Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 8, No. 2, 2016 Article ID IJIM-00612, 7 pages
Research Article
Science and Research Branch (IAU)

Theory of block-pulse functions in numerical solution of Fredholm integral equations of the second kind

A. Abdollahi *†, E. Babolian ‡

Received Date: 2014-11-01 Revised Date: 2015-09-03 Accepted Date: 2015-12-09

Abstract

Recently, the block-pulse functions (BPFs) are used in solving electromagnetic scattering problem, which are modeled as linear Fredholm integral equations (FIEs) of the second kind. But the theoretical aspect of this method has not fully investigated yet. In this article, in addition to presenting a new approach for solving FIE of the second kind, the theory of both methods is investigated as a main part. By providing a new method based on BPFs for solving FIEs of the second kind, the least squares and non-least squares solutions are defined for this problem. First, the convergence of the non-least squares solution is proved by the Nyström method. Then, considering the fact that the set of all invertible matrices is an open set, the convergence of the least squares solution is investigated. The convergence of Nyström method has the main role in proving the basic results. Because the presented convergence trend is independent of the orthogonality of the basis functions, the given method can be applied for any arbitrary method.

Keywords : Block-pulse functions; Fredholm integral equation; Least squares approximation.

1 Introduction

It is necessary to have an effective theoretical aspect for numerical methods. The block-pulse functions, because of their simple structure, are used in numerical solution of linear Fredholm integral equations and other related functional equations [2, 3, 4, 8, 14, 16]. Since in these approaches, finding an approximate solution is led to solve the corresponding linear system of equations, the following statements should be investigated:

i) Existence and uniqueness of solution for the corresponding system.

ii) Convergence of the approximate solution to the exact solution.

In [1, 7, 12], the above propositions are discussed for some numerical methods. Since one- and two-dimensional electromagnetic scattering problems can be modeled by second kind FIE [5, 6, 9, 10]. S. Hatamzadeh-V. and Z. Masouri solved this equation by BPFs [8]. The advantage of this method is the low cost of setting up the equations without applying any projection methods such as collocation or Galerkin. The main focus of this article is to present an effective theoretical aspect for the given method in [8]. Because of the orthogonality of block-pulse functions, the convergence of the presented approach in [8] can be investigated by Galerkin method [12], but a

*Corresponding author. a.abdollahi@iau-maragheh.ac.ir
†Department of Mathematics, Maragheh Branch, Islamic Azad University, Maragheh, Iran.
‡Department of Mathematics, College of Basic Sciences, Tehran Science and Research Branch, Islamic Azad University, Tehran, Iran.

new approach has been used for investigating its convergence, that can be used for non-orthogonal and arbitrary basis.

In this article, beside presenting a new approach for second kind FIE based on BPFs (that coincides with Nyström method based on midpoint rule), the statements i and ii are investigated and then by elementary theorems, the proved lemmas are extended for the given method in [8]. The proofs of lemmas are mostly derived from the property of continuous functions. First of all, we describe briefly some characteristics of BPFs and the applications for solving a second kind FIE.

2 Solving Fredholm integral equations

2.1 Review of block-pulse functions

For each positive integer n, we define

$$h=\frac{1}{n},\quad t_i=ih,\quad i=0,\cdots,n, \tag{2.1}$$

that is, $\{t_i\}_{i=0}^{n}$ are equidistant points in $[0,1]$. With a little change an $n-$set of BPFs is defined over the interval $[0,1]$ as follows [16]:

$$\varphi_i(t)=\begin{cases}1, & t\in[t_i,t_{i+1}),\\ 0, & otherwise,\end{cases}\quad i=0,\cdots,n-2, \tag{2.2}$$

$$\varphi_{n-1}(t)=\begin{cases}1, & t\in[t_{n-1},t_n],\\ 0, & otherwise,\end{cases}\quad (i=n-1).$$

Lemma 2.1 *If $f\in C[0,1]$ and*

$$f(t)\simeq\sum_{i=0}^{n-1}\overline{f}_i\varphi_i(t)=:L_n(t), \tag{2.3}$$

then L_n is a least squares approximation for f in the basis $\{\varphi_i\}_{i=0}^{n-1}$, if and only if,

$$\overline{f}_i=\frac{1}{h}\int_{t_i}^{t_{i+1}}f(t)dt,\quad i=0,\cdots,n-1. \tag{2.4}$$

Proof: *See [16].*

By using the mean value theorem for integrals in 2.4, for each $i\in\{0,\cdots,n-1\}$, there exists $\xi_i\in(t_i,t_{i+1})$ such that

$$\frac{1}{h}\int_{t_i}^{t_{i+1}}f(t)dt=f(\xi_i), \tag{2.5}$$

since $t_i<\xi_i<t_{i+1}$, for sufficiently large n, we have

$$f(\xi_i)\simeq f(t_i), \tag{2.6}$$

from (2.4) - (2.6), we get

$$\overline{f}_i\simeq f(t_i). \tag{2.7}$$

Definition 2.1 *If f is defined on $[0,1]$ and*

$$f_i:=f(t_i),\quad i=0,\cdots,n-1,$$

then from (2.3) and (2.7) we have

$$f(t)\simeq\sum_{i=0}^{n-1}f_i\varphi_i(t)=:I_n(t),$$

and we define I_n as a non-least squares approximation for f in the basis $\{\varphi_i\}_{i=0}^{n-1}$.

Lemma 2.2 *Let $f\in C^1[0,1]$. If L_n and I_n are respectively least squares and non-least squares approximations for f in the basis $\{\varphi_i\}_{i=0}^{n-1}$, then*

$$\|f-L_n\|_\infty\leq\frac{1}{n}\|f'\|_\infty,$$

and

$$\|f-I_n\|_\infty\leq\frac{1}{n}\|f'\|_\infty.$$

Proof: See [15].

Lemma 2.3 *Let L_n and I_n be respectively least squares and non-least squares approximations for f and $t\in[t_i,t_{i+1})$, then*

$$L_n(t)=\overline{f}_i$$

$$I_n(t)=f(t_i)$$

Proof: From the definition of block-pulse functions (2.2), the proof is trivial.

2.2 Review of expansion method for solving FIE based on BPFs

The following approach is recalled from [8]. Consider the second kind FIE

$$x(t)-\int_0^1 k(t,s)x(s)ds=f(t),\quad t\in[0,1], \tag{2.8}$$

with $k\in C^p([0,1]\times[0,1])$ and $f\in C^q[0,1]$, with $p,q\geq 1$.
These provide us to write

$$x(t) \simeq \sum_{i=0}^{n-1} \overline{x}_i \varphi_i(t),$$

$$f(t) \simeq \sum_{i=0}^{n-1} \overline{f}_i \varphi_i(t),$$

$$k(t,s) \simeq \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \overline{k}_{ij} \varphi_i(t)\varphi_j(s),$$

where from lemma 2.4, for each $i,j \in \{0,\cdots,n-1\}$, we have

$$\overline{f}_i = \frac{1}{h}\int_{t_i}^{t_{i+1}} f(t)dt,$$

$$\overline{x}_i = \frac{1}{h}\int_{t_i}^{t_{i+1}} x(t)dt, \tag{2.9}$$

$$\overline{k}_{ij} = \frac{1}{h^2}\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}} k(t,s)dtds. \tag{2.10}$$

By taking

$$\overline{X} = [\overline{x}_0,\cdots,\overline{x}_{n-1}]^T,$$

$$\overline{F} = [\overline{f}_0,\cdots,\overline{f}_{n-1}]^T,$$

$$\overline{K} = [\overline{k}_{ij}], \quad i,j = 0,\cdots,n-1,$$

Eq. (2.8) is discretized as:

$$(I - h\overline{K})\overline{X} = \overline{F}. \tag{2.11}$$

Definition 2.2 *If $\overline{X} = [\overline{x}_0,\cdots,\overline{x}_{n-1}]^T$ be a solution of (2.11), then*

$$\overline{x}_n(t) := \sum_{i=0}^{n-1} \overline{x}_j \varphi_j(t), \tag{2.12}$$

will be called a least squares approximate solution of (2.8) in the basis $\{\varphi_i\}_{i=0}^{n-1}$.

Remark 2.1 For each $i,j \in \{0,\cdots,n-1\}$ there exist constants ξ_i, η_j and α_i such that

$$\overline{x}_i = x(\alpha_i), \quad t_i < \alpha_i < t_{i+1},$$

$$\overline{k}_{ij} = k(\xi_i,\eta_j), t_i < \xi_i < t_{i+1}, t_j < \eta_j < t_{j+1}.$$

Proof: By applying the mean value theorem for integrals in equations (2.9) and (2.10) the proof is obvious.

2.3 New approach based on BPFs for numerical solution of FIE

Consider the second kind FIE in (2.8). For each $t \in [0,1]$, let $I_{n,t}(.)$ be a non-least squares approximation of $k(t,.)x(.)$ in the basis $\{\varphi_j\}_{j=0}^{n-1}$, i.e.,

$$k(t,s)x(s) \simeq I_{n,t}(s), \quad t \in [0,1]. \tag{2.13}$$

Then by definition 2.1, we have

$$I_{n,t}(s) = \sum_{j=0}^{n-1} k(t,t_j)x(t_j)\varphi_j(s), \quad t \in [0,1]. \tag{2.14}$$

From (2.13) the approximate equation of (2.8) is concluded as follows:

$$x(t) - \int_0^1 I_{n,t}(s)ds \simeq f(t), \quad t \in [0,1].$$

Since $\int_0^1 \varphi_j(s)ds = h,\ 0 \le j \le n-1$, we have

$$\int_0^1 I_{n,t}(s)ds = h\sum_{j=0}^{n-1} k(t,t_j)x(t_j), \quad t \in [0,1], \tag{2.15}$$

thus

$$x(t) - h\sum_{j=0}^{n-1} k(t,t_j)x(t_j) \simeq f(t), \quad t \in [0,1]. \tag{2.16}$$

Replacing t by t_i in (2.16) (for $i = 0,\cdots,n-1$), yields

$$x(t_i) - h\sum_{j=0}^{n-1} k(t_i,t_j)x(t_j) = f(t_i), \tag{2.17}$$

which can be written as

$$(I - hK)X = F, \tag{2.18}$$

where

$$X = [x_0,\cdots,x_{n-1}]^T,\ F = [f_0,\cdots,n-1]^T,$$

$$K = [k_{ij}],\ x_i = x(t_i),\ f_i = f(t_i),$$

$$k_{ij} = k(t_i,t_j), \quad i,j = 0,\cdots,n-1,$$

Definition 2.3 *If the system (2.18) is solvable, then*

$$x_n(t) := f(t) + h\sum_{j=0}^{n-1} k(t,t_j)x(t_j), \tag{2.19}$$

and

$$\widetilde{x}_n(t) := \sum_{j=0}^{n-1} x(t_j)\varphi_j(t), \tag{2.20}$$

are approximate solutions of (2.8). We define $\widetilde{x}_n$ as a non-least squares approximate solution of second kind FIE in the basis $\{\varphi_j\}_{j=0}^{n-1}$.

About the approximate solution $x_n(t)$ defined by (2.19), the proof of the following lemma is obvious.

Lemma 2.4 *Let $\hat{x}_n(t)$ be approximate solution of (2.8) by Nyström method with the left-side quadrature rule. Then*

$$x_n(t) \equiv \hat{x}_n(t).$$

Since the convergence of Nyström method is investigated in [12], the following theorem is concluded from [12].

Theorem 2.1 *Let $x_n(t)$ and $x(t)$ be approximate(by (2.19)) and exact solutions of (2.8) respectively. Then*

$$x_n(t) \longrightarrow x(t).$$

Moreover, for sufficiently large n the linear system (2.18) has a unique solution, i.e., $(I - hK)$ is a nonsingular matrix.

3 Convergence

In this section, the approximate operator of the integral operator is defined and some preliminary lemmas are proved.

Lemma 3.1 *Let $k \in C^p([0,1] \times [0,1])$, $f \in C^q[0,1]$ and Eq. (2.8) has the unique solution x, then $x \in C^r[0,1]$, with $r = \min(p,q)$.*

Proof: See [7].

Note: We also denote the non-least squares approximation of $k(t_i,.)x(.)$ in the basis $\{\varphi_j\}_{j=0}^{n-1}$ by $I_{n,i}(.)$ for each $0 \leq i \leq n-1$, i.e.,

$$k(t_i,s)x(s) \simeq I_{n,i}(s), \tag{3.21}$$

and from definition 2.1, we get (for $i = 0, \cdots, n-1$)

$$I_{n,i}(s) = \sum_{j=0}^{n-1} k(t_i,t_j)x(t_j)\varphi_j(s). \tag{3.22}$$

Since $\int_0^1 \varphi_i(s)ds = h$, for each $0 \leq i \leq n-1$, we have (for $i = 0, \cdots, n-1$)

$$\int_0^1 I_{n,i}(s)ds = h\sum_{j=0}^{n-1} k(t_i,t_j)x(t_j). \tag{3.23}$$

Lemma 3.2 *If $t \in [t_i, t_{i+1})$, then for each $\varepsilon > 0$ and sufficiently large n, we have*

$$\left| \int_0^1 I_{n,t}(s)ds - \int_0^1 I_{n,i}(s)ds \right| < \varepsilon. \tag{3.24}$$

Proof: Let $\varepsilon > 0$ be arbitrary, from (2.15) and (3.23), one gets

$$\left| \int_0^1 I_{n,t}(s)ds - \int_0^1 I_{n,i}(s)ds \right| \leq$$

$$h\sum_{j=0}^{n-1} \mid k(t,t_j) - k(t_i,t_j) \mid\mid x(t_j) \mid,$$

since $\{t_i\}_{i=0}^n$ are equidistant points in $[0,1]$ and k is a continuous function, for sufficiently large n, we conclude

$$\mid k(t,t_j) - k(t_i,t_j) \mid < \frac{\varepsilon}{\|x\|+1}.$$

Finally, boundedness of x, and $h = \frac{1}{n}$ complete the proof.

Lemma 3.3 *Let x_n and $\widetilde{x}_n$ be approximate solutions of (2.8) defined in definition 2.3. Then for arbitrary $\varepsilon > 0$ and sufficiently large n, we have*

$$\mid x_n(t) - \widetilde{x}_n(t) \mid < \varepsilon, \quad \forall t \in [0,1].$$

Proof: Let $t \in [t_i, t_{i+1})$. Then by lemma 2.3 we have

$$\widetilde{x}_n(t) = x(t_i),$$

by combining this equation with (2.17), one gets

$$\widetilde{x}_n(t) = f(t_i) + h\sum_{j=0}^{n-1} k(t_i,t_j)x(t_j), \tag{3.25}$$

this equation with (3.23) imply

$$\widetilde{x}_n(t) = f(t_i) + \int_0^1 I_{n,i}(s)ds.$$

Also from (2.15) and (2.19) we have

$$x_n(t) = f(t) + \int_0^1 I_{n,t}(s)ds,$$

thus

$$| x_n(t) - \widetilde{x}_n(t) | \leq | f(t) - f(t_i) | +$$
$$\Big| \int_0^1 I_{n,t}(s)ds - \int_0^1 I_{n,i}(s)ds \Big|.$$

Since f is a continuous function, for sufficiently large n we have

$$| f(t) - f(t_i) | < \frac{\varepsilon}{2},$$

also for sufficiently large n, by lemma 3.2,

$$\Big| \int_0^1 I_{n,t}(s)ds - \int_0^1 I_{n,i}(s)ds \Big| < \frac{\varepsilon}{2}.$$

Hence, the proof is completed.

Lemma 3.4 *Let $\widetilde{x}_n$ and $\overline{x}_n$ be respectively approximate solutions defined by (2.20) and (2.12), then for $\varepsilon > 0$ arbitrary and sufficiently large n, we have*

$$| \widetilde{x}_n(t) - \overline{x}_n(t) | < \varepsilon, \quad \forall t \in [0, 1].$$

Proof: Let $t \in [t_i, t_{i+1})$. Then by lemma 2.3 we get

$$\widetilde{x}_n(t) = x(t_i), \quad \overline{x}_n(t) = \overline{x}_i,$$

and the remark 2.1 suggests existing an $\alpha_i \in (t_i, t_{i+1})$ such that

$$\overline{x}_i = x(\alpha_i).$$

Since $x(t)$ is a continuous function (lemma 3.1), for sufficiently large n, we have

$$| \widetilde{x}_n(t) - \overline{x}_n(t) | = | x(t_i) - x(\alpha_i) | < \varepsilon,$$

and this completes the proof.

Corollary 3.1 *The non-least squares approximate solution $\widetilde{x}_n(t)$, defined by (2.20), converges to the exact solution of Eq. (2.8).*

Proof: Let x be the exact solution of Eq. (2.8). Then

$$\|\widetilde{x}_n - x\| \leq \|\widetilde{x}_n - x_n\| + \|x_n - x\|,$$

for sufficiently large n and arbitrary $\varepsilon > 0$, from lemmas 3.3 and theorem 2.1, one gets

$$\|\widetilde{x}_n - x_n\| < \frac{\varepsilon}{2}, \ \|x_n - x\| < \frac{\varepsilon}{2},$$

which complete the proof.

Finally, we are going to establish the statements i and ii from Introduction for the given method in [8] (section 2.2).

Theorem 3.1 *The set of all invertible matrices is an open set.*

Proof: See [11].

Lemma 3.5 *For sufficiently large n, the linear system (2.11) is nonsingular i.e., for sufficiently large n, $I - h\overline{K}$ is an invertible matrix.*

Proof: By definition of infinity norm of matrices, we have

$$\|(I - hK) - (I - h\overline{K})\|_\infty = h\|\overline{K} - K\|_\infty =$$
$$\frac{1}{n} \max_{0 \leq i \leq n-1} \sum_{j=0}^{n-1} | \overline{k}_{ij} - k_{ij} |,$$

also by remark (2.1), we have

$$| \overline{k}_{ij} - k_{ij} | = | k(\xi_i, \eta_j) - k(t_i, t_j) |,$$

thus

$$\|(I - hK) - (I - h\overline{K})\|_\infty = \tag{3.26}$$
$$\frac{1}{n} \max_{0 \leq i \leq n-1} \sum_{j=0}^{n-1} | k(\xi_i, \eta_j) - k(t_i, t_j) |.$$

For sufficiently large n, it follows from the uniform continuity of k that

$$|k(\xi_i, \eta_j) - k(t_i, t_j)| < \varepsilon, \quad j = 0, \cdots, n-1. \tag{3.27}$$

From (3.26) and (3.27), we conclude

$$\|(I - hK) - (I - h\overline{K})\|_\infty < \varepsilon. \tag{3.28}$$

Since $I - hK$ is an invertible matrix, for sufficiently large n, the proof is completed by theorem 3.1.

Corollary 3.2 *The least squares approximate solution $\overline{x}_n(t)$, defined by (2.12), converges to the exact solution of Eq. (2.8).*

Proof: Let x be the exact solution of Eq. (3.27). Then

$$\|x-\overline{x}_n\| \leq \|x-\widetilde{x}_n\| + \|\widetilde{x}_n-\overline{x}_n\|,$$

and for sufficiently large n, and arbitrary $\varepsilon > 0$, from corollary 3.1 and lemma 3.4, one gets

$$\|x-\widetilde{x}_n\| < \frac{\varepsilon}{2}, \quad \|\widetilde{x}_n-\overline{x}_n\| < \frac{\varepsilon}{2},$$

which complete the proof.

4 Conclusion

The advantage of the presented method is that it is necessary to evaluate n^2+n integrals for setting up the system (2.11), while the elements of the matrices $(I-hK)$ and F in (2.18) are obtained by simple replacements, that is, without computing the related integrals, and so the computational cost of the new approach is less than the presented method in [8].

It should be mentioned that the theory of the new method is the means by which the theoretical aspect of the given method in [8] is established. In this process, the open set of invertible matrices has an important role. The method of this article can be applied for other types of basis functions, and we will extend it for wavelet bases in our future works.

References

[1] K. E. Atkinson, *The Numerical Solution of Integral equations of the Second Kind*, Combridge Uni. Press, Combridge (1997).

[2] E. Babolian, K. Maleknejad, M. Mordad, B. Rahimi, *A numerical method to solve Fredholm-Volterra integral equations in two dimensional spaces using Block Pulse Functions and operational matrix*, Journal of Computational and Applied Mathematics 235 (2011) 3965-3971.

[3] E. Babolian, Z. Masouri, *Direct method to solve Volterra integral equation of the first kind using operationnal matrix with block-pulse functions*, Journal of Computational and Applied Mathematics 220 (2008) 51-57.

[4] E. Babolian, A. Salimi Shamloo, *Numerical solution of Volterra integral and integro-differential equations of convolution type by using operationl matrices of piecewise constant orthogonal functions*, Journal of Computational and Applied Mathematics 214 (2008) 495-508.

[5] C. A. Balanis, *Advanced Engineering Electromagnetics*, Wiley, New York (1989).

[6] R. Bancroft, *Understanding Electromagnetic Scattering Using the Moment Method. Artech House*, London (1996).

[7] L. M. Delves, J. L. Mohamed, *Computational Methods for Integral Equations*, Artech House, London (1985).

[8] S. Hatamzadeh-Varmazyar, Z. Masouri, *Numerical method for analysis of one and two dimensional electromagnetic scattering based on using linear Fredholm integral equation models*, Journal of Mathematical and Computer Modelling 54 (2011) 2199-2210.

[9] S. Hatamzadeh-Varmazyar, M. Naser-Moghadasi, Z. Masouri, *A moment method simulation of electromagnetic scattering from conducting bodies*, Progress in Electromagnetics Research PIER, 81 (2008) 99-119.

[10] S. Hatamzadeh-Varmazyar, M. Naser-Moghadasi, E. Babolian, Z. Masouri, *Calculating the radar cross section of the resistive targets using the Haar wavelets*, Progress in Electromagnetics Research PIER 83 (2008) 55-80.

[11] David R. Kincaid, E. Ward Cheney, *Numerical Analysis: Mathematics of scientific computing*, American Mathematical Society (2002).

[12] R. Kress, *Linear Integral Equations*, Springer-Verlag, New York (1999).

[13] R. Kress, *Numerical Analysis*, Springer-Verlag, New York (1998).

[14] Z. Masouri, E. Babolian, S. Hatamzadeh-Varmazyar, *An expansion-iterative method for numerically solving Volterra integral equation of the first kind*, Computers and

Mathematics with Applications 59 (2010) 1491-1499.

[15] C. P. Rao, *Piecewise constant orthogonal functions and their application to systems and control*, Springer, Berlin (1983).

[16] P. Sannuti, *Analysis and synthesis of dynamic systems via block-pulse functions*, Proc. IEE. 124 (1977) 569-571.

Ali Abdollahi is an assistant professor at the Department of Mathematics, Maragheh Branch, Islamic Azad University, Maragheh, Iran. Interested in numerical analysis and integral Equations. He is a Reviewer in the Journals of Iranian Journal of Fuzzy systems, and others.

Esmail Babolian is Professor at the Department of Mathematics, College of Basic Science, Tehran Science and Research Branch, Islamic Azad University, Tehran, Iran. Interested in numerical solution of functional Equations, numerical linear algebra and mathematical education.