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Application of piezoelectric actuators to active control of composite spherical caps

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Abstract. Dynamics of reinforced shallow spherical caps is considered in this paper. The cap is subjected to a uniform dynamic pressure that results in a globally axisymmetric motion. The reduction of amplitudes and stresses is achieved by using pairs of piezoelectric actuators bonded to the meridional stiffeners (one of the actuators can be bonded to the inner surface of the skin). These pairs of actuators produce dynamic couples that can reduce the amplitude of motion. The analysis is based on the theory of shallow shells (Donnell–Mushtari–Vlasov theory) using a smeared stiffeners technique. The analytical solution is generated for a cap clamped along the boundary using the approximation of the meridional curve suggested by Huang. Numerical results illustrate the feasibility of a significant reduction of deflections and stresses using piezoelectric actuators with weight and energy consumption acceptable in practical design.

1. Introduction

Application of piezoelectric actuators for reducing deformations and stresses in the structures has been considered since the eighties. The concept of piezoelectric stiffening for acoustic noise suppression for aircraft and launch loads has been studied using plate and cylindrical shell models (Knowles *et al* 1997). In particular, the concept of piezoelectric stiffeners has been investigated (Birman 1993, Knowles *et al* 1997). In the present analysis, this concept is applied to the formulation of an active control problem for a composite shallow spherical cap reinforced in both meridional and circumferential directions by closely spaced stiffeners. The cap is subject to an axisymmetric external dynamic pressure and temperature. The analysis is concerned with global deformations (and stresses) that are axisymmetric, while relatively small deformations between the stiffeners are disregarded. The global deformations and stresses are reduced using pairs of piezoelectric actuators bonded to the meridional stiffeners and generating bending moments that reduce the moments produced by external loads.

The approach taken here is based on the theory developed by Simitses and Blackmon (1975) for axisymmetric buckling of shallow reinforced caps. The contribution of the stiffeners is incorporated into the piezo-coupled formulation using the technique of smeared stiffeners. The paper presents a closed-form analytical solution for transverse and meridional displacements that satisfy the boundary and axisymmetry conditions. Numerical results show that, for a representative problem, employing piezoelectric actuators for efficient control of dynamic deformations and stresses is achievable. Note that this paper extends the theory previously developed for piezo-coupled beams and plates (Knowles and Murray 1997, Murray and Knowles 1997).

2. Piezo-coupled shell theory analysis

Consider a spherical cap reinforced by a system of stiffeners in the meridional and circumferential directions that are attached to the inner surface. The structure is manufactured from a symmetrically laminated composite material so that its coupling stiffnesses, extensional stiffnesses A_{16} and A_{26} and bending stiffnesses D_{16} and D_{26} are equal to zero. The meridional stiffeners are designed so that the ratios of the cross sectional area and the first and second moments of area about the middle surface of the skin to the spacing remain constant. The corresponding ratios for the circumferential stiffeners also remain constant throughout the cap. Pairs of piezoelectric actuators are bonded to the meridional stiffeners as shown in figure 1. The elements of these pairs can be activated in antiphase to produce bending couples without applying in-surface stresses. An alternative solution is to employ two elements of each pair with opposite polarities and to activate them by in-phase voltage to achieve the same effect.

As was previously shown (Birman and Adali 1995), generating an active moment using pairs of actuators as

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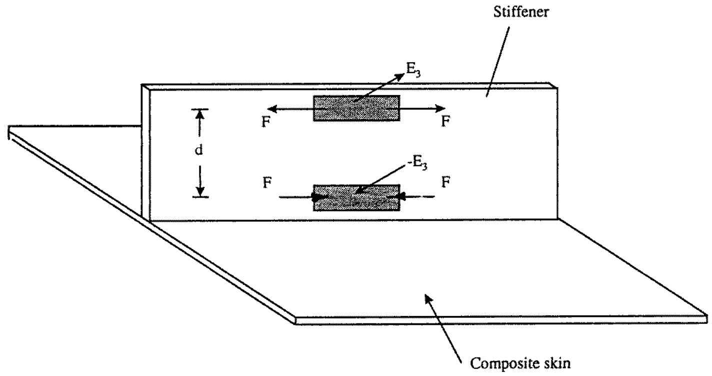

Figure 1. A pair of piezoelectric actuators generating moment. Note that a similar pair should be attached to the opposite surface of the stiffener to avoid introduction of torsion. The force $F = e_{31}E_3A_p/2$ where A_p is the cross sectional area of both actuators.

shown in figure 1 results in a more effective reduction of bending deformations, compared to the case of in-surface force excitation. The stiffeners in the circumferential direction do not have actuators bonded to them, as this would result in undesirable axisymmetric deformations.

A uniform external pressure combined with a uniform temperature result in deformations including global and local components. Although local deformations of the skin between meridional stiffeners imply locally nonsymmetric pattern of displacements, global deflections of the shell are axisymmetric. Therefore, neglecting local deformations that remain negligible, as long as the spacing between the stiffeners is small, the problem is axisymmetric.

The Donnell–Mushtari–Vlasov (DMV) theory (Vol'mir 1976) is employed in this paper to analyse the piezo-coupled shell structure. The stiffeners are assumed closely spaced in both meridional and circumferential directions so that the smeared stiffeners technique can be validly applied. The DMV-based analysis requires the following assumptions (which can be somewhat weakened):

(a) in-plane displacements do not affect bending strains (changes of curvature and twist of the middle surface);
(b) in-plane inertia is negligible;
(c) transverse shear stresses are negligible, i.e., the skin is thin and the height (depth) of the stiffeners is small compared to the radius at the boundary plane, i.e., R.

The variations of temperature affect the properties of both composite and piezoelectric materials (Birman 1996). However, this effect is not considered in the present analysis, temperature is assumed constant and the properties of the materials can be taken corresponding to this value. Deformations of the shell are assumed small compared to the thickness of the skin so that the problem is geometrically linear. Two displacement components are considered in the following analysis, i.e., w, that is perpendicular to the boundary plane, and u, that represents displacements within the meridional planes in the direction parallel to the boundary plane. The latter component is usually much smaller than the former displacement and it may be neglected (Birman and Simitses 1989).

3. Governing equations

The strain–displacement relations are (Simitses and Blackmon 1975):

$$\begin{aligned}
\epsilon_{rr} &= u_{,r} + Z_{,r}w_{,r} + \tfrac{1}{2}w_{,r}^2 \\
\epsilon_{\theta\theta} &= u/r \\
\gamma_{r\theta} &= 0 \\
\kappa_{rr} &= -w_{,rr} \quad \kappa_{\theta\theta} = -w_{,r}/r \quad \kappa_{r\theta} = 0
\end{aligned} \tag{1}$$

where ϵ_{ij} are the middle surface strains, κ_{ij} are the changes of curvature and twist of this surface, Z is a distance from the reference (boundary) plane to the undeformed middle surface and r is the in-plane radial coordinate. Note that the nonlinear term is retained in these equations. Although the subsequent analysis is linear, the nonlinear term is needed to incorporate in-surface thermal stresses that affect the response to the formulation.

The constitutive relations for a composite structure represent stress resultants and stress couples in terms of strains. However, in the case considered here, the stress resultants and couples include several components, i.e., the contributions of the skin, smeared stiffeners, terms associated with the stiffness of piezoelectric actuators (this contribution may be negligible, if the cross sectional area of the actuators is small) and the active moments produced by these actuators. The integration of the stresses in the skin and stiffeners in the direction perpendicular to the middle surface results in the corresponding contributions to the total stress resultants and couples. In particular, the stresses in generally laminated laminae that form the skin are given by:

$$\begin{pmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \tau_{r\theta} \end{pmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ & Q_{22} & Q_{26} \\ \text{sym} & & Q_{66} \end{bmatrix} \cdot \begin{pmatrix} \epsilon_{rr} + z\kappa_{rr} - \alpha_{rr}T \\ \epsilon_{\theta\theta} + z\kappa_{\theta\theta} - \alpha_{\theta\theta}T \\ -\alpha_{r\theta}T \end{pmatrix} \tag{2}$$

where Q_{ij} are transformed reduced stiffness values, α_{ij} are the coefficients of thermal expansion and T is a uniform temperature in excess of a stress-free level. The stiffeners act as narrow beams so that, for example, the stress in the meridional stiffeners is given by the first equation in (2), where the terms proportional to Q_{12} and Q_{16} are disregarded.

The stress in the meridional actuators subjected to an electric field acting perpendicular to the surface is:

$$\sigma_{rr} \;=\; E_p(\in_{rr} \;-\alpha T) \;-\; e_{31}E_3 \qquad (3)$$

where E_p is the modulus of elasticity, e_{31} is a piezoelectric constant and E_3 is the electric field. The directional subscripts 1 and 3 refer to the meridional and actuator-thickness directions, respectively. The integration of equations (2) and (3) over the z-coordinate counted from the middle surface of the cap yields the following constitutive relationship:

$$\begin{pmatrix} N_r \\ N_\theta \\ M_r \\ M_\theta \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} & B_{11} & 0 \\ A_{12} & A_{22} & 0 & B_{22} \\ B_{11} & 0 & D_{11} & D_{12} \\ 0 & B_{22} & D_{12} & D_{22} \end{bmatrix} \cdot \begin{pmatrix} \in_{rr} \\ \in_{\theta\theta} \\ \kappa_{rr} \\ \kappa_{\theta\theta} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ M_r(E) \\ 0 \end{pmatrix} - \begin{pmatrix} N_r^T \\ N_\theta^T \\ M_r^T \\ M_\theta^T \end{pmatrix} \qquad (4)$$

where the extensional, coupling and bending stiffness is given as

$$\begin{aligned}
A_{11} &= \int Q_{11s}\mathrm{d}z + (1/s_1)\left(b_1 \int Q_{11r}\mathrm{d}z \;+\; E_pA_p\right) \\
A_{12} &= \int Q_{12s}\mathrm{d}z \\
A_{22} &= \int Q_{22s}\mathrm{d}z \;+ (1/s_2)b_2 \int Q^*_{11r}\mathrm{d}z \\
B_{11} &= (1/s_1)\left(b_1 \int Q_{11r}z\mathrm{d}z + E_pF_p\right) \\
B_{22} &= (1/s_2)(b_2 \int Q^*_{11r}z\mathrm{d}z \\
D_{11} &= \int Q_{11s}z^2\mathrm{d}z \;+\; (1/s_1)\left(b_1 \int Q_{11r}z^2\mathrm{d}z +\; E_pI_p\right) \\
D_{12} &= \int Q_{12s}z^2\mathrm{d}z \\
D_{22} &= \int Q_{22s}z^2\mathrm{d}z \;+\; (1/s_2)b_2 \int Q^*_{11r}z^2\mathrm{d}z.
\end{aligned} \qquad (5)$$

The integration of equations (5) is understood to be carried out with respect to the thickness of the skin or with respect to the height of the stiffener, where s_1 and s_2 represent the meridional and circumferential stiffener spacings, respectively. The widths of these stiffeners are similarly connoted by b_1 and b_2. The subscripts s and r refer to the contributions of the skin (s) and the stiffeners (r). The cross sectional area of each pair of piezoelectric stiffeners and their first and second moments about the middle surface of the skin of the cap are denoted by A_p, F_p and I_p, respectively. The transformed reduced stiffness of the circumferential stiffeners in the natural coordinate system with the 1-axis oriented in the circumferential direction is denoted by Q^*_{11r}. Note that the design is such that the ratios b_i/s_i $(i \;=\; 1, 2)$ remain constant. The contribution of the stiffness of piezoelectric actuators is also independent of the meridional coordinate; if the cross-sectional area of these actuators is small compared to those of the skin and stiffeners, this contribution is negligible.

The thermal terms that appear in equations (4) are given by

$$\begin{aligned}
N_r^T &= \left\{\int (Q_{11s}\alpha_{rrs} \;+\; Q_{12s}\alpha_{\theta\theta s})\,\mathrm{d}z + \frac{1}{s_1}\left[b_1 \int Q_{11r}\alpha_{rrr}\mathrm{d}z \;+\; E_p\alpha A_p\right]\right\}T \\
N_\theta^T &= \left\{\int (Q_{12s}\alpha_{rrs} + Q_{22s}\alpha_{\theta\theta s})\,\mathrm{d}z + \frac{b_2}{s_2} \int Q^*_{11r}\alpha^*_{\theta\theta r}\mathrm{d}z\right\}T \\
M_r^T &= \frac{1}{s_1}\left(b_1 \int Q_{11r}\alpha_{rrr}z\mathrm{d}z + E_p\alpha F_p\right) \\
M_\theta^T &= \frac{b_2}{s_2} \int Q^*_{11r}\alpha^*_{\theta\theta r}z\mathrm{d}z
\end{aligned} \qquad (6)$$

where α and $\alpha^*_{\theta\theta r}$ are the coefficients of thermal expansion of the piezoelectric actuators and the circumferential stiffeners in their axial direction, respectively. Note that the skin does not affect thermal stress couples due to symmetric lamination. The term

$$M_r(E) \;=\; \frac{e_{31}A_p}{2s_1}E_3d \qquad (7)$$

where d is the distance between two actuators that form a couple (as shown in figure 1) represents the active piezoelectric stress couple applied to the structure. Note that since the ratio A_p/s_1 is kept constant, the expression in equation (7) is independent of the meridional coordinate.

The strain energy of a cap is given by

$$U \;=\; \pi \int_0^R \left(N_r \in_{rr} \;+ N_\theta \in_{\theta\theta} \;+ M_r\kappa_{rr} \;+\; M_\theta\kappa_{\theta\theta}\right) r\mathrm{d}r \qquad (8)$$

where R is the in-plane cap radius and the kinetic energy is represented by $T = \pi \int_0^R mw^2_{,t}r\mathrm{d}r$ where t is time and m is the mass per unit area and can be calculated as

$$m = \rho_s h + (1/s_1)\left(\rho_r A_{rr} \;+\; \rho_p A_p\right) + (1/s_2)\rho_r A_{\theta r}. \qquad (9)$$

In equation (9), ρ_i denotes the mass density of the skin (s), stiffener (r) and piezoelectric (p) materials, h is the thickness of the skin, A_{rr} and $A_{\theta r}$ are the cross sectional areas of the meridional and circumferential stiffeners. The substitution of equations (1)–(4) into equation (8) yields the following expression for the strain energy in terms of displacements:

$$\begin{aligned}
U = \pi \int_0^R \Bigg\{ & A_{11}\left(u_{,r} + Z_{,r}\, w_{,r}\right)^2 + 2A_{12}\frac{u}{r}\left(u_{,r} \;+ Z_{,r}\, w_{,r}\right) \\
& + A_{22}(u/r)^2 - 2B_{11}w_{,rr}\left(u_{,r} + Z_{,r}\, w_{,r}\right) \\
& - 2B_{22}(u/r)w_{,r}/r + D_{11}\left(w_{,rr}\right)^2 + 2D_{12}w_{,rr}\, w_{,r}/r + \\
& D_{22}\left(w_{,r}/r\right)^2 - N_r^T\left(u_{,r} + Z_{,r}\, w_{,r} + \frac{1}{2}w_{,r}^2\right) \\
& - N_\theta^T(u/r) + \left[M_r^T + M_r(E)\right]w_{,rr} + M_\theta^T w_{,r}/r \Bigg\} r\mathrm{d}r
\end{aligned} \qquad (10)$$

Note that the nonlinear term, i.e. $w_{,r}^2/2$ in the expression for the meridional strain in equation (1) would result in higher-order terms in equation (10) and in nonlinear equations of motion. However, the present solution is confined to the linear problem and higher-order terms are disregarded in equation (10). The exception is the product of the meridional thermal stress resultant by $w_{,r}^2/2$. This product is retained since it reflects the in-surface thermally induced stress that is present in the linear formulation. Although the stress couple $M_r(E)$ generated by piezoelectric actuators is, technically speaking, an external load, it is incorporated into the strain energy.

Following the previous approach to the analysis of shallow spherical caps (Huang 1964, Simitses and Blackmon 1975), the meridional curve of the unloaded cap is approximated by a quadratic function

$$Z = f\left[1 - (r/R)^2\right] \tag{11}$$

where f is the rise of the cap over the boundary plane.

The boundary conditions corresponding to a clamped edge, i.e. $w = w_{,r} = 0$, can be satisfied if the deflection of a shallow cap is represented as (Vol'mir 1976):

$$w = a(t)\left[1 - (r/R)^2\right]^2 \tag{12}$$

where t is time and $a(t)$ is a generalized coordinate. If meridional displacements in the boundary plane are precluded, i.e., $u = 0$, one can use

$$u = b(t)(r/R)\,(1 - (r/R)) \tag{13}$$

as an approximation for this component of displacements where $b(t)$ is the second generalized coordinate. This satisfies the symmetry requirement $u(0) = 0$.

Note that the form of deformations given by these equations is similar to that employed by Timoshenko (1940) in his investigation of an axisymmetric static nonlinear problem of circular plates. This similarity reflects on the approach of the theory of shallow shells adopted in the present analysis where the coordinates of the middle surface of the shell are identified with those of a reference plane.

The substitution of these approximations into the expressions (8) and (9) yields the following expressions for the strain and kinetic energies:

$$\begin{aligned} U &= (\pi/2)\big\{\big[\kappa_{11} + \kappa_1(T)\big]a^2 + 2\kappa_{12}ab \\ &\quad + \kappa_{22}b^2 + \kappa_2(T)a + \kappa_3(T)b + 2M_r(E)a\big\} \\ T &= (\pi/2)m_{11}a_{,t}^2 \end{aligned} \tag{14}$$

where κ_{ii}, $\kappa_i(T)$ and m_{11} are numerical coefficients that can easily be evaluated and are given by

$$\begin{aligned} \kappa_{11} &= \left(16/R^2\right)\left(\frac{2}{15}f^2A_{11} - \frac{1}{3}fB_{11} + D_{11} + \frac{1}{3}D_{22}\right) \\ \kappa_1(T) &= -\frac{2}{3}N_r^T \\ m_{11} &= mR^2/5. \end{aligned} \tag{15}$$

The expressions for the coefficients κ_{12} and κ_{22} can easily be derived and are omitted here for brevity. Note that the Lagrange equation used in the following analysis yields constant terms $\kappa_2(T)$ and $\kappa_3(T)$ that do not affect the motion. Therefore, these terms do not have to be considered in linear dynamic problems. The energy of the applied pressure $q = q(R, t)$ is given by

$$U_p = -2\pi\int_0^R qwr\,\mathrm{d}r. \tag{16}$$

If $q = q(t)$, this becomes $U_p = -\frac{1}{3}\pi qaR^2$.

These strain, kinetic and external pressure energy expressions can now be substituted into the Lagrange equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial\left(U + U_p\right)}{\partial q_i} = 0 \tag{17}$$

where the generalized coordinate $q_i = u$ or w. This yields a system of two equations of motion:

$$\begin{aligned} &m_{11}a_{,tt} + [\kappa_{11} + \kappa_1(T)]\,a + \kappa_{12}b = \frac{1}{3}qR^2 - M_r(E) \\ &\kappa_{12}a + \kappa_{22}b = 0. \end{aligned} \tag{18}$$

The solution of equations (18) can now be obtained as a function of $q(t)$. For example, consider the estimate for the effect of piezoelectric actuation when it is possible to reduce the amplitude of deflections of the cap to a uniformly distributed harmonic pressure, i.e. $A = a(E_3)/a(E_3 = 0)$. From consideration of equations (18), it follows that

$$A = 1 - \frac{3M_r(E_{\max})}{q_{\max}R^2} \tag{19}$$

where $E_{\max}$ and $q_{\max}$ are the amplitude values of the electric field and pressure, respectively. Note that equation (19) is obtained by assumption that the fluctuations of the electric field occur with the same frequency and are tuned in phase with those of the external pressure.

4. Numerical results

Consider the effectiveness of the proposed active control method for a spherical cap with piezoelectric actuators characterized by $e_{31} = -11.8$ N m^{-1} V^{-1} typical for PZT-5A.

Referring to figure 1, the expression (19) can be represented in the form

$$A = 1 - 6e_{31}B\phi_{\max}d\big/\left(q_{\max}R^2\right) \tag{20}$$

where $\phi_{\max}$ is the amplitude value of voltage taken equal to 200 V and $B = b_{piezo}/s_1$, b_{piezo} being the width of each piezoelectric actuator. In the following examples, the ratio B is assumed constant throughout the cap. The arm of the piezoelectric couples $d = 0.1$ m.

The results shown in figure 2 illustrate the effects of the amplitude of the dynamic pressure and the parameter B on the effectiveness of the active control. As follows from this figure, higher amplitudes of the dynamic pressure lower the effectiveness of the control. This is natural since the voltage and geometry of piezoelectric elements remain

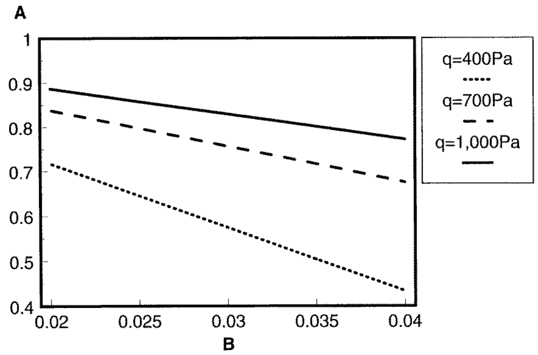

Figure 2. Effects of the dynamic pressure amplitude and the ratio B on active control.

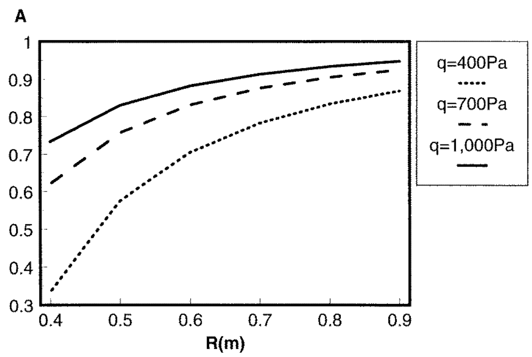

Figure 3. Effect of cap radius on active control.

without change. The second conclusion is that higher values of B result in a higher effectiveness of control. This is explained by the fact that wider actuators produce a higher control moment. The effect of the cap radius shown in figure 3 is also predictable. Larger radius results in a more flexible cap and accordingly, the effectiveness of control decreases.

5. Conclusions

The paper presents the theory of composite shallow spherical caps reinforced by closely spaced meridional and circumferential stiffeners. Global dynamic deformations and stresses are controlled by pairs of piezoelectric actuators that produce bending moments in the meridional direction. The theory employs the Donnell–Mushtari–Vlasov theory of shallow shells and the smeared stiffeners technique. The closed-form solution is obtained for a cap clamped along the boundary. Numerical results generated for a cap subject to a uniformly distributed dynamic pressure illustrate the feasibility of significant reductions of displacements using dynamic moments produced by piezoelectric actuators. It is shown that the effectiveness of active control is negatively affected by an increase of the cap radius or by larger amplitudes of dynamic pressure. Predictably, using piezoelectric actuators of a larger size results in a better control of dynamic displacements.

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