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Research Article

Almost Periodic Solution of a Diffusive Mixed System with Time Delay and Type III Functional Response

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A delayed predator-prey model with diffusion and competition is proposed. Some sufficient conditions on uniform persistence of the model have been obtained. By applying Liapunov-Razumikhin technique, we will point out, under almost periodic circumstances, a set of sufficient conditions that assure the existence and uniqueness of the positive almost periodic solution which is globally asymptotically stable.

1. Introduction

In the nature world, diffusion often occurs in an ecological environment; that is, species can diffuse between patches. The works about autonomous systems in this field were pioneered by Levin, after Levin [1], Kishimoto [2], and Takeuchi [3] studied this kind of model. But all the coefficients in the system they studied are constants. Since biological and environmental parameters are naturally subject to fluctuation in time, the effects of a varying environment are considered as important selective forces on systems in a fluctuating environment. More realistic and interesting models should take into account both the seasonality of the changing environment and the effects of time delays [4–7]. This motivated Chen et al. [8–11], and others to consider nonautonomous predator-prey models with almost periodic coefficients and diffusion. In this paper, we study the almost periodic solution of the delayed predator-prey model with diffusion and competition so as to obtain some conditions under which three species are uniformly persistent. In addition, we obtain that for the almost periodic system there exists a unique almost positive periodic solution which is globally asymptotically stable.

The organization of this paper is as follows. In the next section, we develop our model, establish its important properties, and give several lemmas, which will be a key for our proofs and discussions. In Section 3, sufficient conditions are given for uniform persistence of three species. In Section 4, by applying Liapunov-Razumikhin technique, we prove the existence and uniqueness of the positive almost periodic solution which is globally asymptotically stable. Finally, we give a discussion of our results.

2. Model and preliminaries

It is assumed that the ecosystem is composed of two isolated patches, and the prey population can disperse among the patches instantaneously. The state variables of the models, $x_i = x_i(t)$ $(i = 1, 2)$, describe the densities of the prey population in Patch 1 and Patch 2, respectively. We suppose that the net exchange of the prey population from Patch j to Patch i is proportional to the difference of the concentration between $x_j - x_i$ with $D_i(t)$, $i, j = 1, 2$. The state variables of the models, $x_i = x_i(t)$ $(i = 3, 4)$, describe the densities of the predator population in Patch 1 with competition.

Let us consider the following delayed diffusive predator-prey system with competition and functional response:

$$
\begin{aligned}
x_1' &= x_1\left(a_{10}(t) - a_{11}(t)x_1\right) - \frac{\alpha_1(t)x_1^2x_3}{1+\beta_1(t)x_1^2} - \frac{\alpha_2(t)x_1^2x_4}{1+\beta_2(t)x_1^2} + D_1(t)\left(x_2 - x_1\right),\\
x_2' &= x_2\left(a_{20}(t) - a_{21}(t)x_2\right) + D_2(t)\left(x_1 - x_2\right),\\
x_3' &= x_3\left(-a_{30}(t) + a_{31}(t)\frac{\alpha_1(t)x_1^2(t-\tau_1)}{1+\beta_1(t)x_1^2(t-\tau_1)} - a_{32}(t)x_3 - a_{34}(t)x_4\right),\\
x_4' &= x_4\left(-a_{40}(t) + a_{41}(t)\frac{\alpha_2(t)x_1^2(t-\tau_2)}{1+\beta_2(t)x_1^2(t-\tau_2)} - a_{42}(t)x_4 - a_{43}(t)x_3\right),
\end{aligned}
\tag{2.1}
$$

with the initial condition

$$
x_1(s) = \phi_1(s) \in C\left([-\tau, 0], \mathbf{R}_+\right), \quad s \in [-\tau, 0],\ \phi_1(0) \geq 0,\ x_i(0) = \phi_i \geq 0 \text{ (constants)},\ i = 2, 3, 4.
\tag{2.2}
$$

Here, $a_{i0}(t)$ and $a_{i1}(t)$ $(i = 1, 2)$ represent the intrinsic growth rate and the intraspecific interference coefficient of the prey population x_i $(i = 1, 2)$, respectively. We then assume that the death rate of the predator population x_i $(i = 3, 4)$ in Patch 1 is proportional to both the existing predator population with the proportional functions $a_{30}(t)$ and, respectively, $a_{40}(t)$ and to its square with the proportional functions $a_{32}(t)$ and, respectively, $a_{42}(t)$. The predator consumes the prey according to Holling type III functional response [12, 13], that is, $\alpha_1(t)x_1^2x_3/(1+\beta_1(t)x_1^2)$ and $\alpha_2(t)x_1^2x_4/(1+\beta_2(t)x_1^2)$. τ_i $(i = 1, 2)$ is the time to digest food in the predator organism. $\tau = \max\{\tau_1, \tau_2\}$. $\mathbf{R}_+ \doteq \{z : z \geq 0\}$.

We introduce some notations and definitions, and state some preliminary lemmas which will be useful for establishing our main results.

Let $\mathbf{R}_+^4 = \{X \in R^4 : X = (x_1, x_2, x_3, x_4),\ x_i > 0,\ i = 1, 2, 3, 4\}$. $C = C([-\tau, \infty) \times \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}_+, \mathbf{R}_+^4)$. Assume Ω is a subset of $\mathbf{R}_+ \times C([-\tau, 0], \mathbf{R}_+) \times \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}_+$. Denote by $f = (f_1, f_2, f_3, f_4)^T : \Omega \to \mathbf{R}_+^4$ the map defined by the right-hand side of system (2.1). If $V : \mathbf{R}_+ \times C \to \mathbf{R}_+$ is a continuous function, then the upper right derivative of $V(t,x)$ with respect to system (2.1) is defined as

$$D^+V(t,X) = \lim_{h\to 0^+}\sup \frac{1}{h}\left[V\left(t+h, X+hf(t,X)\right) - V(t,X)\right]. \tag{2.3}$$

Obviously, the global existence and uniqueness of solutions of system (2.1) are guaranteed by the smoothness properties of f (see [14, 15] for details on fundamental properties of retarded functional differential equations).

For convenience, we introduce the following notations:

$$\overline{\varphi} = \sup_{t\geq 0}\{\varphi(t)\}, \qquad \underline{\varphi} = \inf_{t\geq 0}\{\varphi(t)\}. \tag{2.4}$$

In this paper, we need all the coefficients to satisfy

$$\begin{aligned} &\min_{\substack{i=1,2,3,4,\\ j=0,1,2,3,}}\left\{\underline{a_{ij}}, \underline{\alpha_i}, \underline{\beta_i}, \underline{D_i}\right\} > 0, \\ &\max_{\substack{i=1,2,3,4,\\ j=0,1,2,3.}}\left\{\overline{a_{ij}}, \overline{\alpha_i}, \overline{\beta_i}, \overline{D_i}\right\} < \infty. \end{aligned} \tag{2.5}$$

Definition 2.1. System (2.1) is said to be uniformly persistent if there exists a compact region $D \subset R_+^4$ such that every solution (x_1, x_2, x_3, x_4) of system (2.1) with initial conditions (2.2) eventually enters and remains in the region D.

For convenience, the set CIP $= \{u : [0,\infty) \to [0,\infty) \mid u(s)$ is positive and nondecreasing for $s > 0$, $u(0) = 0\}$.

Lemma 2.2 (see [16, 17]). *Consider the following almost periodic equation:*

$$x'(t) = g(t, x_t). \tag{2.6}$$

Let $C_{H^*} = \{x_t \in C : \|x_t\| = \sup_{\theta\in[-\tau,0]}|x_t(\theta)| < H^*\}$, $S_{H^*} = \{x \in \mathbf{R}^n : |x| < H^*\}$, $H^* \in \mathbf{R}_+$ *or* $H^* = +\infty$, $g : \mathbf{R}\times C_{H^*} \to \mathbf{R}^n$, *and* g *is uniformly almost periodic with respect to* t. *Let* $V : \mathbf{R}_+ \times S_{H^*} \times S_{H^*} \to \mathbf{R}_+$. *Assume that the following conditions hold:*

(i) $a(\|x-y\|) \leq V(t,x,y) \leq b(\|x-y\|)$, $a(\cdot), b(\cdot) \in CIP$, $b(0) > 0$;

(ii) $|V(t,x_1,y_1) - V(t,x_2,y_2)| \leq L(\|x_1-x_2\| + \|y_1-y_2\|)$, *where* L *is a positive constant;*

(iii) *there exists a continuous nondecreasing function* $P(S)$ *such that*

$$\begin{gathered} P(S) > S \quad \text{if } S > 0, \\ D^+\left(V\left(t, x_1(t), x_2(t)\right)\right) \leq -\mathcal{C}V\left(t, x_1(t), x_2(t)\right), \quad \mathcal{C} \in \mathbf{R}_+, \\ \text{if } P\left(V\left(t, x_1(t), x_2(t)\right)\right) \geq V\left(t+\theta, x_1(t+\theta), x_2(t+\theta)\right), \quad \theta \in [-\tau, 0]. \end{gathered} \tag{2.7}$$

If system (2.6) *has a solution* $\xi(t) : \|\xi_t\| \leq H < H^*$, $t \geq t_0$, *then system* (2.6) *has a unique positive almost periodic solution* $\eta(t)$ *which is uniformly asymptotically stable, and* $\operatorname{mod}(\eta) \subset \operatorname{mod}(g)$. *Furthermore, if* g *is* ω*-periodic with respect to* t, *then system* (2.6) *has a positive* ω*-periodic solution which is globally asymptotically stable.*

Here, $\mathrm{mod}(\phi)$ denotes the module of $\phi(t)$ which is the set consisting of all real numbers which are finite linear combinations of elements of the set

$$\Lambda = \left\{ \alpha \in \mathbf{R} \mid \lim_{T\to\infty} \frac{1}{T} \int_0^T \phi(t) \exp(-i\alpha t)dt \neq 0 \right\} \tag{2.8}$$

with integer coefficients.

Lemma 2.3. $\mathbf{R}_+^4 = \{(x_1, x_2, x_3, x_4) \mid x_i > 0,\ i = 1,2,3,4\}$ *is a positive invariant set of system* (2.1).

Proof. Let (x_1, x_2, x_3, x_4) be a solution of system (2.1) with initial conditions (2.2). Hence, for $t \in \mathbf{R}$ and $(x_1, x_2, x_3, x_4) \in \mathbf{R}_+^4$, we can derive

$$\begin{aligned}
x_1'|_{x_1=0} &= D_1(t)x_2 > 0 \quad \text{for } x_2 > 0,\\
x_2'|_{x_2=0} &= D_2(t)x_1 > 0 \quad \text{for } x_1 > 0,\\
x_3 > x_3(0)\exp&\left(\int_0^t \big(-a_{30}(s) - a_{32}(s)x_3(s) - a_{34}(s)x_4(s)\big)ds\right) > 0,\\
x_4 > x_4(0)\exp&\left(\int_0^t \big(-a_{40}(s) - a_{42}(s)x_4(s) - a_{43}(s)x_3(s)\big)ds\right) > 0.
\end{aligned} \tag{2.9}$$

Therefore, we obtain the positive invariance of $\mathbf{R}_+^4$. This completes the proof. □

We will focus our discussion on $\mathbf{R}_+^4$ with respect to a biological meaning. This also ensures the solution with positive initial value to be positive all the time.

3. Uniform persistence

In what follows, we want to construct an ultimately bounded region of system (2.1).

Theorem 3.1. *There exist three constants* $M_i > M_i^*$ $(i = 1,2,3)$ *such that* $x_j(t) \le M_1$ $(j = 1,2)$, $x_3(t) \le M_2$, *and* $x_4(t) \le M_3$ *for each positive solution* $(x_1(t), x_2(t), x_3(t), x_4(t))$ *of system* (2.1) *with* t *large enough, where*

$$M_1^* = \max\left\{ \frac{\overline{a_{10}}}{\underline{a_{11}}}, \frac{\overline{a_{20}}}{\underline{a_{21}}} \right\}, \qquad M_2^* = \frac{A}{\underline{a_{32}}}, \qquad M_3^* = \frac{B}{\underline{a_{42}}}, \tag{3.1}$$

$$A \doteq \overline{a_{31}\left(\frac{\alpha_1}{\beta_1}\right)} - \underline{a_{30}} > 0, \qquad B \doteq \overline{a_{41}\left(\frac{\alpha_2}{\beta_2}\right)} - \underline{a_{40}} > 0. \tag{3.2}$$

Proof. Suppose that $(x_1(t), x_2(t), x_3(t), x_4(t))$ is a solution of system (2.1) with initial conditions (2.2). According to the first two equations of (2.1), we have

$$\begin{aligned}
x_1' &\le \overline{a_{10}}x_1 - \underline{a_{11}}x_1^2 + D_1(t)\left[x_2 - x_1\right],\\
x_2' &\le \overline{a_{20}}x_2 - \underline{a_{21}}x_2^2 + D_2(t)\left[x_1 - x_2\right].
\end{aligned} \tag{3.3}$$

We define the following lines in x_1-x_2 plane:

$$\begin{aligned} &\text{Line } L_1\text{: } x_1 = M_1, \quad 0 \le x_2 \le M_1, \\ &\text{Line } L_2\text{: } x_2 = M_1, \quad 0 \le x_1 \le M_1. \end{aligned} \tag{3.4}$$

Then, we have

$$x_1'|_{L_1} < 0, \quad x_2'|_{L_2} < 0. \tag{3.5}$$

Hence, it follows from

$$\max\{x_1(0), x_2(0)\} \le M_1 \tag{3.6}$$

that

$$\max\{x_1(t), x_2(t)\} \le M_1 \quad \text{for } t \ge 0. \tag{3.7}$$

If

$$\max\{x_1(0), x_2(0)\} > M_1, \tag{3.8}$$

we only consider what follows. If $x_i > M_1$, $i = 1, 2$, from the given condition we get

$$\overline{a_{i0}} x_i - \underline{a_{i1}} x_i^2 < M_1 \left(\overline{a_{i0}} - \underline{a_{i1}} M_1 \right) < 0, \quad i = 1, 2. \tag{3.9}$$

Let

$$\begin{aligned} -\alpha &\doteq \max_{i=1,2} \left\{ M_1 \left(\overline{a_{i0}} - \underline{a_{i1}} M_1 \right) \right\}, \\ g(t) &= \max\{x_1(t), x_2(t)\}. \end{aligned} \tag{3.10}$$

Next, we consider the following three cases.

Case 1. $x_1(0) > x_2(0)$, $g(0) = x_1(0) > M_1$. Then, there exists $\varepsilon > 0$ such that $g(t) = x_1(t) > M_1$ for $t \in [0, \varepsilon)$. We also derive that

$$x_1' \le \overline{a_{10}} x_1 - \underline{a_{11}} x_1^2 < -\alpha < 0. \tag{3.11}$$

Hence, if $t_2 > t_1$ and $t_1, t_2 \in [0, \varepsilon)$, we get

$$g(t_2) - g(t_1) < -\alpha(t_2 - t_1). \tag{3.12}$$

Case 2. $x_2(0) > x_1(0)$, $g(0) = x_2(0) > M_1$. Similarly, we could obtain that there exists $[0, \varepsilon)$. If $t_2 > t_1$ and $t_1, t_2 \in [0, \varepsilon)$, we get

$$g(t_2) - g(t_1) < -\alpha(t_2 - t_1). \tag{3.13}$$

Case 3. $x_2(0) = x_1(0) = g(0) > M_1$. We can also find an interval $[0, \varepsilon)$ such that $g(t) = x_1(t) > M_1$ or $g(t) = x_2(t) > M_1$. In the same way, if $t_2 > t_1$ and $t_1, t_2 \in [0, \varepsilon)$, we can obtain

$$g(t_2) - g(t_1) < -\alpha(t_2 - t_1). \tag{3.14}$$

Now, we can know that if $g(0) > M_1$, g(t) will monotonously decrease by speed α. So, there exists $T_1 > 0$. If $t \geq T_1$, we have

$$g(t) < M_1. \tag{3.15}$$

According to the third equation of (2.1), we have

$$\begin{aligned} x_3' &\leq x_3\left(-\underline{a_{30}} + \overline{a_{31}} \frac{\alpha_1(t)x_1^2(t-\tau_1)}{1+\beta_1(t)x_1^2(t-\tau_1)} - \underline{a_{32}}x_3 \right) \\ &< x_3\left(-\underline{a_{30}} + \overline{a_{31}}\overline{\left(\frac{\alpha_1}{\beta_1}\right)} - \underline{a_{32}}x_3 \right), \\ x_3'\big|_{x_3=M_2} &< x_3\left(-\underline{a_{30}} + \overline{a_{31}}\overline{\left(\frac{\alpha_1}{\beta_1}\right)} - \underline{a_{32}}x_3 \right). \end{aligned} \tag{3.16}$$

Hence, it follows from $x_3(0) \leq M_2$ that $x_3(t) \leq M_2$ for $t \geq 0$.

If

$$x_3(0) > M_2, \tag{3.17}$$

we only consider what follows. If $x_3 > M_2$, from the given condition we obtain

$$x_3\left(-\underline{a_{30}} + \overline{a_{31}}\overline{\left(\frac{\alpha_1}{\beta_1}\right)} - \underline{a_{32}}x_3 \right) < M_2\left[-\underline{a_{30}} + \overline{a_{31}}\overline{\left(\frac{\alpha_1}{\beta_1}\right)} - \underline{a_{32}}M_2 \right] < 0. \tag{3.18}$$

Let

$$-\beta = M_2\left[-\underline{a_{30}} + \overline{a_{31}}\overline{\left(\frac{\alpha_1}{\beta_1}\right)} - \underline{a_{32}}M_2 \right]. \tag{3.19}$$

We also derive that

$$x_3' < M_2\left[-\underline{a_{30}} + \overline{a_{31}}\overline{\left(\frac{\alpha_1}{\beta_1}\right)} - \underline{a_{32}}M_2 \right] = -\beta < 0. \tag{3.20}$$

Hence, if $t_2 > t_1$ and $t_1, t_2 \in [0, \varepsilon)$, we get

$$x_3(t_2) - x_3(t_1) < -\beta(t_2 - t_1). \tag{3.21}$$

Now, we can know that if $x_3(0) > M_2$, $x_3(t)$ will monotonously decrease by speed β. So, there exists T_2 such that $x_3(t) < M_2$ for $t \geq T_2$. Similarly, we also get

$$\begin{aligned} x_4' &\leq x_4\left(-\underline{a_{40}} + \overline{a_{41}} \frac{\alpha_2(t)x_1^2(t-\tau_2)}{1+\beta_2(t)x_1^2(t-\tau_2)} - \underline{a_{42}}x_4 \right) \\ &< x_4\left(-\underline{a_{40}} + \overline{a_{41}}\overline{\left(\frac{\alpha_2}{\beta_2}\right)} - \underline{a_{42}}x_4 \right). \end{aligned} \tag{3.22}$$

We can also choose the same M_3. There exists $T_3 > 0$ such that $x_4(t) < M_3$ for $t > T_3$. This completes the proof. □

Theorem 3.2. *Suppose that system* (2.1) *satisfies the following conditions:*

$$\begin{gathered}\underline{a_{10}} - \overline{D_1} > 0, \qquad \underline{a_{20}} - \overline{D_2} > 0, \\ E = \left(\underline{a_{10}} - \overline{D_1}\right)^2 - 4\overline{a_{11}}\overline{\left(\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2}\right)}M_x > 0, \\ \underline{a_{31}}\frac{\underline{\alpha_1}m_1^2}{1 + \overline{\beta_1}m_1^2} - \overline{a_{30}} - \overline{a_{34}}M_x > 0, \\ \underline{a_{41}}\frac{\underline{\alpha_2}m_1^2}{1 + \overline{\beta_2}m_1^2} - \overline{a_{40}} - \overline{a_{43}}M_x > 0\end{gathered} \tag{3.23}$$

in which

$$m_1 = \frac{\underline{a_{10}} - \overline{D_1} + \sqrt{E}}{2\overline{a_{11}}}. \tag{3.24}$$

Then, system (2.1) *is uniformly persistent.*

Proof. Suppose (x_1, x_2, x_3, x_4) is a solution of system (2.1) with the initial condition (2.2). According to the first equation of (2.1), we get

$$\begin{aligned}x_1'(t) &\geq x_1(t)\left(\left(a_{10}(t) - D_1(t)\right) - \frac{\alpha_1(t)x_1^2(t)x_3(t)}{1 + \beta_1(t)x_1^2(t)} - \frac{\alpha_2(t)x_1^2(t)x_4(t)}{1 + \beta_2(t)x_1^2(t)}\right) \\ &\geq -a_{11}(t)x_1^2(t) + \left(a_{10}(t) - D_1(t)\right)x_1(t) - \frac{\alpha_1(t)M_x}{\beta_1(t)} - \frac{\alpha_2(t)M_x}{\beta_2(t)}.\end{aligned} \tag{3.25}$$

So,

$$\liminf_{t \to \infty} x_1(t) \geq m_1 > 0. \tag{3.26}$$

Then, there exists a $T_5 > 0$ such that

$$x_1(t) \geq m_1 \quad \text{for } t \geq T_5. \tag{3.27}$$

Similarly,

$$\liminf_{t \to \infty} x_2(t) \geq m_2 \doteq \frac{\underline{a_{20}} - \overline{D_2}}{\overline{a_{21}}} > 0. \tag{3.28}$$

Then, there exists a $T_6 > 0$ such that

$$x_2(t) \geq m_2 \quad \text{for } t \geq T_6. \tag{3.29}$$

From the third equation of (2.1), we obtain

$$x_3'(t) \geq x_3(t)\left(- a_{30}(t) + a_{31}(t)\frac{\alpha_1(t)m_1^2}{1 + \beta_1(t)m_1^2} - a_{32}(t)x_3(t) - a_{34}(t)M_x\right). \tag{3.30}$$

So,

$$\liminf_{t\to\infty} x_3(t) \ge m_3 \doteq \frac{\underline{a_{31}}\left(\underline{\alpha_1} m_1^2/(1+\overline{\beta_1} m_1^2)\right) - \overline{a_{30}} - \overline{a_{34}} M_x}{\overline{a_{32}}} > 0. \tag{3.31}$$

Then, there exists a $T_7 > 0$ such that

$$x_3(t) \ge m_3 \quad \text{for } t \ge T_7. \tag{3.32}$$

Similarly, we also get

$$\liminf_{t\to\infty} x_4(t) \ge m_4 \doteq \frac{\underline{a_{41}}\left(\underline{\alpha_2} m_1^2/(1+\overline{\beta_2} m_1^2)\right) - \overline{a_{40}} - \overline{a_{43}} M_x}{\overline{a_{42}}} > 0. \tag{3.33}$$

Then, there exists a $T_8 > 0$ such that

$$x_4(t) \ge m_4 \quad \text{for } t \ge T_8. \tag{3.34}$$

Finally, let

$$D = \left\{ (x_1, x_2, x_3, x_4) \mid m_x < x_i < M_x,\ i = 1,2,3,4 \right\}, \tag{3.35}$$

where $m_x = \min_{i=1,2,3,4}\{m_i\}$ and $M_x = \max\{M_1^*, M_2^*, M_3^*\}$; M_i^* $(i = 1,2,3)$ is given in Theorem 3.1. From Theorem 3.1 and the above analysis, we see that D is a bounded compact region in $\mathbf{R}_+^4$ which has positive distance from coordinate hyperplanes. Let $\widehat{T} = \max\{T_i,\ i = 1,\dots,8\}$, then we obtain that if $t > \widehat{T}$, then every positive solution of system (2.1) with initial conditions (2.2) eventually enters and remains in the region D. This completes the proof. □

4. Almost periodic solution

In this section, we derive sufficient conditions which guarantee that the periodic solution of periodic system (2.2) is globally attractive.

Theorem 4.1. *In addition to* (2.5), (3.2), *and* (3.23), *assume further that all the coefficients of system* (2.1) *are continuous and positive almost periodic functions and*

$$\begin{aligned}
&\left(\underline{a_{11}} + \frac{\underline{D_1} m_2}{M_1^2} + \frac{\underline{\alpha_1} m_3}{1+\overline{\beta_1} M_1^2} + \frac{\underline{\alpha_2} m_4}{1+\overline{\beta_2} M_1^2}\right) m_1 \\
&\quad > \left(\frac{2\overline{\alpha_1}\overline{\beta_1} M_1^2 M_3}{(1+\underline{\beta_1} m_1^2)^2} + \frac{2\overline{\alpha_2}\overline{\beta_2} M_1^2 M_4}{(1+\underline{\beta_2} m_1^2)^2} + \frac{\overline{D_2}}{m_2}\right) M_1 + \frac{M_x^2}{m_x}\left(\frac{2\overline{\alpha_1 a_{31}} M_1}{(1+\underline{\beta_1} m_1^2)^2} + \frac{2\overline{\alpha_2 a_{41}} M_1}{(1+\underline{\beta_2} m_1^2)^2}\right), \\
&\left(\underline{a_{21}} + \frac{\underline{D_2} m_1}{M_2^2}\right) m_2 > \frac{\overline{D_1}}{m_1} M_2 + \frac{M_x^2}{m_x}\left(\frac{2\overline{\alpha_1 a_{31}} M_1}{(1+\underline{\beta_1} m_1^2)^2} + \frac{2\overline{\alpha_2 a_{41}} M_1}{(1+\underline{\beta_2} m_1^2)^2}\right), \\
&\underline{a_{32}} m_3 > \left(\frac{\overline{\alpha_1} M_1}{1+\underline{\beta_1} m_1^2} + \overline{a_{43}}\right) M_3 + \frac{M_x^2}{m_x}\left(\frac{2\overline{\alpha_1 a_{31}} M_1}{(1+\underline{\beta_1} m_1^2)^2} + \frac{2\overline{\alpha_2 a_{41}} M_1}{(1+\underline{\beta_2} m_1^2)^2}\right), \\
&\underline{a_{42}} m_4 > \left(\frac{\overline{\alpha_2} M_1}{1+\underline{\beta_2} m_1^2} + \overline{a_{34}}\right) M_3 + \frac{M_x^2}{m_x}\left(\frac{2\overline{\alpha_1 a_{31}} M_1}{(1+\underline{\beta_1} m_1^2)^2} + \frac{2\overline{\alpha_2 a_{41}} M_1}{(1+\underline{\beta_2} m_1^2)^2}\right).
\end{aligned} \tag{4.1}$$

Then, system (2.1) *has a unique positive almost periodic solution which is globally asymptotically stable. Furthermore, if system* (2.1) *is an* ω*-periodic system, then system* (2.1) *has a positive* ω*-periodic solution which is globally asymptotically stable.*

Proof. Consider the product system of (2.1):

$$\begin{aligned}
x_1' &= x_1\left(a_{10}(t) - a_{11}(t)x_1\right) - \frac{\alpha_1(t)x_1^2x_3}{1+\beta_1(t)x_1^2} - \frac{\alpha_2(t)x_1^2x_4}{1+\beta_2(t)x_1^2} + D_1(t)\left(x_2 - x_1\right),\\
x_2' &= x_2\left(a_{20}(t) - a_{21}(t)x_2\right) + D_2(t)\left(x_1 - x_2\right),\\
x_3' &= x_3\left(-a_{30}(t) + a_{31}(t)\frac{\alpha_1(t)x_1^2(t-\tau_1)}{1+\beta_1(t)x_1^2(t-\tau_1)} - a_{32}(t)x_3 - a_{34}(t)x_4\right),\\
x_4' &= x_4\left(-a_{40}(t) + a_{41}(t)\frac{\alpha_2(t)x_1^2(t-\tau_2)}{1+\beta_2(t)x_1^2(t-\tau_2)} - a_{42}(t)x_4 - a_{43}(t)x_3\right),\\
y_1' &= y_1\left(a_{10}(t) - a_{11}(t)y_1\right) - \frac{\alpha_1(t)y_1^2y_3}{1+\beta_1(t)y_1^2} - \frac{\alpha_2(t)y_1^2y_4}{1+\beta_2(t)y_1^2} + D_1(t)\left(y_2 - y_1\right),\\
y_2' &= y_2\left(a_{20}(t) - a_{21}(t)y_2\right) + D_2(t)\left(y_1 - y_2\right),\\
y_3' &= y_3\left(-a_{30}(t) + a_{31}(t)\frac{\alpha_1(t)y_1^2(t-\tau_1)}{1+\beta_1(t)y_1^2(t-\tau_1)} - a_{32}(t)y_3 - a_{34}(t)y_4\right),\\
y_4' &= y_4\left(-a_{40}(t) + a_{41}(t)\frac{\alpha_2(t)y_1^2(t-\tau_2)}{1+\beta_2(t)y_1^2(t-\tau_2)} - a_{42}(t)y_4 - a_{43}(t)y_3\right).
\end{aligned} \tag{4.2}$$

It is easily noted that the existence and uniqueness of the positive almost periodic solution of system (2.1) are equivalent to the existence and uniqueness of the positive almost periodic solution of system (4.2). Then, choose the following function:

$$V(t) = V(t, x_i, y_i) = \sum_{i=1}^{4} \left| \ln x_i(t) - \ln y_i(t) \right|. \tag{4.3}$$

Obviously, $V(t)$ satisfies conditions (i) and (ii) of Lemma 2.2. Next, we will prove that $V(t)$ satisfies condition (iii) of Lemma 2.2. It follows that

$$\frac{x_1'}{x_1} - \frac{y_1'}{y_1} = -a_{11}(x_1 - y_1) - D_1\left(\frac{x_2}{x_1} - \frac{y_2}{y_1}\right) - \left(\frac{\alpha_1 x_1 x_3}{1+\beta_1 x_1^2} - \frac{\alpha_1 y_1 y_3}{1+\beta_1 y_1^2}\right) - \left(\frac{\alpha_2 x_1 x_4}{1+\beta_2 x_1^2} - \frac{\alpha_2 y_1 y_4}{1+\beta_2 y_1^2}\right) \tag{4.4}$$

in which

$$\frac{\alpha_1 x_1 x_3}{1+\beta_1 x_1^2} - \frac{\alpha_1 y_1 y_3}{1+\beta_1 y_1^2} = \left(\frac{\alpha_1 x_3}{1+\beta_1 x_1^2} - \frac{\alpha_1 \beta_1 y_1 y_3 (x_1 + y_1)}{(1+\beta_1 x_1^2)(1+\beta_1 y_1^2)}\right)(x_1 - y_1) + \frac{\alpha_1 y_1}{1+\beta_1 x_1^2}(x_3 - y_3); \tag{4.5}$$

also,

$$\begin{aligned}
\frac{x_2'}{x_2}-\frac{y_2'}{y_2} &= \left(-a_{21}-\frac{D_2 y_1}{x_2 y_2}\right)(x_2-y_2)+\frac{D_2}{x_1}(x_1-y_1),\\
\frac{x_3'}{x_3}-\frac{y_3'}{y_3} &= a_{31}\frac{\alpha_1\left[x_1(t-\tau_1)+y_1(t-\tau_1)\right]}{\left[1+\beta_1 x_1^2(t-\tau_1)\right]\left[1+\beta_1 y_1^2(t-\tau_1)\right]}\left[x_1(t-\tau_1)-y_1(t-\tau_1)\right]\\
&\quad - a_{32}(x_3-y_3)-a_{34}(x_4-y_4),\\
\frac{x_4'}{x_4}-\frac{y_4'}{y_4} &= a_{41}\frac{\alpha_2\left[x_1(t-\tau_2)+y_1(t-\tau_2)\right]}{\left[1+\beta_1 x_1^2(t-\tau_2)\right]\left[1+\beta_1 y_1^2(t-\tau_2)\right]}\left[x_1(t-\tau_2)-y_1(t-\tau_2)\right]\\
&\quad - a_{42}(x_4-y_4)-a_{43}(x_3-y_3).
\end{aligned}\tag{4.6}$$

In this regard, after few computations, it is noted that

$$\begin{aligned}
D^+V(t,x_i,y_i) &= \sum_{i=1}^{4}\operatorname{sgn}(x_i(t)-y_i(t))\left(\frac{x_i'(t)}{x_i(t)}-\frac{y_i'(t)}{y_i(t)}\right)\\
&= \left[-a_{11}-\frac{D_1 y_2}{x_1 y_1}-\frac{\alpha_1 x_3}{(1+\beta_1 x_1^2)}-\frac{\alpha_2 x_4}{(1+\beta_2 x_1^2)}+\frac{\alpha_1\beta_1 y_1 y_3(x_1+y_1)}{(1+\beta_1 x_1^2)(1+\beta_1 y_1^2)}\right.\\
&\qquad\left.+\frac{\alpha_2\beta_1 y_1 y_4(x_1+y_1)}{(1+\beta_2 x_1^2)(1+\beta_2 y_1^2)}\right]|x_1-y_1|+\left[-a_{21}-\frac{D_2 y_1}{x_2 y_2}\right]|x_2-y_2|\\
&\quad - a_{32}|x_3-y_3|-a_{42}|x_4-y_4|+\operatorname{sgn}(x_1-y_1)\frac{D_1(x_2-y_2)}{x_1}\\
&\quad - \operatorname{sgn}(x_1-y_1)\frac{\alpha_1 y_1}{(1+\beta_1 x_1^2)}(x_3-y_3)-\operatorname{sgn}(x_1-y_1)\frac{\alpha_2 y_1}{(1+\beta_2 x_1^2)}(x_4-y_4)\\
&\quad + \operatorname{sgn}(x_2-y_2)\frac{D_2(x_1-y_1)}{x_2}\\
&\quad + \operatorname{sgn}(x_3-y_3)\frac{a_{31}\alpha_1\left[x_1(t-\tau_1)+y_1(t-\tau_1)\right]}{\left[1+\beta_1 x_1^2(t-\tau_1)\right]\left[1+\beta_1 y_1^2(t-\tau_1)\right]}\left[x_1(t-\tau_1)-y_1(t-\tau_1)\right]\\
&\quad - a_{34}\operatorname{sgn}(x_3-y_3)(x_4-y_4)\\
&\quad + \operatorname{sgn}(x_4-y_4)\frac{a_{41}\alpha_2\left[x_1(t-\tau_2)+y_1(t-\tau_2)\right]}{\left[1+\beta_1 x_1^2(t-\tau_2)\right]\left[1+\beta_1 y_1^2(t-\tau_2)\right]}\left[x_1(t-\tau_2)-y_1(t-\tau_2)\right]\\
&\quad - a_{43}\operatorname{sgn}(x_4-y_4)(x_3-y_3)\\
&\le -\left(\underline{a_{11}}+\frac{\underline{D_1}m_2}{M_1^2}+\frac{\underline{\alpha_1}m_3}{1+\overline{\beta_1}M_1^2}+\frac{\underline{\alpha_2}m_4}{1+\overline{\beta_2}M_1^2}-\frac{2\overline{\alpha_1}\overline{\beta_1}M_1^2M_3}{(1+\underline{\beta_1}m_1^2)^2}\right.\\
&\qquad\left.-\frac{2\overline{\alpha_2}\overline{\beta_2}M_1^2M_4}{(1+\underline{\beta_2}m_1^2)^2}-\frac{\overline{D_2}}{m_2}\right)|x_1-y_1|+\left(-\underline{a_{21}}-\frac{\underline{D_2}m_1}{M_2^2}+\frac{\overline{D_1}}{m_1}\right)|x_2-y_2|
\end{aligned}$$

$$+\left(-\underline{a_{32}}+\frac{\overline{\alpha_1}M_1}{1+\underline{\beta_1}m_1^2}+\overline{a_{43}}\right)|x_3-y_3|+\left(-\underline{a_{42}}+\frac{\overline{\alpha_2}M_1}{1+\underline{\beta_2}m_1^2}+\overline{a_{34}}\right)|x_4-y_4|$$
$$+\frac{2\overline{\alpha_1 a_{31}}M_1}{(1+\underline{\beta_1}m_1^2)^2}|x_1(t-\tau_1)-y_1(t-\tau_1)|+\frac{2\overline{\alpha_2 a_{41}}M_1}{(1+\underline{\beta_2}m_1^2)^2}|x_1(t-\tau_2)-y_1(t-\tau_2)|. \tag{4.7}$$

It follows from (4.1) that

$$\frac{1}{M_x}\sum_{i=1}^{4}\|x_i-y_i\|\le V(t,x_i,y_i)\le\frac{1}{m_x}\sum_{i=1}^{4}\|x_i-y_i\|. \tag{4.8}$$

Choose $P(s)=(M_x/m_x)s>s>0$, $a(s)=(1/M_x)s>0$, $b(s)=(1/m_x)s>0$. When

$$P(V(t,x_i(t),y_i(t)))\ge V(t+\theta,x_i(t+\theta),y_i(t+\theta)),\quad \theta\in[-\tau,0],\ i=1,2,3,4,$$
$$\begin{aligned}|x_1(t-\tau)-y_1(t-\tau)|&\le M_x|\ln x_1(t-\tau)-\ln y_1(t-\tau)|\\&\le M_xV(t-\tau,x_i(t-\tau),y_i(t-\tau))\\&\le M_x\frac{M_x}{m_x}V(t,x_i(t),y_i(t));\end{aligned} \tag{4.9}$$

then

$$\begin{aligned}\frac{2\overline{\alpha_1 a_{31}}M_1}{(1+\underline{\beta_1}m_1^2)^2}|x_1(t-\tau_1)-y_1(t-\tau_1)|&\le\frac{M_x^2}{m_x}\frac{2\overline{\alpha_1 a_{31}}M_1}{(1+\underline{\beta_1}m_1^2)^2}V(t,x_i(t),y_i(t)),\\ \frac{2\overline{\alpha_2 a_{41}}M_1}{(1+\underline{\beta_2}m_1^2)^2}|x_1(t-\tau_2)-y_1(t-\tau_2)|&\le\frac{M_x^2}{m_x}\frac{2\overline{\alpha_2 a_{41}}M_1}{(1+\underline{\beta_2}m_1^2)^2}V(t,x_i(t),y_i(t)).\end{aligned} \tag{4.10}$$

Hence,

$$\begin{aligned}D^+V(t,x_i,y_i)\le\Bigg[&-\left(\underline{a_{11}}+\frac{\underline{D_1}m_2}{M_1^2}+\frac{\underline{\alpha_1}m_3}{1+\overline{\beta_1}M_1^2}+\frac{\underline{\alpha_2}m_4}{1+\overline{\beta_2}M_1^2}\right)m_1\\&+\left(\frac{2\overline{\alpha_1}\overline{\beta_1}M_1^2M_3}{(1+\underline{\beta_1}m_1^2)^2}+\frac{2\overline{\alpha_2}\overline{\beta_2}M_1^2M_4}{(1+\underline{\beta_2}m_1^2)^2}+\frac{\overline{D_2}}{m_2}\right)M_x\Bigg]|\ln x_1-\ln y_1|\\&+\left[-\left(\underline{a_{21}}+\frac{\underline{D_2}m_1}{M_2^2}\right)m_2+\frac{\overline{D_1}}{m_1}M_2\right]|\ln x_2-\ln y_2|\\&+\left[-\underline{a_{32}}m_3+\left(\frac{\overline{\alpha_1}M_1}{1+\underline{\beta_1}m_1^2}+\overline{a_{43}}\right)M_3\right]|\ln x_3-\ln y_3|\\&+\left[-\underline{a_{42}}m_4+\left(\frac{\overline{\alpha_2}M_1}{1+\underline{\beta_2}m_1^2}+\overline{a_{34}}\right)M_4\right]|\ln x_4-\ln y_4|\\&+\frac{M_x^2}{m_x}\left(\frac{2\overline{\alpha_1 a_{31}}M_1}{(1+\underline{\beta_1}m_1^2)^2}+\frac{2\overline{\alpha_2 a_{41}}M_1}{(1+\underline{\beta_2}m_1^2)^2}\right)V(t,x_i(t),y_i(t))\le -CV(t,x_i(t),y_i(t)),\end{aligned} \tag{4.11}$$

where

$$
\begin{aligned}
-\mathcal{C} = \max\Bigg\{ & -\left(\underline{a_{11}} + \frac{\underline{D_1} m_2}{M_1^2} + \frac{\underline{\alpha_1} m_3}{1+\overline{\beta_1}M_1^2} + \frac{\underline{\alpha_2} m_4}{1+\overline{\beta_2}M_1^2}\right) m_1 \\
& + \left(\frac{2\overline{\alpha_1}\,\overline{\beta_1} M_1^2 M_3}{\left(1+\underline{\beta_1} m_1^2\right)^2} \frac{2\overline{\alpha_2}\,\overline{\beta_2} M_1^2 M_4}{\left(1+\underline{\beta_2} m_1^2\right)^2} + \frac{\overline{D_2}}{m_2}\right) M_x + \frac{M_x^2}{m_x}\left(\frac{2\overline{\alpha_1}\,\overline{a_{31}} M_1}{\left(1+\underline{\beta_1} m_1^2\right)^2} + \frac{2\overline{\alpha_2}\,\overline{a_{41}} M_1}{\left(1+\underline{\beta_2} m_1^2\right)^2}\right) \\
& - \left(\underline{a_{21}} + \frac{\underline{D_2} m_1}{M_2^2}\right) m_2 + \frac{\overline{D_1}}{m_1} M_2 + \frac{M_x^2}{m_x}\left(\frac{2\overline{\alpha_1}\,\overline{a_{31}} M_1}{\left(1+\underline{\beta_1} m_1^2\right)^2} + \frac{2\overline{\alpha_2}\,\overline{a_{41}} M_1}{\left(1+\underline{\beta_2} m_1^2\right)^2}\right) \\
& - \underline{a_{32}} m_3 + \left(\frac{\overline{\alpha_1} M_1}{1+\underline{\beta_1} m_1^2} + \overline{a_{43}}\right) M_3 + \frac{M_x^2}{m_x}\left(\frac{2\overline{\alpha_1}\,\overline{a_{31}} M_1}{\left(1+\underline{\beta_1} m_1^2\right)^2} + \frac{2\overline{\alpha_2}\,\overline{a_{41}} M_1}{\left(1+\underline{\beta_2} m_1^2\right)^2}\right) \\
& - \underline{a_{42}} m_4 + \left(\frac{\overline{\alpha_2} M_1}{1+\underline{\beta_2} m_1^2} + \overline{a_{34}}\right) M_4 + \frac{M_x^2}{m_x}\left(\frac{2\overline{\alpha_1}\,\overline{a_{31}} M_1}{\left(1+\underline{\beta_1} m_1^2\right)^2} + \frac{2\overline{\alpha_2}\,\overline{a_{41}} M_1}{\left(1+\underline{\beta_2} m_1^2\right)^2}\right)\Bigg\}.
\end{aligned}
\tag{4.12}
$$

This completes the proof. □

5. Discussion

In this work, we consider a nonautonomous delayed predator-prey model with competition and diffusion. Some sufficient conditions on uniform persistence of the model have been given. By means of the Liapunov-Razumikhin technique, it is also seen that, under almost periodic circumstances, the existence and uniqueness of the positive almost periodic solution which is globally asymptotically stable are governed by several inequalities.

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