journal of applied mechanics

briefs notes

On the recommendation of the Executive Committee of the ASME Applied Mechanics Division, it was decided to initiate a section devoted to Brief Notes on technical matters in mechanics. These notes must not be longer than 1500 words (about 5 double-spaced typewritten pages, plus 1 figure or equivalent) and will be subject to the usual review procedures prior to publication. After approval such Notes will be published as soon as possible. The Notes should be submitted to the Technical Editors of the Journal of Applied Mechanics.

A Refined Laminated Plate Theory

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Introduction

Elasticity solutions on layered composite plates [1, 2][2] indicate the inadequacy of the classical laminated plate theory [3, 4], in which the Kirchhoff-Love kinematic assumptions are adopted and the effects of transverse shear deformation are neglected. Accurate modeling of the laminated plate behavior is possible only if the assumptions of nondeformable normals is abandoned as was done in the derivation of a laminated cylindrical shell theory [5]. More recently, laminated plate theories are derived using separate assumptions for the displacements of each layer [6]. In this Note, a refined laminated plate theory similar to one of those in reference [6] is derived. In the present theory, the interlamina shear stresses are introduced as additional unknown variables. Numerical results show excellent agreement with elasticity solutions.

Laminated Plate Equations

For each layer of a plate, the following displacement field is assumed:

$$\begin{aligned} u_i &= u_i^0(x, y) + z_i[\psi_x(x, y)]_i \\ v_i &= v_i^0(x, y) + z_i[\psi_y(x, y)]_i \\ w_i &= w(x, y) \end{aligned} \qquad (1)$$

where z_i is measured from the middle surface of the ith layer. The continuity conditions are, with h_i representing the thickness of the ith layer,

$$\begin{aligned} f_i &= u_{i+1}^0 - u_i^0 - [h_{i+1}(\psi_x)_{i+1} + h_i(\psi_x)_i]/2 = 0 \\ g_i &= v_{i+1}^0 - v_i^0 - [h_{i+1}(\psi_y)_{i+1} + h_i(\psi_y)_i]/2 = 0 \end{aligned} \qquad (2)$$

Introducing the interlayer shear stresses $(\lambda_x)_i$ and $(\lambda_y)_i$ as Lagrange multipliers, the governing equations can be obtained by minimizing the following modified potential energy functional:

$$\pi' = \sum_n \iiint \frac{1}{2} c_{ij}\epsilon_i\epsilon_j dV - \iint pw dA + \sum_{n-1} [(\lambda_x)_i f_i + (\lambda_y)_i g_i] \qquad (3)$$

where c_{ij} is the elastic modulus matrix and p is the normal load.

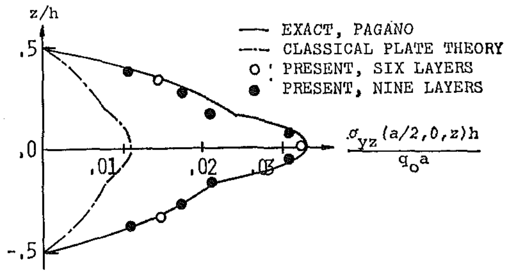

Fig. 1 Dimensionless transverse shear stress of a three-layer rectangular plate ($a/h = 4$)

Both the original c_{ij} and the reduced modulus Q_{ij} [6] are used in this study. The resulting equations include, in addition to equation (2), the following set of equations:

$$\begin{aligned} &[A_{11}u^0_{,xx} + 2A_{16}u^0_{,xy} + A_{66}u^0_{,yy} + A_{16}v^0_{,xx} + (A_{12} + A_{66})v^0_{,xy} \\ &\qquad + A_{26}v^0_{,yy}]_i + (\lambda_x)_i - (\lambda_x)_{i-1} = 0 \\ &[A_{16}u^0_{,xx} + (A_{12} + A_{66})u^0_{,xy} + A_{26}u^0_{,yy} + A_{66}v^0_{,xx} + 2A_{26}v^0_{,xy} \\ &\qquad + A_{22}v^0_{,yy}]_i + (\lambda_y)_i - (\lambda_y)_{i-1} = 0 \\ &[D_{11}\psi_{x,xx} + 2D_{16}\psi_{x,xy} + D_{66}\psi_{x,yy} + D_{16}\psi_{y,xx} \\ &\qquad + (D_{12} + D_{66})\psi_{y,xy} + D_{26}\psi_{y,yy} - A_{55}(\psi_x + w_{,x}) \\ &\qquad - A_{45}(\psi_y + w_{,y})]_i + [(\lambda_x)_i + (\lambda_x)_{i-1}]\cdot h_i/2 = 0 \qquad (4) \\ &[D_{16}\psi_{x,xx} + (D_{12} + D_{66})\psi_{x,xy} + D_{26}\psi_{x,yy} + D_{66}\psi_{y,xx} \\ &\quad + 2D_{26}\psi_{y,xy} + D_{22}\psi_{y,yy} - A_{45}(\psi_x + w_{,x}) - A_{44}(\psi_y + w_{,y})]_i \\ &\qquad + [(\lambda_y)_i + (\lambda_y)_{i-1}]\cdot h_i/2 = 0 \qquad i = 1, 2, \ldots, n \\ &\sum_{i=1}^{n} [A_{55}(\psi_{x,x} + w_{,xx}) + A_{45}(\psi_{x,y} + \psi_{y,x} + 2w_{,xy}) \\ &\qquad + A_{44}(\psi_{y,y} + w_{,yy}) + A_{44}(\psi_{y,y} + w_{,yy})] + p = 0 \end{aligned}$$

with

$$(\lambda_x)_0 = (\lambda_y)_0 = (\lambda_x)_n = (\lambda_y)_n = 0$$

and

$$[A_{jk}, D_{jk}]_i = [c_{jk}(h, h^3/12)]_i.$$

If λ_x and λ_y are eliminated from equation (4), then the present theory reduces to one of those in reference [6]. By retaining λ_x and λ_y, however, not only does one obtain a more convenient recursive form, but also the interlamina shear stresses are now among the direct solutions of equations (2) and (4). Thus they need not be calculated indirectly from the displacement function (1), which actually lead to discontinuous transverse shear stresses across the interfaces.

Numerical Example

A three-layer crossply (0 deg/90 deg/0 deg) laminated rectangular plate ($b = 3a$) simply supported along all edges and sub-

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[2] Numbers in brackets designate References at end of Note.

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Table 1 Maximum transverse deflection of a three-layer rectangular plate, $\bar{w} = 100E_T wh^3/(q_0 a^4)$

No. of layers	Case	a/h 4	10	20	50	100
3	a	2.736	0.898	0.602	0.517	0.505
	b	2.741	0.901	0.605	0.520	0.508
6	a	2.797	0.912	0.606	0.518	0.505
	b	2.802	0.915	0.609	0.520	0.508
9	a	2.820	0.915	0.607	0.518	0.505
	b	2.825	0.918	0.609	0.521	0.508
Exact [2]		2.82	0.919	0.610	0.520	0.508

Note: a: Original modulus, b: reduced modulus.

Table 2 Maximum transverse shear stresses at the middle surface of the plate,* $(\bar{\sigma}_{xz}, \bar{\sigma}_{yz}) = 10h(\sigma_{xz}, \sigma_{yz})/(q_0 \cdot a)$

	Case	a/h 4	10	20	50	100
	a	3.48	4.20	4.34	4.39	4.39
$\bar{\sigma}_{xz}$	b	3.49	4.20	4.34	4.39	4.39
	Exact†	3.51	4.20	4.34	4.39	4.39
	a	0.317	0.153	0.124	0.115	0.114
$\bar{\sigma}_{yz}$	b	0.312	0.147	0.117	0.118	0.107
	Exact†	0.334	0.152	0.119	0.110	0.108

Note: a: Original modulus, b: reduced modulus.
* From the 6-layer division.
† Reference [2].

jected to the loading $p = q_0 \sin(\pi x/a) \sin(\pi y/b)$ is analyzed. The material properties are: $E_L = 25 \times 10^6$ psi; $E_T = 10^6$ psi; $G_{LT} = 0.5 \times 10^6$ psi; $G_{TT} = 0.2 \times 10^6$ psi; and $\nu_{LT} = \nu_{TT} = 0.25$. The plate is equally divided into 3, 6, and 9 sublayers. The solutions are

$$[u_i^0, (\psi_x)_i, (\lambda_x)_i] = [A_i, B_i, (S_x)_i] \cos(\pi x/a) \sin(\pi y/b)$$
$$[v_i^0, (\psi_y)_i, (\lambda_y)_i] = [C_i, D_i, (S_y)_i] \sin(\pi x/a) \cos(\pi y/b) \quad (5)$$
$$w = E \sin(\pi x/a) \sin(\pi y/b)$$

The calculated dimensionless transverse deflections and transverse shear stresses are listed in Tables 1 and 2. The transverse shear stresses at the midpoint of an edge is depicted in Fig. 1. It is seen that the results are in very good agreement with those of the exact theory [2].

Discussion and Conclusion

The present theory represents a more precise approach to the problem of thick laminated plates. The governing equations are recursive and thus numerical techniques such as the finite-difference method can be applied easily. Also, a finite-element model [7] can be derived by using separate assumptions for each layer. Using reduced moduli in the present theory yields slightly different results. Some improvements, though negligible in the example given, are observed for thin plates (large a/h). This seems to confirm the findings of reference [8] in which the two approaches are compared on plate theories with nondeformable normal assumptions.

Acknowledgment

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On the Natural Frequencies of Transverse Vibrations of an Elastic Plate (With In-Plane Forces) Resting on a Winkler Foundation

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The present Note describes a method of obtaining natural frequencies of transverse vibrations of an elastic plate, with in-plane time-invariant forces, resting on a constant modulus Winkler foundation. The method used here consists of transforming the system equation into a 2×2 matrix equation. This matrix equation is transformed into Banach space through double finite Fourier sine transform and then natural frequencies of vibration are deduced.

The equation of motion of free transverse vibrations of an elastic plate resting on a Winkler foundation and subjected to the action of uniformly distributed and constant in-plane force, Q, may be given by

$$D\nabla^4 w(x, y, t) - Q\nabla^2 w(x, y, t) + \rho h \frac{\partial^2 w(x, y, t)}{\partial t^2} + kw(x, y, t) = 0 \quad (1)$$

where

$D = \dfrac{Eh^3}{12(1 - \nu^2)}$ = plate flexural rigidity

Q = forces acting along the edges of the plate, having dimensions force/length

ρ = plate density

h = plate thickness

k = foundation modulus, having dimensions force/length^3

$w(x, y, t)$ = transverse deflection of the middle surface of the elastic plate

$\nabla^4 = \nabla^2 \cdot \nabla^2$

$\nabla^2 = \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)$

For the case when the plate is executing simple harmonic motion, the transverse deflection $w(x, y, t)$ can be expressed as

$$w(x, y, t) = u(x, y)e^{i\omega_{mn}t} \quad (2)$$

Since equation (1) is defined in a linear vector space having only linear differential operators we get the following with the help of equation (2):

$$D\nabla^2(\nabla^2 - Q/D)u(x, y) = (\rho h\omega_{mn}^2 - k)u(x, y) \quad (3)$$

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