X_p \cap X_r \subseteq X_q$; and
(e) each internal node X_i of T has exactly two children, say X_L and X_R, and either $X_i = X_L$ or $X_i = X_R$.

We will use notions leaf, node, child, and root in their usual meaning. Fig. 3 illustrates a tree-decomposition T of the partial 3-tree in Fig. 1. Note that $V_T = \{X_0, X_1, \ldots, X_6\}$. We always denote by X_0 the root of a tree-decomposition T.

Since a tree-decomposition T of a partial k-tree G can be found in linear time [6], we may assume that a partial k-tree G and its tree-decomposition T are given. The number of nodes of T constructed by the algorithm in [6] is $O(n)$.

By the condition (c) of a tree-decomposition, for every edge $e = (u, v) \in E$, there is at least one leaf X of T such that $u, v \in X$. We choose one of such leaves as the *representative* of the edge e, and denote it by $\text{rep}(e)$. Each node X_i of T corresponds to a subgraph $G_i = (V_i, E_i)$ of G. The vertex set V_i and edge set E_i of G_i are recursively defined as follows:

(i) if X_i is a leaf of T, then $V_i = X_i$ and $E_i = \{e \in E \mid \text{rep}(e) = X_i\}$; and

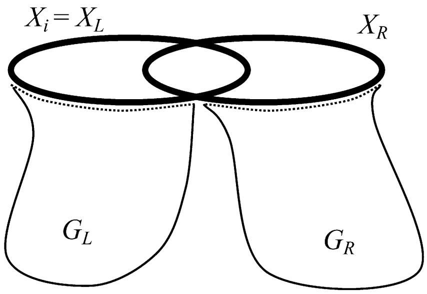

Fig. 4. Graph $G_i = G_L \cup G_R$.

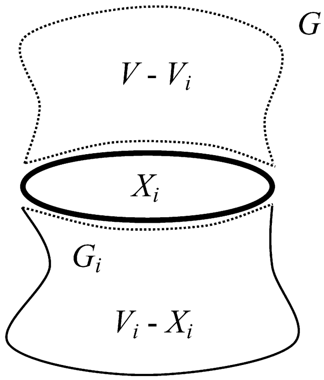

Fig. 5. Graphs G and G_i.

(ii) if X_i is an internal node of T, the left child X_L of X_i corresponds to a subgraph $G_L = (V_L, E_L)$ of G, and the right child X_R corresponds to $G_R = (V_R, E_R)$, then $V_i = V_L \cup V_R$ and $E_i = E_L \cup E_R$, and hence G_i is a union of two graphs G_L and G_R as illustrated in Fig. 4, where X_i $(= X_L)$ and X_R are indicated by ovals drawn by thick lines.

Note that $E_L \cap E_R = \emptyset$. Clearly $G = G_0$ for the root X_0 of T. The condition (d) of a tree-decomposition implies that

$$V_L \cap V_R = X_L \cap X_R \subseteq X_i,$$

and that no edge of G joins a vertex in $V_i - X_i$ and a vertex in $V - V_i$ for each node X_i of T [11]. (See Fig. 5.)

The root $X_0 = \{v_1, v_2, v_3, v_4\}$ of the tree-decomposition T in Fig. 3 has two children X_1 $(= X_0)$ and $X_2 = \{v_1, v_3, v_4, v_6\}$. Figs. 6(a), (b) and (c) illustrate the subgraphs G_0, G_1 and G_2 corresponding to X_0, X_1 and X_2, respectively.

3. Algorithm for partial *k*-trees

The main result of this section is the following theorem.

Theorem 1. *Let G be a partial k-tree, let ℓ be a bounded nonnegative integer, and let α be a positive integer. Then it can be determined in time* $O(n(\alpha+1)^{2^{2(k+1)(\ell+1)+1}})$ *whether G has an ℓ-edge-coloring with α colors.*

The number α is not assumed to be a fixed constant, but can be assumed to be smaller than the number m of edges in G. Therefore, using a binary search technique, one can compute the ℓ-chromatic index $\chi'_\ell(G)$ of G by applying Theorem 1 for at most $\log_2 m$ values of α, $1 \leqslant \alpha < m$. We thus have the following corollary.

Corollary 1. *The ℓ-chromatic index $\chi'_\ell(G)$ of a partial k-tree G can be computed in polynomial time.*

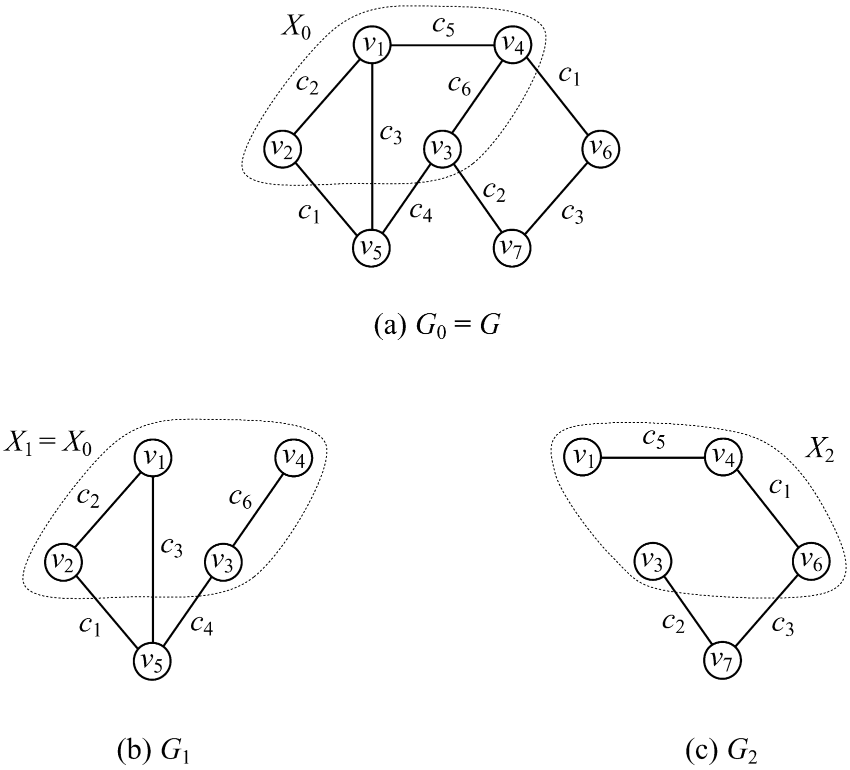

Fig. 6. (a) Colorings f_0 of $G = G_0$, (b) f_1 of G_1, and (c) f_2 of G_2.

In the remainder of this section we give a proof of Theorem 1.

3.1. Idea and terms

From now on we call an ℓ-edge-coloring simply a *coloring*. Although we give an algorithm to determine whether a partial k-tree G has a coloring with α colors, it can be easily modified so that it actually finds a coloring of G with α colors if G has. Our idea is to extend techniques developed for the ordinary edge-coloring problem [5,18] and the distance-vertex-coloring problem [17] to the ℓ-edge-coloring problem and is to reduce the size of a Dynamic Programming (DP) table to $O((\alpha+1)^{2^{2(k+1)(\ell+1)}})$ by considering "counts" and "pair-counts".

Let $G = (V, E)$ be a partial k-tree, and let $T = (V_T, E_T)$ be a tree- decomposition of G. Let C be a set of α colors $c_1, c_2, \ldots, c_\alpha$. For a node X_i of T, a mapping $f : E_i \to C$ is called an *entire coloring of* $G_i = (V_i, E_i)$ if $f(e) \neq f(e')$ for any pair of edges $e, e' \in E_i$ with $\mathrm{dist}(e, e') \leqslant \ell$. Remember that $\mathrm{dist}(e, e')$ is the distance between e and e' in the entire graph G, not in the subgraph G_i. Thus an entire coloring of G_i is a coloring of G_i, while a coloring of G_i is not always an entire coloring of G_i. However, a coloring of G_0 $(= G)$ is an entire coloring of G. Figs. 6(a), (b) and (c) illustrate entire colorings of G_0, G_1 and G_2, respectively, for the case $\ell = 1$.

For a vertex u and an edge $e = (v, w)$, the *distance between* u *and* e in G is defined as follows:

$$\mathrm{dist}(u, e) = \min\{\mathrm{dist}(u, v), \mathrm{dist}(u, w)\}.$$

Thus $\mathrm{dist}(u, e) = 0$ if u is an end-vertex of e.

For an entire coloring f of G_i, an integer j, $0 \leqslant j \leqslant \ell$, and a vertex $v \in X_i$, we define a set $D(f, j, v) \subseteq C$ as follows:

$$D(f, j, v) = \{c \in C \mid G_i \text{ has an edge } e \text{ such that } f(e) = c \text{ and } \mathrm{dist}(v, e) = j\}. \tag{1}$$

Thus $D(f, j, v)$ consists of all colors c that are assigned to edges e of G_i with $\text{dist}(v, e) = j$. For example, $D(f_1, 0, v_1) = \{c_2, c_3\}$ and $D(f_1, 1, v_1) = \{c_1, c_4, c_6\}$ for the entire coloring f_1 of the graph G_1 in Fig. 6(b). Note that $\text{dist}(v_1, v_4) = 1$ for the entire graph G depicted in Fig. 6(a) although there is no edge joining v_1 and v_4 in G_1.

For a node $X_i \in V_T$ of T, an entire coloring f of G_i, an integer j, $0 \leqslant j \leqslant \ell$, and a color $c \in C$, we define a set $Y(X_i; f, j, c) \subseteq X_i$ as follows:

$$Y(X_i; f, j, c) = \{v \in X_i \mid c \in D(f, j, v)\}. \tag{2}$$

Thus $Y(X_i; f, j, c)$ consists of all vertices v in X_i for which G_i has an edge e such that $f(e) = c$ and $\text{dist}(v, e) = j$. For example, $Y(X_1; f_1, 0, c_6) = \{v_3, v_4\}$ and $Y(X_1; f_1, 1, c_6) = \{v_1\}$ for the entire coloring f_1 in Fig. 6(b).

We denote by 2^{X_i} the power set of X_i, and by $(2^{X_i})^{\ell+1}$ the direct product of $\ell + 1$ copies of 2^{X_i}. Thus, if $\mathbb{A} \in (2^{X_i})^{\ell+1}$, then $\mathbb{A}$ is an $(\ell + 1)$-tuple $(A^0, A^1, \ldots, A^\ell)$ of sets $A^0, A^1, \ldots, A^\ell \subseteq X_i$. For an entire coloring f of G_i, we define a mapping $C_f : (2^{X_i})^{\ell+1} \to 2^C$ as follows:

$$C_f(\mathbb{A}) = \{c \in C \mid A^j = Y(X_i; f, j, c) \text{ for each } j, 0 \leqslant j \leqslant \ell\}, \tag{3}$$

where $\mathbb{A} = (A^0, A^1, \ldots, A^\ell) \in (2^{X_i})^{\ell+1}$. For example, $C_{f_1}(\{v_3, v_4\}, \{v_1\}) = \{c_6\}$ and $C_{f_1}(\{v_3, v_4\}, \{v_1, v_2\}) = \emptyset$ for the entire coloring f_1 in Fig. 6(b). Probably $C_f(\mathbb{A}) = \emptyset$ for many $\mathbb{A} \in (2^{X_i})^{\ell+1}$. We call the mapping C_f the *color function of f on X_i*. We write

$$\mathcal{F}_f = \{C_f(\mathbb{A}) \mid \mathbb{A} \in (2^{X_i})^{\ell+1}\},$$

then $\mathcal{F}_f$ is clearly a partition of the set C.

For a node X_i of T, we say that an entire coloring of G_i is *extendable* if it can be extended to a coloring of $G = G_0$ without changing the entire coloring of G_i. Both the entire coloring f_1 of G_1 in Fig. 6(b) and the entire coloring f_2 of G_2 in Fig. 6(c) are extendable because both can be extended to the coloring f_0 of G_0 in Fig. 6(a).

A mapping $\gamma : (2^{X_i})^{\ell+1} \to \{0, 1, \ldots, \alpha\}$ is called a *count on node X_i*. A count γ on X_i is defined to be *active* if G_i has an entire coloring f whose color function C_f satisfies

$$|C_f(\mathbb{A})| = \gamma(\mathbb{A}) \tag{4}$$

for each $\mathbb{A} \in (2^{X_i})^{\ell+1}$. Such a count γ is called the *count of the entire coloring f*. Since $|C| = \alpha$ and $\mathcal{F}_f$ is a partition of C, an active count γ satisfies

$$\sum \gamma(\mathbb{A}) = \alpha, \tag{5}$$

where the summation above is taken over all $\mathbb{A} \in (2^{X_i})^{\ell+1}$.

Example. For the entire coloring f_1 of G_1 in Fig. 6(b), the color function C_{f_1} on X_1 satisfies

$$\begin{aligned}
&C_{f_1}(\{v_2\}, \{v_1, v_3\}) = \{c_1\},\\
&C_{f_1}(\{v_1, v_2\}, \{v_4\}) = \{c_2\},\\
&C_{f_1}(\{v_1\}, \{v_2, v_3, v_4\}) = \{c_3\},\\
&C_{f_1}(\{v_3\}, \{v_1, v_2, v_4\}) = \{c_4\},\\
&C_{f_1}(\emptyset, \emptyset) = \{c_5\},\\
&C_{f_1}(\{v_3, v_4\}, \{v_1\}) = \{c_6\},
\end{aligned}$$

and

$$C_{f_1}(\mathbb{A}) = \emptyset$$

for any other $\mathbb{A} \in (2^{X_1})^2$. Therefore the count γ_1 of f_1 satisfies

$$\begin{aligned}
&\gamma_1(\{v_2\}, \{v_1, v_3\}) = 1,\\
&\gamma_1(\{v_1, v_2\}, \{v_4\}) = 1,
\end{aligned}$$

$$\gamma_1(\{v_1\},\{v_2,v_3,v_4\})=1,$$
$$\gamma_1(\{v_3\},\{v_1,v_2,v_4\})=1,$$
$$\gamma_1(\emptyset,\emptyset)=1,$$
$$\gamma_1(\{v_3,v_4\},\{v_1\})=1,$$

and

$$\gamma_1(\mathbb{A})=0$$

for any other $\mathbb{A}\in(2^{X_1})^2$.

We then have the following lemma.

Lemma 1. *Assume that f and g are entire colorings of G_i for a node X_i of T, and that f and g have the same count. Then f is extendable if and only if g is extendable.*

Proof. See Appendix A. □

Define an equivalence relation $\cong$ on the set of all entire colorings of G_i, as follows: $f\cong g$ if the entire colorings f and g of G_i have the same (active) count. Then each active count on X_i characterizes an equivalence class of entire colorings of G_i. Lemma 1 implies that either all the entire colorings in an equivalence class are extendable or none of them is extendable. Since $|X_i|\leqslant k+1$, there are at most $(\alpha+1)^{2^{(k+1)(\ell+1)}}$ distinct counts $\gamma:(2^{X_i})^{\ell+1}\to\{0,1,\ldots,\alpha\}$ on X_i.

The main step of our algorithm is to compute a table of all active counts on each node of T from the leaves to the root X_0 of T by means of dynamic programming. From the table on the root X_0 one can easily know whether G has a coloring with α colors, as follows.

Lemma 2. *A partial k-tree G has a coloring with α colors if and only if the table on the root X_0 has at least one active count.*

3.2. Algorithm

We first outline an algorithm to determine whether a partial k-tree G has a coloring with α colors $c_1,c_2,\ldots,c_\alpha$ in C, as follows.

Step 1: We compute the table of all active counts on each leaf X_i of T as follows:
- (i) enumerate all mappings $f:E_i\to\{c_1,c_2,\ldots,c_j\}$, where $j=\min\{\alpha,|E_i|\}$;
- (ii) remove mappings that are not entire colorings of G_i; and
- (iii) compute all the active counts corresponding to entire colorings of G_i;

Step 2: We compute the table of all active counts on each internal node X_i of T from all active counts on its children X_L and X_R, as follows:
- (i) enumerate all "pair-counts" on X_i, and find "active" ones by using Lemma 3 below (a "pair-count" and an "active pair-count" will be defined later); and
- (ii) compute all active counts on X_i from all active pair-counts on X_i by using Lemma 4(b) below; and

Step 3: Using Lemma 2, we determine whether $G=G_0$ has a coloring with α colors.

We then explain the details of Steps 1–3, and analyze the computation time of each step.

Step 1. As preprocessing, for all pairs of vertices u and v in the same leaf of T, we determine whether $\mathrm{dist}(u,v)\leqslant\ell$ or not, and compute $\mathrm{dist}(u,v)$ if $\mathrm{dist}(u,v)\leqslant\ell$. This preprocessing can be done in linear time as follows. For a partial k-tree G, one can construct a data structure which allows to determine in time O(1) whether $\mathrm{dist}(u,v)\leqslant\ell$ for two given vertices u and v in G and if so $\mathrm{dist}(u,v)$ is returned. Such a data structure can be constructed in linear time [14].

Since each leaf contains at most $k+1$ vertices and T has $O(n)$ leaves, the preprocessing can be done in linear time by using the data structure.

Since G_i is a simple graph, $j \leqslant |E_i| \leqslant k(k+1)/2$. Therefore the number of distinct mappings f enumerated in Step 1(i) above is at most $j^{k(k+1)/2} = O(1)$. Since the distances $\mathrm{dist}(u, v)$ have been computed for any two vertices u and v with $\mathrm{dist}(u, v) \leqslant \ell$, Step 1(ii) above can be done in time $O(1)$ for each mapping f. Clearly Step 1(iii) above can be done in time $O(1)$ for each entire coloring f. Thus one can compute the table on a leaf X_i of T in time $O(1)$. Since T has $O(n)$ leaves, the tables for all leaves can be computed in time $O(n)$.

Step 2. We first define a "pair-count" and an "active pair-count" on an internal node X_i of T, and then explain how to compute all active pair-counts on X_i from all active counts on its children X_L and X_R in Step 2(i). We finally explain how to compute all active counts on X_i from all active pair-counts on X_i in Step 2(ii).

Either $X_i = X_L$ or $X_i = X_R$ by the condition (e) of a tree-decomposition. Therefore, one may assume without loss of generality that $X_i = X_L$. A mapping

$$\rho : \left(2^{X_L}\right)^{\ell+1} \times \left(2^{X_R}\right)^{\ell+1} \to \{0, 1, \ldots, \alpha\}$$

is called a *pair-count on* X_i. There are at most $(\alpha+1)^{2^{2(k+1)(\ell+1)}}$ distinct pair-counts. For an entire coloring f of G_i, we denote by $f_L = f|G_L$ the *restriction* of f to G_L: $f_L(e) = f(e)$ for each edge e of G_L. Similarly, we denote by $f_R = f|G_R$ the restriction of f to G_R. We denote by C_{f_L} the color function of f_L on X_L, and by C_{f_R} the color function of f_R on X_R. Then we define a pair-count ρ to be *active* if G_i has an entire coloring f such that

$$\rho(\mathbb{A}_L, \mathbb{A}_R) = \left|C_{f_L}(\mathbb{A}_L) \cap C_{f_R}(\mathbb{A}_R)\right| \tag{6}$$

for each pair of $\mathbb{A}_L \in (2^{X_L})^{\ell+1}$ and $\mathbb{A}_R \in (2^{X_R})^{\ell+1}$. Such a pair-count ρ is called the *pair-count of the entire coloring* f *of* G_i. Thus, $\rho(\mathbb{A}_L, \mathbb{A}_R)$ is the number of colors $c \in C$ such that $A_L^j = Y(X_L; f_L, j, c)$ and $A_R^j = Y(X_R; f_R, j, c)$ for each j, $0 \leqslant j \leqslant \ell$, where $\mathbb{A}_L = (A_L^0, A_L^1, \ldots, A_L^\ell)$ and $\mathbb{A}_R = (A_R^0, A_R^1, \ldots, A_R^\ell)$. For example,

$$\begin{aligned} \rho\big(\{v_2\}, \{v_1, v_3\}, \{v_4, v_6\}, \{v_1, v_3\}\big) &= \left|C_{f_1}\big(\{v_2\}, \{v_1, v_3\}\big) \cap C_{f_2}\big(\{v_4, v_6\}, \{v_1, v_3\}\big)\right| \\ &= \left|\{c_1\}\right| \\ &= 1 \end{aligned}$$

for the entire coloring f_0 in Fig. 6(a), where $f_1 = f_0|G_1$ and $f_2 = f_0|G_2$ as illustrated in Figs. 6(b) and (c), respectively. We now have the following lemma.

Lemma 3. *Let X_i be an internal node of T, and let X_L and X_R be the children of X_i. Then a pair-count ρ on X_i is active if and only if ρ satisfies the following conditions* (a) *and* (b):

(a) *if $\rho(\mathbb{A}_L, \mathbb{A}_R) \geqslant 1$ for a pair of $\mathbb{A}_L = (A_L^0, A_L^1, \ldots, A_L^\ell) \in (2^{X_L})^{\ell+1}$ and $\mathbb{A}_R = (A_R^0, A_R^1, \ldots, A_R^\ell) \in (2^{X_R})^{\ell+1}$, then $A_L^{j_1} \cap A_R^{j_2} = \emptyset$ for every pair of nonnegative integers j_1 and j_2 with $j_1 + j_2 \leqslant \ell$; and*

(b) *there is an active count γ_L on X_L such that*

$$\gamma_L(\mathbb{A}_L) = \sum \rho(\mathbb{A}_L, \mathbb{A}), \tag{7}$$

for each $\mathbb{A}_L \in (2^{X_L})^{\ell+1}$ where the summation above is taken over all $\mathbb{A} \in (2^{X_R})^{\ell+1}$, and there is an active count γ_R on X_R such that

$$\gamma_R(\mathbb{A}_R) = \sum \rho(\mathbb{A}, \mathbb{A}_R), \tag{8}$$

for each $\mathbb{A}_R \in (2^{X_R})^{\ell+1}$ where the summation above is taken over all $\mathbb{A} \in (2^{X_L})^{\ell+1}$.

Proof. See Appendix B. □

Using Lemma 3, we compute all active pair-counts ρ on X_i from all pairs of active counts γ_L on X_L and γ_R on X_R, as follows. There are at most $(\alpha+1)^{2^{2(k+1)(\ell+1)}}$ distinct pair-counts ρ on X_i. For each ρ of them, we determine whether ρ satisfies conditions (a) and (b) in Lemma 3. For each pair-count ρ, one can know in time O(1) whether ρ satisfies condition (a), because there are at most $(\ell+1)^2 2^{2(k+1)(\ell+1)} = \mathrm{O}(1)$ distinct pairs $(A_L^{j_1}, A_R^{j_2})$. On the other hand, for each pair-count ρ, one can know in time $\mathrm{O}((\alpha+1)^{2^{(k+1)(\ell+1)+1}})$ whether ρ satisfies condition (b), because there are at most

$$\bigl((\alpha+1)^{2^{(k+1)(\ell+1)}}\bigr)^2 = (\alpha+1)^{2^{(k+1)(\ell+1)+1}}$$

pairs of active counts γ_L and γ_R, and one can know in time O(1) for each of them whether it satisfies Eqs. (7) and (8). Thus all active pair-counts ρ on X_i can be found in time $\mathrm{O}((\alpha+1)^{2^{2(k+1)(\ell+1)+1}})$, since there are at most $(\alpha+1)^{2^{2(k+1)(\ell+1)}}$ distinct pair-counts ρ on X_i and

$$(\alpha+1)^{2^{(k+1)(\ell+1)+1}}(\alpha+1)^{2^{2(k+1)(\ell+1)}} \leqslant (\alpha+1)^{2^{2(k+1)(\ell+1)+1}}.$$

In Step 2(ii) we compute all active counts on an internal node X_i from all active pair-counts on X_i, as in the following Lemma 4(b).

Lemma 4. *Assume that X_i is an internal node of T, X_L and X_R are the two children of X_i, and $X_i = X_L$. Then the following* (a) *and* (b) *hold.*

(a) *If f is an entire coloring of G_i, then the color functions C_f on X_i, C_{f_L} on X_L and C_{f_R} on X_R satisfy*

$$C_f(\mathbb{A}) = \bigcup \bigl(C_{f_L}(\mathbb{A}_L) \cap C_{f_R}(\mathbb{A}_R)\bigr) \tag{9}$$

for every $\mathbb{A} = (A^0, A^1, \ldots, A^\ell) \in (2^{X_i})^{\ell+1}$, where the union above is taken over all pairs of $\mathbb{A}_L = (A_L^0, A_L^1, \ldots, A_L^\ell) \in (2^{X_L})^{\ell+1}$ and $\mathbb{A}_R = (A_R^0, A_R^1, \ldots, A_R^\ell) \in (2^{X_R})^{\ell+1}$ such that

$$A^j = \bigl(A_L^j \cup A_R^j\bigr) \cap X_i \tag{10}$$

for each integer j, $0 \leqslant j \leqslant \ell$.

(b) *A count γ on X_i is active if and only if there exists an active pair-count ρ on X_i such that*

$$\gamma(\mathbb{A}) = \sum \rho(\mathbb{A}_L, \mathbb{A}_R) \tag{11}$$

for each $\mathbb{A} = (A^0, A^1, \ldots, A^\ell) \in (2^{X_i})^{\ell+1}$, where the summation above is taken over all pairs of $\mathbb{A}_L = (A_L^0, A_L^1, \ldots, A_L^\ell) \in (2^{X_L})^{\ell+1}$ and $\mathbb{A}_R = (A_R^0, A_R^1, \ldots, A_R^\ell) \in (2^{X_R})^{\ell+1}$ satisfying Eq. (10).

Proof. See Appendix C. □

Using Lemma 4(b), we compute all active counts γ on X_i from all active pair-counts ρ on X_i. There are at most $(\alpha+1)^{2^{2(k+1)(\ell+1)}}$ distinct active pair-counts ρ. From each ρ of them we compute an active count γ by Eq. (11). This can be done in time O(1) since $|A^j|, |A_L^j|, |A_R^j| \leqslant k+1 = \mathrm{O}(1)$ for each integer j, $0 \leqslant j \leqslant \ell$. We have thus shown that all active counts γ on X_i can be computed in time $\mathrm{O}((\alpha+1)^{2^{2(k+1)(\ell+1)}})$ from all active pair-counts ρ on X_i.

One can thus compute the DP table for an internal node X_i from the tables of the children X_L and X_R in time

$$\mathrm{O}\bigl((\alpha+1)^{2^{2(k+1)(\ell+1)+1}} + (\alpha+1)^{2^{2(k+1)(\ell+1)}}\bigr) = \mathrm{O}\bigl((\alpha+1)^{2^{2(k+1)(\ell+1)+1}}\bigr).$$

Since T has O(n) internal nodes, one can compute the DP tables for all internal nodes in time $\mathrm{O}(n(\alpha+1)^{2^{2(k+1)(\ell+1)+1}})$.

Step 3. From the DP table for the root X_0 one can know in time O(1) by Lemma 2 whether G has a coloring with α colors.

This completes a proof of Theorem 1.

4. 2-approximation algorithm for planar graphs

The main result of this section is the following theorem.

Theorem 2. *There is a polynomial-time* 2*-approximation algorithm for the distance-edge-coloring problem on planar graphs.*

In the remainder of this section, as a proof of Theorem 2, we give a polynomial-time algorithm to find an ℓ-edge-coloring of a given planar graph G with at most $2\chi'_\ell(G)$ colors. The approximation algorithm can be obtained by combining our algorithm in Section 3 with a general method for obtaining approximation algorithms for NP-complete problems on planar graphs [4].

The method [4] partitions the vertex set V of a planar graph $G = (V, E)$ into a number p of subsets $V_0, V_1, \dots, V_{p-1}$ for some integer p so that every edge is between adjacent subsets or within the same subset, that is, if $(u, v) \in E$ and $u \in V_i$ then $v \in V_{i-1} \cup V_i \cup V_{i+1}$. Clearly, $\mathrm{dist}(u, v) \geqslant |i - j|$ if $u \in V_i$ and $v \in V_j$. Let

$$V' = \bigcup \{ V_i \mid i \bmod 2(\ell+1) \leqslant \ell + 1 \}$$

and

$$V'' = (V - V') \cup \left(\bigcup \{ V_i \mid i \bmod 2(\ell+1) = 0 \text{ or } \ell + 1 \} \right),$$

then both of the ends u and v of each edge $(u, v) \in E$ are contained in either V' or V''. Let $G' = (V', E')$ be the subgraph of G induced by V', and let $G'' = (V'', E'')$ be the subgraph of G such that $E'' = E - E'$. Then G' is a vertex-disjoint union of subgraphs H'_j, $0 \leqslant j \leqslant \lfloor p/(2(\ell+1)) \rfloor$; H'_j corresponds to $V_{2(\ell+1)j} \cup V_{2(\ell+1)j+1} \cup \cdots \cup V_{2(\ell+1)j+(\ell+1)}$. Every subgraph H'_j, $0 \leqslant j \leqslant \lfloor p/(2(\ell+1)) \rfloor$, is an $(\ell+2)$-outerplanar graph and hence is a partial $(3\ell+5)$-tree [7]. Since G' is a vertex-disjoint union of H'_j, $0 \leqslant j \leqslant \lfloor p/(2(\ell+1)) \rfloor$, G' is a partial $(3\ell+5)$-tree. Similarly, G'' is a vertex-disjoint union of subgraphs H''_j, $0 \leqslant j \leqslant \lfloor p/(2(\ell+1)) \rfloor$, and is a partial $(3\ell+5)$-tree; H''_j corresponds to $V_{2(\ell+1)j+\ell+1} \cup V_{2(\ell+1)j+\ell+2} \cup \cdots \cup V_{2(\ell+1)j+2(\ell+1)}$.

We now describe the approximation algorithm. Using a data structure in [14], one can determine in time O(1) whether the distance $\mathrm{dist}(u, v)$ is at most ℓ for two given vertices u and v in a planar graph G and, if so, return $\mathrm{dist}(u, v)$. We find an entire ℓ-edge-coloring of G' with the minimum number $\chi^*_\ell(G')$ of colors by using the polynomial-time algorithm in Section 3. In the entire ℓ-edge-coloring of G', any two edges e and e' with $\mathrm{dist}(e, e') \leqslant \ell$ must have different colors, where $\mathrm{dist}(e, e')$ is the distance between e and e' in the entire graph G, not in G'. Thus $\chi^*_\ell(G') \leqslant \chi'_\ell(G)$. Similarly, we find an entire ℓ-edge-coloring of G'' with the minimum number $\chi^*_\ell(G'')$ of colors, where $\chi^*_\ell(G'') \leqslant \chi'_\ell(G)$. One may assume that the colors for G' are different from the colors for G''. Combining the colorings of G' and G'', we finally obtain an ℓ-edge-coloring of G with $\chi^*_\ell(G') + \chi^*_\ell(G'') \leqslant 2\chi'_\ell(G)$ colors. This completes the proof of Theorem 2.

5. Conclusions

In this paper, we obtained two algorithms. The first algorithm is to determine whether a given partial k-tree G has an ℓ-edge-coloring with α colors in time $O(n(\alpha+1)^{2^{2(k+1)(\ell+1)+1}})$, where n is the number of vertices in G and α is an arbitrary positive integer. Using the algorithm, one can compute the ℓ-chromatic index $\chi'_\ell(G)$ of G in polynomial time. Our algorithm takes linear time if α is a fixed constant. It is easy to modify the algorithm so that it actually finds an ℓ-edge-coloring of G with $\chi'_\ell(G)$ colors. The second algorithm is a polynomial-time 2-approximation algorithm for the distance-edge-coloring problem on planar graphs.

Many variants of the distance-edge-coloring problem can be solved for partial k-trees in polynomial time. Consider for example a problem in which, for a given set $L \subseteq \{0, 1, \dots, \ell\}$, one wishes to color all edges of a graph G with the minimum number of colors so that every pair of edges e and e' with $\mathrm{dist}(e, e') \in L$ have different colors. Such a problem can be solved in polynomial time for partial k-trees similarly as the ℓ-edge-coloring problem.

Replace some of the edges in a partial k-tree by multiple edges. The resulting multigraph is called a *partial k-multitree*. One can easily extend our algorithms for partial k-trees and planar simple graphs to those for partial k-multitrees and planar multigraphs.

Appendix A. Proof of Lemma 1

Let C_f be a color function of f on X_i, and let C_g be a color function of g on X_i, then

$$C_f(\mathbb{A}) = \{c \in C \mid A^j = Y(X_i; f, j, c) \text{ for each } j, 0 \leqslant j \leqslant \ell\} \tag{A.1}$$

and

$$C_g(\mathbb{A}) = \{c \in C \mid A^j = Y(X_i; g, j, c) \text{ for each } j, 0 \leqslant j \leqslant \ell\} \tag{A.2}$$

for each $\mathbb{A} = (A^0, A^1, \ldots, A^\ell) \in (2^{X_i})^{\ell+1}$. Since f and g have the same count, we have

$$|C_f(\mathbb{A})| = |C_g(\mathbb{A})| \tag{A.3}$$

for each $\mathbb{A} \in (2^{X_i})^{\ell+1}$. Since both $\mathcal{F}_f = \{C_f(\mathbb{A}) \mid \mathbb{A} \in (2^{X_i})^{\ell+1}\}$ and $\mathcal{F}_g = \{C_g(\mathbb{A}) \mid \mathbb{A} \in (2^{X_i})^{\ell+1}\}$ are partitions of C, by Eq. (A.3) there exists a bijection $\xi : C \to C$ such that

$$c \in C_g(\mathbb{A}) \quad \text{if and only if} \quad \xi(c) \in C_f(\mathbb{A}) \tag{A.4}$$

for every color $c \in C$.

Clearly

$$Y(X_i; g, j, c) = Y(X_i; f, j, \xi(c)) \tag{A.5}$$

holds for each color $c \in C$ and each $j, 0 \leqslant j \leqslant \ell$.

Let f_ξ be an entire coloring of G_i such that

$$f_\xi(e) = c \quad \text{if and only if} \quad f(e) = \xi(c) \tag{A.6}$$

for each edge $e \in E_i$. Then by Eqs. (1) and (A.6) we have

$$\begin{aligned} D(f_\xi, j, v) &= \{c \in C \mid G_i \text{ has an edge } e \text{ such that } f_\xi(e) = c \text{ and } \mathrm{dist}(v, e) = j\} \\ &= \{c \in C \mid G_i \text{ has an edge } e \text{ such that } f(e) = \xi(c) \text{ and } \mathrm{dist}(v, e) = j\} \\ &= \{c \in C \mid \xi(c) \in D(f, j, v)\} \end{aligned} \tag{A.7}$$

for each vertex $v \in X_i$ and each $j, 0 \leqslant j \leqslant \ell$. By Eqs. (2) and (A.7) we have

$$\begin{aligned} Y(X_i; f_\xi, j, c) &= \{v \in X_i \mid c \in D(f_\xi, j, v)\} \\ &= \{v \in X_i \mid \xi(c) \in D(f, j, v)\} \\ &= Y(X_i; f, j, \xi(c)) \end{aligned} \tag{A.8}$$

for each color $c \in C$ and each $j, 0 \leqslant j \leqslant \ell$. By Eqs. (A.5) and (A.8) we have

$$Y(X_i; g, j, c) = Y(X_i; f_\xi, j, c) \tag{A.9}$$

for each color $c \in C$ and each $j, 0 \leqslant j \leqslant \ell$.

We are now ready to prove that f is extendable if and only if g is extendable. It suffices to show that if f is extendable then g is extendable. Suppose that f is extendable. Then G has a coloring f^* which is an extension of f, and hence

$$f^*(e) = f(e) \tag{A.10}$$

for each edge $e \in E_i$. Let f^*_ξ be a coloring of G such that

$$f^*_\xi(e) = c \quad \text{if and only if} \quad f^*(e) = \xi(c) \tag{A.11}$$

for each edge $e \in E$.

Let $G' = (V', E')$ be a graph with $V' = (V - V_i) \cup X_i$ and $E' = E - E_i$. Then G can be partitioned into two edge-disjoint subgraphs G_i and G', and $X_i = V_i \cap V'$. Let f' be an entire coloring of G' such that

$$f'(e) = f^*_\xi(e) \tag{A.12}$$

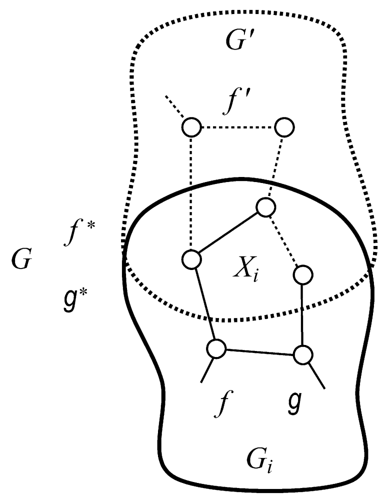

Fig. A.1. Graphs G, G_i and G'.

for each edge $e \in E'$. (See Fig. A.1 where G_i is indicated by solid lines and G' by dotted lines.)

Let $g^*: E \to C$ be a mapping constructed from g and f' as follows:

$$g^*(e) = \begin{cases} g(e) & \text{if } e \in E_i, \\ f'(e) & \text{if } e \in E' = E - E_i. \end{cases}$$

Since g^* is an extension of g, it suffices to show that g^* is a coloring of G. Since g is an entire coloring of G_i, f' is an entire coloring of G', no edge of G joins a vertex in $V_i - X_i$ and a vertex in $V - V_i$ (see Figs. 5 and A.1), it suffices to verify, for every color $c \in C$,

$$j_1 + j_2 \leqslant \ell \quad \Longrightarrow \quad Y(X_i; g, j_1, c) \cap Y(X_i; f', j_2, c) = \emptyset, \tag{A.13}$$

where $Y(X_i; f', j, c)$ is defined to be a set of all vertices v in X_i for which G' has an edge e such that $f'(e) = c$ and $\text{dist}(v, e) = j$.

Let c be a color in C, and let v be a vertex in $Y(X_i; g, j_1, c)$ for an integer j_1, $0 \leqslant j_1 \leqslant \ell$. Then, by Eq. (A.9) we have $v \in Y(X_i; f_\xi, j_1, c)$, and hence G_i has an edge e such that $f_\xi(e) = c$ and $\text{dist}(v, e) = j_1$. Since $e \in E_i$, by Eq. (A.10) we have $f^*(e) = f(e)$. Therefore, by Eqs. (A.6) and (A.11) we have $f^*_\xi(e) = f_\xi(e) = c$. Since f^*_ξ is a coloring of G, G' has no edge e' such that $f^*_\xi(e') = c$ and $\text{dist}(v, e') = j_2$ for every integer j_2 with $j_1 + j_2 \leqslant \ell$. Therefore, by Eq. (A.12) G' has no edge e' such that $f'(e') = c$ and $\text{dist}(v, e') = j_2$. Hence we have $v \notin Y(X_i; f', j_2, c)$, and consequently $Y(X_i; g, j_1, c) \cap Y(X_i; f', j_2, c) = \emptyset$. We have thus verified Eq. (A.13). □

Appendix B. Proof of Lemma 3

Necessity: Assume that a pair-count ρ on X_i is active. Then G_i has an entire coloring f with pair-count ρ satisfying Eq. (6). We show that ρ satisfies conditions (a) and (b), as follows.

(a) Assume that $\rho(\mathbb{A}_L, \mathbb{A}_R) \geqslant 1$ for a pair of $\mathbb{A}_L \in (2^{X_L})^{\ell+1}$ and $\mathbb{A}_R \in (2^{X_R})^{\ell+1}$. Then we shall prove that

$$A_L^{j_1} \cap A_R^{j_2} = \emptyset \tag{B.1}$$

for every pair of nonnegative integers j_1 and j_2 with $j_1 + j_2 \leqslant \ell$.

Since $\rho(\mathbb{A}_L, \mathbb{A}_R) \geqslant 1$, by Eq. (6) there exists a color $c^* \in C$ such that

$$c^* \in C_{f_L}(\mathbb{A}_L) \cap C_{f_R}(\mathbb{A}_R).$$

By Eq. (3) we have

$$C_{f_L}(\mathbb{A}_L) = \{c \in C \mid A_L^j = Y(X_L; f_L, j, c) \text{ for each } j, 0 \leqslant j \leqslant \ell\}. \tag{B.2}$$

Since $c^* \in C_{f_L}(\mathbb{A}_L)$, by Eq. (B.2) we have

$$A_L^j = Y(X_L; f_L, j, c^*) \tag{B.3}$$

for each j, $0 \leqslant j \leqslant \ell$. Similarly, we have

$$A_R^j = Y(X_R; f_R, j, c^*) \tag{B.4}$$

for each j, $0 \leqslant j \leqslant \ell$.

Let j_1 and j_2 be nonnegative integers with $j_1 + j_2 \leqslant \ell$. Let v be an arbitrary vertex in $A_L^{j_1}$, and hence by Eq. (B.3) we have $v \in A_L^{j_1} = Y(X_L; f_L, j_1, c^*)$. Then G_L has an edge e such that $f_L(e) = c^*$ and $\mathrm{dist}(v, e) = j_1$. Therefore, G_R has no edge e' such that $f_R(e') = c^*$ and $\mathrm{dist}(v, e') = j_2$; otherwise, f would not be an entire coloring of G_i. Hence by Eq. (B.4) we have $v \notin Y(X_R; f_R, j_2, c^*) = A_R^{j_2}$. We have thus proved that Eq. (B.1) holds.

(b) Let γ_L be the count of f_L, and let γ_R be the count of f_R. Then γ_L and γ_R are active. Hence it suffices to show that γ_L satisfies Eq. (7) and γ_R satisfies Eq. (8). However, we show only that γ_L satisfies Eq. (7) because one can similarly show that γ_R satisfies Eq. (8).

Let $\mathbb{A}_L = (A_L^0, A_L^1, \ldots, A_L^\ell) \in (2^{X_L})^{\ell+1}$. Since γ_L is the count of f_L, by Eq. (4) we have

$$\gamma_L(\mathbb{A}_L) = \left|C_{f_L}(\mathbb{A}_L)\right|. \tag{B.5}$$

Since $C_{f_L}(\mathbb{A}_L) \subseteq C$ and $\mathcal{F}_{f_R} = \{C_{f_R}(\mathbb{A}) \mid \mathbb{A} \in (2^{X_R})^{\ell+1}\}$ is a partition of C, we have

$$\begin{aligned} C_{f_L}(\mathbb{A}_L) &= C_{f_L}(\mathbb{A}_L) \cap C \\ &= C_{f_L}(\mathbb{A}_L) \cap \Big(\bigcup C_{f_R}(\mathbb{A})\Big) \\ &= \bigcup C_{f_L}(\mathbb{A}_L) \cap C_{f_R}(\mathbb{A}), \end{aligned} \tag{B.6}$$

where the unions above are taken over all $\mathbb{A} \in (2^{X_R})^{\ell+1}$. By Eqs. (6), (B.5) and (B.6) we have

$$\begin{aligned} \gamma_L(\mathbb{A}_L) &= \left|C_{f_L}(\mathbb{A}_L)\right| \\ &= \sum \left|C_{f_L}(\mathbb{A}_L) \cap C_{f_R}(\mathbb{A})\right| \\ &= \sum \rho(\mathbb{A}_L, \mathbb{A}), \end{aligned}$$

where the summations above are taken over all $\mathbb{A} \in (2^{X_R})^{\ell+1}$. We have thus shown that γ_L satisfies Eq. (7).

Sufficiency: Assume that a pair-count ρ on X_i satisfies conditions (a) and (b). Then we shall prove that ρ is active.

Since condition (b) holds, there is an active count γ_L on X_L satisfying Eq. (7). Therefore G_L has an entire coloring f' with count γ_L. Similarly, G_R has an entire coloring f'' with count γ_R, and γ_R satisfies Eq. (8). It suffices to show that the following (i), (ii) and (iii) hold.

(i) There exists a bijection $\xi : C \to C$ such that, for every color $c \in C$,

$$j_1 + j_2 \leqslant \ell \Longrightarrow Y(X_L \cap X_R; f', j_1, c) \cap Y(X_L \cap X_R; f''_\xi, j_2, c) = \emptyset, \tag{B.7}$$

where f''_ξ is an entire coloring of G_R such that, for each edge $e \in E_R$,

$$f''_\xi(e) = c \quad \text{if and only if} \quad f''(e) = \xi(c); \tag{B.8}$$

$Y(X_L \cap X_R; f', j, c)$ is defined to be a set of all vertices v in $X_L \cap X_R$ for which G_L has an edge e such that $f'(e) = c$ and $\mathrm{dist}(v, e) = j$; and $Y(X_L \cap X_R; f''_\xi, j, c)$ is defined to be a set of all vertices v in $X_L \cap X_R$ for which G_R has an edge e such that $f''_\xi(e) = c$ and $\mathrm{dist}(v, e) = j$;

(ii) Let $f : E_i \to C$ be a mapping constructed from f' and f''_ξ as follows:

$$f(e) = \begin{cases} f'(e) & \text{if } e \in E_L, \\ f''_\xi(e) & \text{if } e \in E_R. \end{cases} \tag{B.9}$$

Then f is an entire coloring of G_i; and

(iii) ρ is the pair-count of f, and hence ρ is active.

(i) Let $C_{f'}$ be the color function of f' on X_L, and let $C_{f''}$ be the color function of f'' on X_R. Since γ_L is the count of f', by Eq. (4) we have

$$\gamma_L(\mathbb{A}_L) = \left|C_{f'}(\mathbb{A}_L)\right|$$

for each $\mathbb{A}_L \in (2^{X_L})^{\ell+1}$. By Eq. (5) we have

$$\sum \gamma_L(\mathbb{A}_L) = \alpha,$$

where the summation above is taken over all $\mathbb{A}_L \in (2^{X_L})^{\ell+1}$. Furthermore γ_L satisfies Eq. (7). Therefore, to all pairs of $\mathbb{A}_L \in (2^{X_L})^{\ell+1}$ and $\mathbb{A}_R \in (2^{X_R})^{\ell+1}$, one can assign pairwise disjoint subsets $S(\mathbb{A}_L, \mathbb{A}_R)$ of C so that the following (L-a), (L-b) and (L-c) hold:

(L-a) for each pair $(\mathbb{A}_L, \mathbb{A}_R)$,

$$\rho(\mathbb{A}_L, \mathbb{A}_R) = \left|S(\mathbb{A}_L, \mathbb{A}_R)\right|; \tag{B.10}$$

(L-b) for each $\mathbb{A}_L$, $\{S(\mathbb{A}_L, \mathbb{A}) \mid \mathbb{A} \in (2^{X_R})^{\ell+1}\}$ is a partition of the set $C_{f'}(\mathbb{A}_L)$; and
(L-c) $\{S(\mathbb{A}, \mathbb{B}) \mid \mathbb{A} \in (2^{X_L})^{\ell+1}, \mathbb{B} \in (2^{X_R})^{\ell+1}\}$ is a partition of C.

Similarly, to all pairs of $\mathbb{A}_L \in (2^{X_L})^{\ell+1}$ and $\mathbb{A}_R \in (2^{X_R})^{\ell+1}$, one can assign pairwise disjoint subsets $U(\mathbb{A}_L, \mathbb{A}_R)$ of C so that the following (R-a), (R-b) and (R-c) hold:

(R-a) for each pair $(\mathbb{A}_L, \mathbb{A}_R)$,

$$\rho(\mathbb{A}_L, \mathbb{A}_R) = \left|U(\mathbb{A}_L, \mathbb{A}_R)\right|; \tag{B.11}$$

(R-b) for each $\mathbb{A}_R$, $\{U(\mathbb{A}, \mathbb{A}_R) \mid \mathbb{A} \in (2^{X_L})^{\ell+1}\}$ is a partition of the set $C_{f''}(\mathbb{A}_R)$; and
(R-c) $\{U(\mathbb{A}, \mathbb{B}) \mid \mathbb{A} \in (2^{X_L})^{\ell+1}, \mathbb{B} \in (2^{X_R})^{\ell+1}\}$ is a partition of C.

By Eqs. (B.10) and (B.11) we have

$$\left|S(\mathbb{A}_L, \mathbb{A}_R)\right| = \left|U(\mathbb{A}_L, \mathbb{A}_R)\right|$$

for each pair $(\mathbb{A}_L, \mathbb{A}_R)$. Therefore, by (L-c) and (R-c) there exists a bijection $\xi : C \to C$ such that

$$c \in S(\mathbb{A}_L, \mathbb{A}_R) \text{ if and only if } \xi(c) \in U(\mathbb{A}_L, \mathbb{A}_R) \tag{B.12}$$

for each color $c \in C$. We claim that Eq. (B.7) holds for the bijection ξ.

We now show that

$$S(\mathbb{A}_L, \mathbb{A}_R) = \left\{c \in C_{f'}(\mathbb{A}_L) \mid \xi(c) \in C_{f''}(\mathbb{A}_R)\right\} \tag{B.13}$$

for each pair $(\mathbb{A}_L, \mathbb{A}_R)$. We first show that

$$S(\mathbb{A}_L, \mathbb{A}_R) \subseteq \left\{c \in C_{f'}(\mathbb{A}_L) \mid \xi(c) \in C_{f''}(\mathbb{A}_R)\right\}. \tag{B.14}$$

Let c be an arbitrary color in $S(\mathbb{A}_L, \mathbb{A}_R)$. Since $\mathbb{A}_R \in (2^{X_R})^{\ell+1}$, by (L-b) we have $c \in C_{f'}(\mathbb{A}_L)$. Since $c \in S(\mathbb{A}_L, \mathbb{A}_R)$, by Eq. (B.12) we have $\xi(c) \in U(\mathbb{A}_L, \mathbb{A}_R)$. Therefore by (R-b) we have $\xi(c) \in C_{f''}(\mathbb{A}_R)$. We have thus verified Eq. (B.14). We next show that

$$S(\mathbb{A}_L, \mathbb{A}_R) \supseteq \left\{c \in C_{f'}(\mathbb{A}_L) \mid \xi(c) \in C_{f''}(\mathbb{A}_R)\right\}. \tag{B.15}$$

Let c be an arbitrary color such that

$$c \in C_{f'}(\mathbb{A}_L) \tag{B.16}$$

and

$$\xi(c) \in C_{f''}(\mathbb{A}_R). \tag{B.17}$$

By (L-b) and Eq. (B.16) there exists an $(\ell+1)$-tuple $\mathbb{B} \in (2^{X_R})^{\ell+1}$ such that

$$c \in S(\mathbb{A}_L, \mathbb{B}). \tag{B.18}$$

Then, by Eq. (B.12) we have $\xi(c) \in U(\mathbb{A}_L, \mathbb{B})$, and hence by (R-b) we have

$$\xi(c) \in C_{f''}(\mathbb{B}). \tag{B.19}$$

Since $\mathcal{F}_{f''} = \{C_{f''}(\mathbb{A}) \mid \mathbb{A} \in (2^{X_R})^{\ell+1}\}$ is a partition of C, by Eqs. (B.17) and (B.19) we have $\mathbb{B} = \mathbb{A}_R$, and hence by Eq. (B.18) we have $c \in S(\mathbb{A}_L, \mathbb{A}_R)$. We have thus verified Eq. (B.15).

We are now ready to show that Eq. (B.7) holds for the bijection ξ. Let c be an arbitrary color in C. Since $\mathcal{F}_{f'} = \{C_{f'}(\mathbb{A}) \mid \mathbb{A} \in (2^{X_L})^{\ell+1}\}$ is a partition of C, there exists an $(\ell+1)$-tuple $\mathbb{A}_L = (A_L^0, A_L^1, \ldots, A_L^\ell)$ such that

$$c \in C_{f'}(\mathbb{A}_L). \tag{B.20}$$

Therefore, by Eq. (3) we have

$$A_L^j = Y(X_L; f', j, c) \tag{B.21}$$

for each j, $0 \leqslant j \leqslant \ell$. By (L-b) and Eq. (B.20) there exists an $(\ell+1)$-tuple $\mathbb{A}_R = (A_R^0, A_R^1, \ldots, A_R^\ell)$ such that

$$c \in S(\mathbb{A}_L, \mathbb{A}_R). \tag{B.22}$$

By Eqs. (B.13) and (B.22) we have $\xi(c) \in C_{f''}(\mathbb{A}_R)$ and hence by Eq. (3) we have

$$A_R^j = Y\big(X_R; f'', j, \xi(c)\big) \tag{B.23}$$

for each j, $0 \leqslant j \leqslant \ell$. By Eqs. (1) and (B.8) we have

$$\begin{aligned} D(f''_\xi, j, v) &= \big\{c \in C \mid G_R \text{ has an edge } e \text{ such that } f''_\xi(e) = c \text{ and } \operatorname{dist}(v, e) = j\big\} \\ &= \big\{c \in C \mid G_R \text{ has an edge } e \text{ such that } f''(e) = \xi(c) \text{ and } \operatorname{dist}(v, e) = j\big\} \\ &= \big\{c \in C \mid \xi(c) \in D(f'', j, v)\big\} \end{aligned} \tag{B.24}$$

for each vertex $v \in X_R$ and each j, $0 \leqslant j \leqslant \ell$. By Eqs. (2), (B.23) and (B.24) we have

$$\begin{aligned} A_R^j &= Y\big(X_R; f'', j, \xi(c)\big) \\ &= \big\{v \in X_R \mid \xi(c) \in D(f'', j, v)\big\} \\ &= \big\{v \in X_R \mid c \in D(f''_\xi, j, v)\big\} \\ &= Y(X_R; f''_\xi, j, c) \end{aligned} \tag{B.25}$$

for each j, $0 \leqslant j \leqslant \ell$. By Eqs. (B.21) and (B.25) we have

$$Y(X_L; f', j'_1, c) \cap Y(X_R; f''_\xi, j'_2, c) = A_L^{j'_1} \cap A_R^{j'_2} \tag{B.26}$$

for every pair of integers j'_1 and j'_2, $0 \leqslant j'_1, j'_2 \leqslant \ell$. Let j_1 and j_2 be nonnegative integers with $j_1 + j_2 \leqslant \ell$. By Eqs. (B.10) and (B.22) we have

$$\rho(\mathbb{A}_L, \mathbb{A}_R) = \big|S(\mathbb{A}_L, \mathbb{A}_R)\big| \geqslant 1.$$

Therefore by condition (a) we have

$$A_L^{j_1} \cap A_R^{j_2} = \emptyset. \tag{B.27}$$

By Eqs. (B.26) and (B.27) we have

$$Y(X_L; f', j_1, c) \cap Y(X_R; f''_\xi, j_2, c) = \emptyset. \tag{B.28}$$

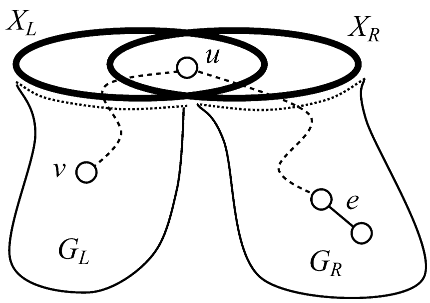

Fig. B.1. Graph $G_i = G_L \cup G_R$.

Since $X_L \cap X_R \subseteq X_L$ and $X_L \cap X_R \subseteq X_R$, we have

$$Y(X_L \cap X_R; f', j, c) \subseteq Y(X_L; f', j, c)$$

and

$$Y(X_L \cap X_R; f''_\xi, j, c) \subseteq Y(X_R; f''_\xi, j, c)$$

for each j, $0 \leqslant j \leqslant \ell$. Therefore, by Eq. (B.28) we have

$$\begin{aligned} &Y(X_L \cap X_R; f', j_1, c) \cap Y(X_L \cap X_R; f''_\xi, j_2, c) \\ &\quad \subseteq Y(X_L; f', j_1, c) \cap Y(X_R; f''_\xi, j_2, c) \\ &\quad = \emptyset. \end{aligned}$$

We have thus proved that Eq. (B.7) holds.

(ii) No edge of G joins a vertex in $V_L - (X_L \cap X_R)$ and a vertex in $V_R - (X_L \cap X_R)$. Therefore, for every vertex v in $V_L - (X_L \cap X_R)$ and every edge e in E_R, any path between v and e passes through a vertex u in $X_L \cap X_R$, and hence

$$\operatorname{dist}(u, e) < \operatorname{dist}(v, e). \tag{B.29}$$

(See Fig. B.1.) Similarly, for every vertex v' in $V_R - (X_L \cap X_R)$ and every edge e' in E_L, there exists a vertex u' in $X_L \cap X_R$ such that

$$\operatorname{dist}(u', e') < \operatorname{dist}(v', e'). \tag{B.30}$$

Let $f : E_i \to C$ be the mapping defined by Eq. (B.9). Since f' is an entire coloring of G_L and f''_ξ is an entire coloring of G_R, by Eqs. (B.7), (B.29) and (B.30) one can easily observe that f is an entire coloring of G_i.

(iii) Let ρ_f be the pair-count of the entire coloring f of G_i, then ρ_f is active. Thus it suffices to show that $\rho = \rho_f$. Eq. (B.9) implies that $f_L = f|G_L = f'$ and $f_R = f|G_R = f''_\xi$. Therefore by Eq. (6) we have

$$\rho_f(\mathbb{A}_L, \mathbb{A}_R) = \left|C_{f'}(\mathbb{A}_L) \cap C_{f''_\xi}(\mathbb{A}_R)\right| \tag{B.31}$$

for each pair of $\mathbb{A}_L = (A_L^0, A_L^1, \ldots, A_L^\ell) \in (2^{X_L})^{\ell+1}$ and $\mathbb{A}_R = (A_R^0, A_R^1, \ldots, A_R^\ell) \in (2^{X_R})^{\ell+1}$. By Eq. (3) we have

$$C_{f''}(\mathbb{A}_R) = \left\{c \in C \mid A_R^j = Y(X_R; f'', j, c) \text{ for each } j, 0 \leqslant j \leqslant \ell\right\}, \tag{B.32}$$

and

$$C_{f''_\xi}(\mathbb{A}_R) = \left\{c \in C \mid A_R^j = Y(X_R; f''_\xi, j, c) \text{ for each } j, 0 \leqslant j \leqslant \ell\right\}. \tag{B.33}$$

By Eqs. (2) and (B.24) we have

$$Y(X_R; f''_\xi, j, c) = \left\{v \in X_R \mid c \in D(f''_\xi, j, v)\right\} = \left\{v \in X_R \mid \xi(c) \in D(f'', j, v)\right\} = Y\big(X_R; f'', j, \xi(c)\big) \tag{B.34}$$

for each color $c \in C$ and each j, $0 \leqslant j \leqslant \ell$. By Eqs. (B.32)–(B.34) we have

$$\begin{aligned} C_{f'}(\mathbb{A}_L) \cap C_{f''_\xi}(\mathbb{A}_R) &= \{c \in C_{f'}(\mathbb{A}_L) \mid A_R^j = Y(X_R; f''_\xi, j, c) \text{ for each } j, 0 \leqslant j \leqslant \ell\} \\ &= \{c \in C_{f'}(\mathbb{A}_L) \mid A_R^j = Y(X_R; f'', j, \xi(c)) \text{ for each } j, 0 \leqslant j \leqslant \ell\} \\ &= \{c \in C_{f'}(\mathbb{A}_L) \mid \xi(c) \in C_{f''}(\mathbb{A}_R)\}. \end{aligned} \tag{B.35}$$

By Eqs. (B.10), (B.13), (B.31) and (B.35) we have

$$\begin{aligned} \rho_f(\mathbb{A}_L, \mathbb{A}_R) &= |C_{f'}(\mathbb{A}_L) \cap C_{f''_\xi}(\mathbb{A}_R)| \\ &= |\{c \in C_{f'}(\mathbb{A}_L)\colon \xi(c) \in C_{f''}(\mathbb{A}_R)\}| \\ &= |S(\mathbb{A}_L, \mathbb{A}_R)| \\ &= \rho(\mathbb{A}_L, \mathbb{A}_R) \end{aligned}$$

for each pair $(\mathbb{A}_L, \mathbb{A}_R)$. We have thus verified $\rho = \rho_f$. □

Appendix C. Proof of Lemma 4

(a) Let $\mathcal{Z}$ be the right side of Eq. (9), that is,

$$\mathcal{Z} = \bigcup \bigl(C_{f_L}(\mathbb{A}_L) \cap C_{f_R}(\mathbb{A}_R)\bigr)$$

where the union above is taken over all pairs of $\mathbb{A}_L \in (2^{X_L})^{\ell+1}$ and $\mathbb{A}_R \in (2^{X_R})^{\ell+1}$ satisfying Eq. (10). We first verify

$$C_f(\mathbb{A}) \supseteq \mathcal{Z} \tag{C.1}$$

in (i) below, and then verify

$$C_f(\mathbb{A}) \subseteq \mathcal{Z} \tag{C.2}$$

in (ii) below.

(i) We verify Eq. (C.1). Let c be an arbitrary color in $\mathcal{Z}$. Then

$$c \in C_{f_L}(\mathbb{A}_L) \cap C_{f_R}(\mathbb{A}_R) \tag{C.3}$$

for a pair of $\mathbb{A}_L = (A_L^0, A_L^1, \ldots, A_L^\ell) \in (2^{X_L})^{\ell+1}$ and $\mathbb{A}_R = (A_R^0, A_R^1, \ldots, A_R^\ell) \in (2^{X_R})^{\ell+1}$ such that

$$A^j = \bigl(A_L^j \cup A_R^j\bigr) \cap X_i \tag{C.4}$$

for each j, $0 \leqslant j \leqslant \ell$. It suffices to verify

$$A^j = Y(X_i; f, j, c)$$

for each j, $0 \leqslant j \leqslant \ell$, because then we have $c \in C_f(\mathbb{A})$. By Eq. (C.3) we have $c \in C_{f_L}(\mathbb{A}_L)$, and hence by Eq. (3) we have

$$A_L^j = Y(X_L; f_L, j, c) \tag{C.5}$$

for each j, $0 \leqslant j \leqslant \ell$. Similarly, we have

$$A_R^j = Y(X_R; f_R, j, c) \tag{C.6}$$

for each j, $0 \leqslant j \leqslant \ell$.

We first verify $A^j \subseteq Y(X_i; f, j, c)$ for each j, $0 \leqslant j \leqslant \ell$. Let v be an arbitrary vertex in A^j. Then we shall show that $v \in Y(X_i; f, j, c)$. Since $v \in A^j$, by Eq. (C.4) we have $v \in X_i$ and $v \in A_L^j \cup A_R^j$. Since $v \in A_L^j \cup A_R^j$, by Eqs. (C.5) and (C.6) either $v \in Y(X_L; f_L, j, c)$ or $v \in Y(X_R; f_R, j, c)$. Hence either G_L has an edge e such that $f_L(e) = c$ and $\mathrm{dist}(v, e) = j$, or G_R has an edge e such that $f_R(e) = c$ and $\mathrm{dist}(v, e) = j$. Therefore, in either case, G_i has an edge e such that $f(e) = c$ and $\mathrm{dist}(v, e) = j$, and hence $v \in Y(X_i; f, j, c)$. We have thus shown that $A^j \subseteq Y(X_i; f, j, c)$.

We next verify $A^j \supseteq Y(X_i; f, j, c)$ for each j, $0 \leqslant j \leqslant \ell$. Let v be an arbitrary vertex in $Y(X_i; f, j, c)$. Then we shall show that $v \in A^j$. Since $v \in Y(X_i; f, j, c)$, $v \in X_i$ and G_i has an edge e such that $f(e) = c$ and $\mathrm{dist}(v, e) = j$. If $e \in E_L$, then by the definition of f_L we have $f_L(e) = f(e) = c$, and hence by Eq. (C.5) we have $v \in Y(X_L; f_L, j, c) = A_L^j$. Similarly, if $e \in E_R$, then we have $v \in Y(X_R; f_R, j, c) = A_R^j$. Therefore, by Eq. (C.4) we have $v \in (A_L^j \cup A_R^j) \cap X_i = A^j$. We have thus verified $A^j \supseteq Y(X_i; f, j, c)$.

(ii) We then verify Eq. (C.2). Let c be an arbitrary color in $C_f(\mathbb{A})$, and let $\mathbb{A} = (A^0, A^1, \ldots, A^\ell)$. Then we shall show that

$$c \in \mathcal{Z}. \tag{C.7}$$

Since $c \in C_f(\mathbb{A})$, we have

$$A^j = Y(X_i; f, j, c) \tag{C.8}$$

for each j, $0 \leqslant j \leqslant \ell$. For each j, $0 \leqslant j \leqslant \ell$, let

$$A_L^j = Y(X_L; f_L, j, c) \tag{C.9}$$

and

$$A_R^j = Y(X_R; f_R, j, c). \tag{C.10}$$

Then we have $c \in C_{f_L}(\mathbb{A}_L)$ and $c \in C_{f_R}(\mathbb{A}_R)$. Therefore, it suffices to show that the pair $\mathbb{A}_L = (A_L^0, A_L^1, \ldots, A_L^\ell)$ and $\mathbb{A}_R = (A_R^0, A_R^1, \ldots, A_R^\ell)$ satisfies Eq. (10), because then Eq. (C.7) holds.

We first show that $A^j \subseteq (A_L^j \cup A_R^j) \cap X_i$ for each j, $0 \leqslant j \leqslant \ell$. Let v be an arbitrary vertex in A^j. Then we shall verify $v \in (A_L^j \cup A_R^j) \cap X_i$. Since $v \in A^j$, by Eq. (C.8) we have $v \in Y(X_i; f, j, c)$, and hence $v \in X_i$ and G_i has an edge e such that $f(e) = c$ and $\mathrm{dist}(v, e) = j$. If $e \in E_L$, then we have $f_L(e) = f(e) = c$, and hence by Eq. (C.9) we have $v \in Y(X_L; f_L, j, c) = A_L^j$. Similarly, if $e \in E_R$, then we have $v \in Y(X_R; f_R, j, c) = A_R^j$. Therefore, in either case, we have $v \in (A_L^j \cup A_R^j) \cap X_i$. We have thus shown that $A^j \subseteq (A_L^j \cup A_R^j) \cap X_i$.

We next show that $A^j \supseteq (A_L^j \cup A_R^j) \cap X_i$ for each j, $0 \leqslant j \leqslant \ell$. Let v be an arbitrary vertex in $(A_L^j \cup A_R^j) \cap X_i$. Then we shall show that $v \in A^j$. Since $v \in A_L^j \cup A_R^j$, by Eqs. (C.9) and (C.10) we have $v \in Y(X_L; f_L, j, c) \cup Y(X_R; f_R, j, c)$. Therefore either G_L has an edge e such that $f_L(e) = c$ and $\mathrm{dist}(v, e) = j$, or G_R has an edge e such that $f_R(e) = c$ and $\mathrm{dist}(v, e) = j$. In either case, G_i has an edge e such that $f(e) = c$ and $\mathrm{dist}(v, e) = j$, and hence by Eq. (C.8) we have $v \in Y(X_i; f, j, c) = A^j$. We have thus shown that $A^j \supseteq (A_L^j \cup A_R^j) \cap X_i$.

(b) *Necessity*: Suppose that a count γ on X_i is active. Then G_i has an entire coloring f with count γ. Let ρ be the pair-count of f, then ρ is active. It suffices to show that ρ satisfies Eq. (11).

Since f has the count γ, we have

$$\gamma(\mathbb{A}) = \left|C_f(\mathbb{A})\right| \tag{C.11}$$

for each $\mathbb{A} \in (2^{X_i})^{\ell+1}$. Let $f_L = f|G_L$ and $f_R = f|G_R$. Since ρ is the pair-count of f, we have

$$\rho(\mathbb{A}_L, \mathbb{A}_R) = \left|C_{f_L}(\mathbb{A}_L) \cap C_{f_R}(\mathbb{A}_R)\right| \tag{C.12}$$

for each pair of $\mathbb{A}_L \in (2^{X_L})^{\ell+1}$ and $\mathbb{A}_R \in (2^{X_R})^{\ell+1}$. By Eqs. (9), (C.11) and (C.12) we have

$$\begin{aligned}\gamma(\mathbb{A}) &= \left|\bigcup\big(C_{f_L}(\mathbb{A}_L) \cap C_{f_R}(\mathbb{A}_R)\big)\right| \\ &= \sum\left|\big(C_{f_L}(\mathbb{A}_L) \cap C_{f_R}(\mathbb{A}_R)\big)\right| \\ &= \sum \rho(\mathbb{A}_L, \mathbb{A}_R)\end{aligned}$$

for every $\mathbb{A} \in (2^{X_i})^{\ell+1}$, where the union and summations above are taken over all pairs $(\mathbb{A}_L, \mathbb{A}_R)$ satisfying Eq. (10). Thus ρ satisfies Eq. (11).

Sufficiency: Suppose that γ is a count on X_i and there exists an active pair-count ρ on X_i satisfying Eq. (11). Since ρ is an active pair-count on X_i, G_i has an entire coloring f with the pair-count ρ. It suffices to show that γ is the count of f, because then γ would be active. By Eq. (9) we have, for each $\mathbb{A} = (A^0, A^1, \ldots, A^\ell) \in (2^{X_i})^{\ell+1}$,

$$\left|C_f(\mathbb{A})\right| = \left|\bigcup\left(C_{f_L}(\mathbb{A}_L) \cap C_{f_R}(\mathbb{A}_R)\right)\right| \tag{C.13}$$

where the union above is taken over all pairs of $\mathbb{A}_L = (A_L^0, A_L^1, \ldots, A_L^\ell) \in (2^{X_L})^{\ell+1}$ and $\mathbb{A}_R = (A_R^0, A_R^1, \ldots, A_R^\ell) \in (2^{X_R})^{\ell+1}$ satisfying Eq. (10). Since ρ is the pair-count of f, Eq. (6) holds. By Eqs. (6), (11) and (C.13) we have, for each $\mathbb{A}$,

$$\begin{aligned}\left|C_f(\mathbb{A})\right| &= \sum\left|C_{f_L}(\mathbb{A}_L) \cap C_{f_R}(\mathbb{A}_R)\right| \\ &= \sum \rho(\mathbb{A}_L, \mathbb{A}_R) \\ &= \gamma(\mathbb{A}),\end{aligned}$$

where the summations above are taken over all pairs $(\mathbb{A}_L, \mathbb{A}_R)$ satisfying Eq. (10). We have thus shown that γ is the count of f, and hence γ is active. □

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