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INVERSE PROBLEM FOR LAGRANGIAN DYNAMICS FOR MULTI-DEGREE-OF-FREEDOM SYSTEMS WITH LINEAR DAMPING

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ABSTRACT

This paper deals with the inverse problem for Lagrangian dynamics for linear multi-degree-of-freedom systems. New results for linearly damped systems are obtained using extensions of results for single-degree-of-freedom systems. First, for a two-degree-of-freedom linear system with linear damping, the conditions for the existence of a Lagrangian are explicitly obtained by solving the Helmholtz conditions. Next, since the Helmholtz conditions are near-impossible to solve for general n -degree-of-freedom systems, a new simple procedure that does not require the use of the Helmholtz conditions and that is easily extended to n -degree-of-freedom linear systems, is developed. The emphasis is on obtaining the Lagrangians for these multi-degree-of-freedom systems in a simple manner, using insights obtained from our understanding of the inverse problem for single- and two-degree-of-freedom systems. Specifically we include systems that commonly arise in linear vibration theory with positive definite mass matrices, and symmetric stiffness and damping matrices. This method yields several new Lagrangians for linear multi-degree-of-freedom systems. Finally, conservation laws for these damped multi-degree-of-freedom systems are found using the Lagrangians obtained.

INTRODUCTION

The inverse problem for Lagrangian dynamics--also known as the inverse problem of the calculus of variations-- is to obtain, for a system described by a given set of differential equations, a Lagrangian function such that the corresponding Euler-Lagrange equations obtained using the calculus of variations yield the given set of equations that describe the system. This problem has attracted many researchers in

various fields of study for its usefulness. Bolza [1] gave a general procedure for finding a Lagrangian for a single-degree-of-freedom dissipative system. This was followed by Leitmann [2] who provided some examples of non-potential forces and the corresponding Lagrangians for which a variational principle exists. This method was extended by Udwadia et al. [3] who used a more systematic derivation to obtain several classes of non-potential forces which could be used to obtain the equations of motion via variational calculus. Recently, the semi-inverse method [4] has been considered due to its simplicity and applicability to many cases. However, Refs. [1]-[4] consider only single-degree-of-freedom (SDOF) systems, and the analysis of this case is relatively easy because Darboux [5] proved in the nineteenth century that a Lagrangian can be always found for the inverse problem for such SDOF systems. For multi-degree-of-freedom (MDOF) systems the configuration variables are coupled with one another and this makes it difficult to solve the inverse problem. The general conditions for the existence of Lagrangians were apparently first obtained by Helmholtz [6,7] and are usually referred to as the Helmholtz conditions. Later, Douglas [8] analyzed in great detail the case of two degrees of freedom and obtained the necessary and sufficient conditions for their existence without utilizing these conditions. Using Douglas' results Hojman and Ramos [9] proposed a simpler method to determine the existence of a Lagrangian for two-dimensional problems in which the potential function does not explicitly contain the generalized velocities. Mestdag et al. [10] derived the conditions under which there exists a Lagrangian and a dissipation function on the right hand side of the more general form of the Euler-Lagrange equation. They also provided some non-conservative systems to which their approach can be applied.

In the present paper, the findings obtained in Refs. [2] and [3] are extended to dissipative, constant coefficient, linear

MDOF systems. The emphasis is on obtaining the Lagrangians for these MDOF systems in a simple manner, using insights obtained from our understanding of the inverse problem for SDOF and 2-DOF systems. It is shown that the solution to a gyroscopically damped linear system is easily found as a special case of the linearly damped case. Lagrangians for special linearly damped MDOF systems with symmetric stiffness and damping matrices are also obtained along with the corresponding Jacobi integrals, which are conserved over time.

LAGRANGIANS FOR DAMPED LINEAR TWO-DEGREE-OF-FREEDOM SYSTEMS

We begin with the problem of finding a Lagrangian function for a linear mass-spring-damper system in single degree of freedom whose governing equation of motion is given by

$$m\ddot{x} + 2b\dot{x} + kx = 0, \quad m > 0, b, k \geq 0 \quad (1)$$

where $x(t)$ is a generalized displacement of the mass, the dot denotes the differentiation with respect to time t , and m , b , and k are the mass, damping, and stiffness coefficients, respectively, which are all assumed to be constants. Unfortunately, Eq. (1) cannot be directly derived as the Euler-Lagrange equation from a variational principle because it does not satisfy the Helmholtz conditions [6,7] (see Eqs. (11)-(14)). However, in Refs. [2,3] it is shown that the following Lagrangian function results in Eq. (1):

$$L = e^{\frac{2b}{m}t} \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \right). \quad (2)$$

More precisely, substituting Eq. (2) into the Euler-Lagrange equation of the standard form $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0$ yields

$$e^{\frac{2b}{m}t} [m\ddot{x}(t) + 2b\dot{x}(t) + kx(t)] = 0. \quad (3)$$

Since the exponential factor in Eq. (3) is always positive in time, we can say that Eq. (3) is 'equivalent' to Eq. (1). This exponential factor, however, plays a significant role, because Eq. (3) does satisfy the Helmholtz conditions.

Taking a hint from Eq. (2), we next consider a two-degree-of-freedom mass-spring-damper system using the Lagrangian given by

$$L = e^{\gamma t} \left[\frac{1}{2}(m_1\dot{x}_1^2 - k_1x_1^2) + \frac{1}{2}(m_2\dot{x}_2^2 - k_2x_2^2) + b_1\dot{x}_1x_2 + b_2x_1\dot{x}_2 + dx_1x_2 \right], \quad (4)$$

where m_i , k_i , b_i , $i=1,2$, and γ and d are constants. Then, the corresponding Euler-Lagrange equations of motion are given by

$$m_1\ddot{x}_1 + \gamma m_1\dot{x}_1 + (b_1 - b_2)\dot{x}_2 + k_1x_1 + (b_1\gamma - d)x_2 = 0, \quad (5a)$$

$$m_2\ddot{x}_2 + \gamma m_2\dot{x}_2 + (b_2 - b_1)\dot{x}_1 + k_2x_2 + (b_2\gamma - d)x_1 = 0. \quad (5b)$$

Depending on the choice of the constants, Eqs. (5) can represent various systems. For example, if we choose $m_1 = m$, $m_2 = 2m$, $\gamma = 2$, $b_1 = b_2 = 0$, $k_1 = k_2 = 2k$, and $d = k$, Eq. (4) becomes

$$L = e^{2t} \left[\frac{1}{2}(m\dot{x}_1^2 - 2kx_1^2) + (m\dot{x}_2^2 - kx_2^2) + kx_1x_2 \right], \quad (6)$$

and the corresponding equations of motion become

$$m\ddot{x}_1 + 2m\dot{x}_1 + 2kx_1 - kx_2 = 0, \quad (7a)$$

$$2m\ddot{x}_2 + 4m\dot{x}_2 + 2kx_2 - kx_1 = 0, \quad (7b)$$

which describe a classically damped 2-DOF system. In fact, Eqs. (7) describe the mechanical system shown in Fig. 1.

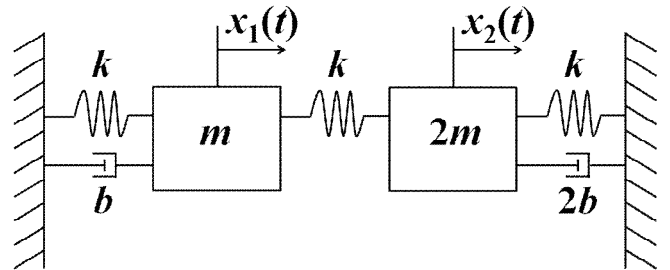


Fig. 1: Linear 2-DOF mass-spring-damper system with $b = 2m$

In order to get a more systematic approach to the inverse problem for a constant coefficient linear 2-DOF system we consider the following equations of motion:

$$f^1 = \ddot{x}_1 + a_1\dot{x}_1 + b_1\dot{x}_2 + c_1x_1 + d_1x_2 = 0, \quad (8a)$$

$$f^2 = \ddot{x}_2 + a_2\dot{x}_1 + b_2\dot{x}_2 + c_2x_1 + d_2x_2 = 0, \quad (8b)$$

where all coefficients are constants and we have divided each equation by the corresponding masses m_1 and m_2 . More generally, we consider the following set of equations:

$$\alpha_{11}f^1 = \alpha_{11}(\ddot{x}_1 + a_1\dot{x}_1 + b_1\dot{x}_2 + c_1x_1 + d_1x_2) = 0, \quad (9a)$$

$$\alpha_{22}f^2 = \alpha_{22}(\ddot{x}_2 + a_2\dot{x}_1 + b_2\dot{x}_2 + c_2x_1 + d_2x_2) = 0, \quad (9b)$$

where α_{ii} ($i=1,2$) are non-zero functions of t , x_1 , x_2 , \dot{x}_1 , and \dot{x}_2 , namely $\alpha_{11} = \alpha_{11}(t, x_1, x_2, \dot{x}_1, \dot{x}_2)$ and $\alpha_{22} = \alpha_{22}(t, x_1, x_2, \dot{x}_1, \dot{x}_2)$. Next, we define functions β_i ($i=1,2$) by

$$\beta_1 := \alpha_{11}(a_1\dot{x}_1 + b_1\dot{x}_2 + c_1x_1 + d_1x_2), \quad (10a)$$

$$\beta_2 := \alpha_{22}(a_2\dot{x}_1 + b_2\dot{x}_2 + c_2x_1 + d_2x_2). \quad (10b)$$

Now let us consider the n differential equations $\alpha_{ij}(t, \mathbf{q}, \dot{\mathbf{q}})\ddot{q}^j + \beta_i(t, \mathbf{q}, \dot{\mathbf{q}}) = 0$ ($i, j = 1, 2, \dots, n$), where $\mathbf{q} = [q^1 \ q^2 \ \dots \ q^n]^T$ is a generalized displacement n -vector that describes the motion of a mechanical system in an n -dimensional configuration space. The superscript "T" is used to denote the transpose of a vector (or a matrix), and the summation convention is used for repeated indices. The question of whether such a system can be obtained from a suitable Lagrangian, $L(t, \mathbf{q}, \dot{\mathbf{q}})$, through the use of the Euler-

Lagrange equations $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0$ appears to have been

first investigated by Helmholtz [6,7]. The necessary and sufficient conditions for the so-called *ordered direct analytic representations* are [7]

$$\alpha_{ij} = \alpha_{ji}, \quad (11)$$

$$\frac{\partial \alpha_{ij}}{\partial \dot{q}^k} = \frac{\partial \alpha_{ik}}{\partial \dot{q}^j}, \quad (12)$$

$$\frac{\partial \beta_i}{\partial \dot{q}^j} + \frac{\partial \beta_j}{\partial \dot{q}^i} = 2\left(\frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k}\right)\alpha_{ij}, \quad (13)$$

$$\frac{\partial \beta_i}{\partial q^j} - \frac{\partial \beta_j}{\partial q^i} = \frac{1}{2}\left(\frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k}\right)\left(\frac{\partial \beta_i}{\partial \dot{q}^j} - \frac{\partial \beta_j}{\partial \dot{q}^i}\right) \quad (14)$$

where the summation convention is applied for repeated indices. Eqs. (11)-(14) are a set of partial differential equations that need to be satisfied by the $2n+1$ independent variables t , \mathbf{q} , and $\dot{\mathbf{q}}$ everywhere in R^{2n+1} . Throughout this paper we shall be dealing with Lagrangians that provide the so-called ordered direct analytic representations of the equations of motion [7].

After solving the Helmholtz conditions Eqs. (11)-(14), we have only the three possible cases summarized in Table 1. Also, corresponding to each case it includes one Lagrangian, which shall be obtained later. It is noted that there are many other possible Lagrangians which are different from the ones given in Table 1.

Case	Conditions	Lagrangian
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I	$\alpha_{11} = \gamma_1 e^{a_1 t}, \quad \alpha_{22} = -\frac{b_1}{a_2} \gamma_1 e^{a_1 t},$ $a_1 = b_2, \quad a_2 d_1 + b_1 c_2 = a_1 a_2 b_1,$ $a_2 \neq 0, \quad b_1 \neq 0, \quad \gamma_1 \neq 0$	$L = e^{a_1 t} \left(\begin{aligned} &\frac{1}{2} \dot{x}_1^2 - \frac{b_1}{2a_2} \dot{x}_2^2 \\ & - \frac{c_1}{2} x_1^2 + \frac{b_1 c_2}{a_2} x_1 x_2 \\ & + \frac{b_1 d_2}{2a_2} x_2^2 + b_1 \dot{x}_1 x_2 \end{aligned} \right)$
II	$\alpha_{11} = \gamma_1 e^{a_1 t}, \quad \alpha_{22} = \frac{d_1}{c_2} \gamma_1 e^{a_1 t},$ $a_2 = 0, \quad b_1 = 0, \quad a_1 = b_2,$ $c_2 \neq 0, \quad d_1 \neq 0, \quad \gamma_1 \neq 0$	$L = e^{a_1 t} \left(\begin{aligned} &\frac{1}{2} \dot{x}_1^2 + \frac{d_1}{2c_2} \dot{x}_2^2 - \frac{c_1}{2} x_1^2 \\ & - d_1 x_1 x_2 - \frac{d_1 d_2}{2c_2} x_2^2 \end{aligned} \right)$
III	$\alpha_{11} = \gamma_1 e^{a_1 t}, \quad \alpha_{22} = \gamma_2 e^{b_2 t},$ $a_2 = 0, \quad b_1 = 0,$ $c_2 = 0, \quad d_1 = 0, \quad \gamma_1 \neq 0, \quad \gamma_2 \neq 0$ <p style="text-align: center;">(uncoupled)</p>	$L = e^{a_1 t} \left(\begin{aligned} &\frac{1}{2} \dot{x}_1^2 - \frac{1}{2} c_1 x_1^2 \end{aligned} \right)$ $+ e^{b_2 t} \left(\begin{aligned} &\frac{1}{2} \dot{x}_2^2 - \frac{1}{2} d_2 x_2^2 \end{aligned} \right)$

Table 1 Three cases when a Lagrangian function of the system described by Eqs. (8) exists and a corresponding Lagrangian for each case

Up to now we have derived the necessary and sufficient conditions for which there exists a Lagrangian function of the 2-DOF system given by Eqs. (8), and obtained three cases summarized in Table 1. We next address the question of finding a Lagrangian for each of these cases. Since a Lagrangian for Case III is already known in Refs. [2,3] (also in Eq. (2)), we focus on obtaining a Lagrangian function for Cases I and II.

For Case I, first, from Table 1, we know that the equations of motion are given by

$$e^{a_1 t} \left(\ddot{x}_1 + a_1 \dot{x}_1 + b_1 \dot{x}_2 + c_1 x_1 + \left(a_1 b_1 - \frac{b_1 c_2}{a_2} \right) x_2 \right) = 0, \quad (15a)$$

$$-\frac{b_1}{a_2} e^{a_1 t} (\ddot{x}_2 + a_2 \dot{x}_1 + a_1 \dot{x}_2 + c_2 x_1 + d_2 x_2) = 0, \quad (15b)$$

where $\gamma_1 = 1$ is used. We next search for the conditions under which a Lagrangian $L(t, \mathbf{x}, \dot{\mathbf{x}})$ exists such that the corresponding Euler-Lagrange equations yield Eqs. (15), that is,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_i} \right) - \frac{\partial L}{\partial x_i} = \alpha_{ij}(t, \mathbf{x}, \dot{\mathbf{x}}) \ddot{x}_j + \beta_i(t, \mathbf{x}, \dot{\mathbf{x}}). \quad (16)$$

Expanding the total time derivative, we have the following identities [7]:

$$\frac{\partial^2 L}{\partial \dot{x}_i \partial \dot{x}_j} = \alpha_{ij}, \quad (17a)$$

$$\frac{\partial^2 L}{\partial \dot{x}_i \partial x_j} \dot{x}_j + \frac{\partial^2 L}{\partial \dot{x}_i \partial t} - \frac{\partial L}{\partial x_i} = \beta_i, \quad (17b)$$

where $i, j=1,2$. Knowing α_{ij} and β_i from Eqs. (15), (16), and Table 1, we obtain the following Lagrangian using Eqs. (17):

$$L = e^{a_1 t} \left(\frac{1}{2} \dot{x}_1^2 - \frac{b_1}{2a_2} \dot{x}_2^2 \right) + \dot{x}_1 p(t, x_1, x_2) + \dot{x}_2 q(t, x_1, x_2) + r(t, x_1, x_2), \quad (18)$$

where $p(t, x_1, x_2)$, $q(t, x_1, x_2)$, and $r(t, x_1, x_2)$ are arbitrary functions of their arguments within the requirements that

$$\begin{aligned} \frac{\partial p}{\partial x_2} - \frac{\partial q}{\partial x_1} &= b_1 e^{a_1 t}, \\ \frac{\partial p}{\partial t} - \frac{\partial r}{\partial x_1} &= e^{a_1 t} \left(c_1 x_1 + \left(a_1 b_1 - \frac{b_1 c_2}{a_2} \right) x_2 \right), \\ \frac{\partial r}{\partial x_2} - \frac{\partial q}{\partial t} &= \frac{b_1}{a_2} e^{a_1 t} (c_2 x_1 + d_2 x_2). \end{aligned} \quad (19)$$

For example, if we choose

$$\begin{aligned} p(t, x_1, x_2) &= b_1 x_2 e^{a_1 t}, \\ q(t, x_1, x_2) &= 0, \\ r(t, x_1, x_2) &= e^{a_1 t} \left(-\frac{c_1}{2} x_1^2 + \frac{b_1 c_2}{a_2} x_1 x_2 + \frac{b_1 d_2}{2a_2} x_2^2 \right), \end{aligned} \quad (20)$$

then the Lagrangian in Eq. (18) becomes

$$L = e^{a_1 t} \left(\frac{1}{2} \dot{x}_1^2 - \frac{b_1}{2a_2} \dot{x}_2^2 - \frac{c_1}{2} x_1^2 + \frac{b_1 c_2}{a_2} x_1 x_2 + \frac{b_1 d_2}{2a_2} x_2^2 + b_1 \dot{x}_1 x_2 \right), \quad (21)$$

which is shown in Table 1.

For Case II, following the same procedure shown in the previous Case I, we can again obtain Lagrangian functions and one possible Lagrangian is

$$L = e^{a_1 t} \left(\frac{1}{2} \dot{x}_1^2 + \frac{d_1}{2c_2} \dot{x}_2^2 \right) + \dot{x}_1 u(t, x_1, x_2) + \dot{x}_2 v(t, x_1, x_2) + w(t, x_1, x_2), \quad (22)$$

where $u(t, x_1, x_2)$, $v(t, x_1, x_2)$, and $w(t, x_1, x_2)$ are arbitrary functions of their arguments within the requirements that

$$\frac{\partial u}{\partial x_2} = \frac{\partial v}{\partial x_1},$$

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial w}{\partial x_1} &= e^{a_1 t} (c_1 x_1 + d_1 x_2), \\ \frac{\partial v}{\partial t} - \frac{\partial w}{\partial x_2} &= e^{a_1 t} \left(d_1 x_1 + \frac{d_1 d_2}{c_2} x_2 \right). \end{aligned} \quad (23)$$

For example, if we choose

$$\begin{aligned} u(t, x_1, x_2) &= 0, \\ v(t, x_1, x_2) &= 0, \\ w(t, x_1, x_2) &= -e^{a_1 t} \left(\frac{c_1}{2} x_1^2 + d_1 x_1 x_2 + \frac{d_1 d_2}{2c_2} x_2^2 \right), \end{aligned} \quad (24)$$

then the Lagrangian Eq. (22) becomes

$$L = e^{a_1 t} \left(\frac{1}{2} \dot{x}_1^2 + \frac{d_1}{2c_2} \dot{x}_2^2 - \frac{c_1}{2} x_1^2 - d_1 x_1 x_2 - \frac{d_1 d_2}{2c_2} x_2^2 \right), \quad (25)$$

which is listed in Table 1. Comparing Eqs. (7), which are the equations of motion of the system shown in Fig. 1, with Case II, we see that

$$\begin{aligned} a_2 &= 0, \quad b_1 = 0, \quad a_1 = b_2 = 2, \quad c_2 = -\frac{k}{2m}, \\ d_1 &= -\frac{k}{m}, \quad \gamma_1 = 1, \quad c_1 = \frac{2k}{m}, \quad d_2 = \frac{k}{m}, \end{aligned} \quad (26)$$

and the Lagrangian given in Eq. (25) then reduces to the one given in Eq. (6). The obtained Lagrangians are summarized in Table 1.

LAGRANGIANS FOR SPECIAL CONSTANT COEFFICIENT MULTI-DEGREE-OF-FREEDOM LINEAR SYSTEMS

If the number of degrees of freedom of a system is much greater than two, then solving the Helmholtz conditions becomes quite complex, and hence obtaining Lagrangians for ordered direct analytic representations of n -degree-of-freedom systems by solving these conditions becomes, in general, extremely difficult, if not nearly impossible. In this section we therefore use ideas from SDOF and 2-DOF systems to expand our thinking to MDOF systems and hence all together bypass the need for solving the Helmholtz conditions. For 2-DOF systems, we used the Lagrangian given by Eq. (4), which we now generalize for an MDOF system as

$$L = e^{\gamma t} \left(\frac{1}{2} \dot{\mathbf{x}}^T \mathbf{M} \dot{\mathbf{x}} - \frac{1}{2} \mathbf{x}^T \mathbf{K} \mathbf{x} - \mathbf{x}^T \mathbf{B} \dot{\mathbf{x}} \right), \quad (27)$$

where $\mathbf{x} = [x_1 \ x_2 \ \cdots \ x_n]^T$ is a generalized displacement n -vector, and \mathbf{M} and \mathbf{K} are the n by n symmetric mass and stiffness matrices, respectively. Also, the scalar γ and the n by n matrix \mathbf{B} are determined according to the requirements of the problem as shall be shown shortly. Then the equation of motion obtained via the Euler-Lagrange equation is

$$\mathbf{M}\ddot{\mathbf{x}} + (\gamma\mathbf{M} + \mathbf{B} - \mathbf{B}^T)\dot{\mathbf{x}} + (\mathbf{K} - \gamma\mathbf{B}^T)\mathbf{x} = \mathbf{0}. \quad (28)$$

First, if we choose $\gamma = 0$ and $\mathbf{B}^T = -\mathbf{B}$ (skew-symmetric), Eq. (28) becomes

$$\mathbf{M}\ddot{\mathbf{x}} + 2\mathbf{B}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}, \quad (29)$$

which is the general equation of motion for an n -degree-of-freedom linear mechanical system with *gyroscopic* damping. The corresponding Lagrangian is

$$L = \frac{1}{2}\dot{\mathbf{x}}^T\mathbf{M}\dot{\mathbf{x}} - \frac{1}{2}\mathbf{x}^T\mathbf{K}\mathbf{x} - \mathbf{x}^T\mathbf{B}\dot{\mathbf{x}} = \frac{1}{2}\dot{\mathbf{x}}^T\mathbf{M}\dot{\mathbf{x}} - \frac{1}{2}\mathbf{x}^T\mathbf{K}\mathbf{x} + \dot{\mathbf{x}}^T\mathbf{B}\mathbf{x}, \quad (30)$$

where \mathbf{M} and \mathbf{K} are symmetric matrices, and \mathbf{B} is a skew-symmetric matrix.

As stated before, Eq. (29) has a skew-symmetric damping matrix, \mathbf{B} . However, in many practical applications, and especially in the theory of linear vibrations, the equations of motion have symmetric stiffness and damping matrices. In order to encompass such systems, we choose $\gamma \neq 0$ and $\mathbf{B}^T = -\mathbf{B}$. Then, with the Lagrangian given in Eq. (27), the Euler-Lagrange equation yields

$$\mathbf{M}\ddot{\mathbf{x}} + (\gamma\mathbf{M} + 2\mathbf{B})\dot{\mathbf{x}} + (\gamma\mathbf{B} + \mathbf{K})\mathbf{x} = \mathbf{0}. \quad (31)$$

When the skew-symmetric matrix $\mathbf{B} = \mathbf{0}$, Eq. (31) reduces to the equation

$$\mathbf{M}\ddot{\mathbf{x}} + \gamma\mathbf{M}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}, \quad (32)$$

which describes a proportionally damped system. The Lagrangian from which this equation is obtainable is simply given, using Eq. (27), by

$$L = e^{\gamma t} \left(\frac{1}{2}\dot{\mathbf{x}}^T\mathbf{M}\dot{\mathbf{x}} - \frac{1}{2}\mathbf{x}^T\mathbf{K}\mathbf{x} \right), \quad (33)$$

for any matrix $\mathbf{M} > \mathbf{0}$, and any symmetric matrix \mathbf{K} . Having disposed off the case $\mathbf{B} = \mathbf{0}$, from here on we shall concentrate then on the case when the skew-symmetric matrix $\mathbf{B} \neq \mathbf{0}$.

We would thus want the matrices $\gamma\mathbf{M} + 2\mathbf{B}$ and $\gamma\mathbf{B} + \mathbf{K}$ in Eq. (31) to be symmetric, where $\mathbf{B} (\neq \mathbf{0})$ is a skew-symmetric matrix. The required conditions are not obvious, so let us consider a 3-DOF system, which, by extension, will help us to adduce the general procedure for handling linearly damped MDOF systems. We start by considering diagonal mass matrices. If we have

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 0 & b_{12} & b_{13} \\ -b_{12} & 0 & b_{23} \\ -b_{13} & -b_{23} & 0 \end{bmatrix}, \quad (34)$$

then, Eq. (31) becomes

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} \gamma m_1 & 2b_{12} & 2b_{13} \\ -2b_{12} & \gamma m_2 & 2b_{23} \\ -2b_{13} & -2b_{23} & \gamma m_3 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} k_1 & \gamma b_{12} & \gamma b_{13} \\ -\gamma b_{12} & k_2 & \gamma b_{23} \\ -\gamma b_{13} & -\gamma b_{23} & k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (35)$$

and the damping and stiffness matrices of this system are not symmetric. However, noting the negative sign in the term that involves \dot{x}_2^2 in Eq. (21), if we choose to use the same \mathbf{B} matrix as in Eq. (34) and

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & -m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & -k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}, \quad (36)$$

in our Lagrangian given in Eq. (27), then the equations of motion that we obtain are

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & -m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} \gamma m_1 & 2b_{12} & 2b_{13} \\ -2b_{12} & -\gamma m_2 & 2b_{23} \\ -2b_{13} & -2b_{23} & \gamma m_3 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} k_1 & \gamma b_{12} & \gamma b_{13} \\ -\gamma b_{12} & -k_2 & \gamma b_{23} \\ -\gamma b_{13} & -\gamma b_{23} & k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (37)$$

or equivalently,

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} \gamma m_1 & 2b_{12} & 2b_{13} \\ 2b_{12} & \gamma m_2 & -2b_{23} \\ -2b_{13} & -2b_{23} & \gamma m_3 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} k_1 & \gamma b_{12} & \gamma b_{13} \\ \gamma b_{12} & k_2 & -\gamma b_{23} \\ -\gamma b_{13} & -\gamma b_{23} & k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (38)$$

in which the damping and stiffness matrices are now both symmetric if we set $b_{13} = 0$.

We now generalize this observation to general n -DOF systems. Our aim is to find the appropriate matrices \mathbf{M} , \mathbf{K} , and \mathbf{B} which when inserted in the Lagrangian given in Eq. (27) will result in the Euler-Lagrange equation given in Eq. (31) that is of the form

$$\bar{\mathbf{M}}\ddot{\mathbf{x}} + \bar{\mathbf{B}}\dot{\mathbf{x}} + \bar{\mathbf{K}}\mathbf{x} = \mathbf{0}, \quad (39)$$

where $\bar{\mathbf{M}}$ is a positive definite diagonal matrix, and the matrices $\bar{\mathbf{B}}$ and $\bar{\mathbf{K}}$ are both symmetric matrices, such as in Eq. (38).

Consider the diagonal matrices $\mathbf{M} = \text{diag}(m_1, m_2, \dots, m_n)$ and $\mathbf{K} = \text{diag}(k_1, k_2, \dots, k_n)$ where $n \geq 2$. We propose to: (i) change the signs of some of the elements of these matrices and (ii) provide a procedure to make the matrix $\bar{\mathbf{M}}$ positive definite, and the matrices $\bar{\mathbf{B}}$ and $\bar{\mathbf{K}}$ symmetric. We do this as follows. If we place negative signs on m_i , m_j , m_k , \dots , k_i , k_j , k_k , \dots ($i, j, k, \dots = 1, 2, \dots, n$, and $i < j < k < \dots$), then the elements of the skew-symmetric matrix \mathbf{B} which is given by

$$\mathbf{B} = \begin{bmatrix} 0 & b_{12} & b_{13} & \dots & b_{1n} \\ -b_{12} & 0 & b_{23} & \dots & b_{2n} \\ -b_{13} & -b_{23} & 0 & \dots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -b_{1n} & -b_{2n} & -b_{3n} & \dots & 0 \end{bmatrix} \quad (40)$$

should have its elements altered by the following rule:

- (1) The elements are set so that $b_{ij} = 0$, $b_{ik} = 0$, $b_{jk} = 0$, \dots , and
- (2) After deleting the i th, j th, k th, \dots rows and i th, j th, k th, \dots columns of the \mathbf{B} matrix in Eq. (40), the remaining elements of \mathbf{B} are set to zero.

Clearly, if we want to place only one negative sign, say, on m_i and k_i ($i = 1, 2, \dots, n$), then only the second rule (2) applies, since \mathbf{B} is skew-symmetric. Also, changing the signs of all the m_i 's and k_i 's ($i = 1, 2, \dots, n$), i.e., m_1, m_2, \dots, m_n and

k_1, k_2, \dots, k_n , will result in $\mathbf{B} = \mathbf{0}$, a case already considered in Eqs. (32) and (33), though more generally.

For example, in a 4-DOF system if we have

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & -m_2 & 0 & 0 \\ 0 & 0 & -m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & -k_2 & 0 & 0 \\ 0 & 0 & -k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 0 & b_{12} & b_{13} & b_{14} \\ -b_{12} & 0 & b_{23} & b_{24} \\ -b_{13} & -b_{23} & 0 & b_{34} \\ -b_{14} & -b_{24} & -b_{34} & 0 \end{bmatrix}, \quad (41)$$

that is, $i = 2$ and $j = 3$ in this case, then we should choose the elements of the \mathbf{B} matrix by the rule:

- (1) $b_{23} = 0$, and
- (2) After we delete the 2nd and 3rd rows and columns, the remaining elements should be set to zero, i.e., set $b_{14} = 0$.

In brief, we should choose the following \mathbf{B} matrix:

$$\mathbf{B} = \begin{bmatrix} 0 & b_{12} & b_{13} & 0 \\ -b_{12} & 0 & 0 & b_{24} \\ -b_{13} & 0 & 0 & b_{34} \\ 0 & -b_{24} & -b_{34} & 0 \end{bmatrix}. \quad (42)$$

Now using the Lagrangian given by Eq. (27), with \mathbf{M} and \mathbf{K} defined in Eq. (41) and \mathbf{B} defined in Eq. (42), the equations of motion given by Eq. (31) become

$$\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & -m_2 & 0 & 0 \\ 0 & 0 & -m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} + \begin{bmatrix} \gamma m_1 & 2b_{12} & 2b_{13} & 0 \\ -2b_{12} & -\gamma m_2 & 0 & 2b_{24} \\ -2b_{13} & 0 & -\gamma m_3 & 2b_{34} \\ 0 & -2b_{24} & -2b_{34} & \gamma m_4 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} + \begin{bmatrix} k_1 & \gamma b_{12} & \gamma b_{13} & 0 \\ -\gamma b_{12} & -k_2 & 0 & \gamma b_{24} \\ -\gamma b_{13} & 0 & -k_3 & \gamma b_{34} \\ 0 & -\gamma b_{24} & -\gamma b_{34} & k_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (43)$$

or

$$\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} + \begin{bmatrix} \gamma m_1 & 2b_{12} & 2b_{13} & 0 \\ 2b_{12} & \gamma m_2 & 0 & -2b_{24} \\ 2b_{13} & 0 & \gamma m_3 & -2b_{34} \\ 0 & -2b_{24} & -2b_{34} & \gamma m_4 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} + \begin{bmatrix} k_1 & \gamma b_{12} & \gamma b_{13} & 0 \\ \gamma b_{12} & k_2 & 0 & -\gamma b_{24} \\ \gamma b_{13} & 0 & k_3 & -\gamma b_{34} \\ 0 & -\gamma b_{24} & -\gamma b_{34} & k_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (44)$$

and Eq. (44) has symmetric damping and stiffness matrices as well as a positive definite mass matrix. The corresponding Lagrangian is given by Eq. (27).

As a special case, in an n -DOF system ($n \geq 2$) if one chooses

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -m_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & m_3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -m_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (-1)^{n+1} m_n \end{bmatrix},$$

$$\mathbf{K} = \begin{bmatrix} k_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -k_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & k_3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -k_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (-1)^{n+1} k_n \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 0 & b_1 & 0 & 0 & \cdots & 0 \\ -b_1 & 0 & b_2 & 0 & \cdots & 0 \\ 0 & -b_2 & 0 & b_3 & \cdots & 0 \\ 0 & 0 & -b_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad (45)$$

using the matrices given by Eq. (45) in the Lagrangian given by Eq. (27), the Euler-Lagrange equation of motion given by Eq. (31) yields

$$\begin{bmatrix} m_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & m_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & m_3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & m_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & m_n \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \\ \vdots \\ \ddot{x}_n \end{bmatrix} + \begin{bmatrix} \gamma m_1 & 2b_1 & 0 & 0 & \cdots & 0 \\ 2b_1 & \gamma m_2 & -2b_2 & 0 & \cdots & 0 \\ 0 & -2b_2 & \gamma m_3 & 2b_3 & \cdots & 0 \\ 0 & 0 & 2b_3 & \gamma m_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \gamma m_n \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \vdots \\ \dot{x}_n \end{bmatrix} + \begin{bmatrix} k_1 & \gamma b_1 & 0 & 0 & \cdots & 0 \\ \gamma b_1 & k_2 & -\gamma b_2 & 0 & \cdots & 0 \\ 0 & -\gamma b_2 & k_3 & \gamma b_3 & \cdots & 0 \\ 0 & 0 & \gamma b_3 & k_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & k_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad (46)$$

which is the equation for an MDOF system with *tridiagonal* symmetric damping and stiffness matrices of the form of Eq. (39). It can give the equations of motion of certain systems consisting of a chain of masses in which each mass is connected to its neighbors by linear dashpots and springs. The strength of the procedure introduced in this section is that it totally bypasses the Helmholtz conditions which are near-impossible to solve for arbitrarily large, finite values of n .

As the last application of the results obtained so far, we propose a Jacobi integral that is conserved at all times for the types of linear MDOF systems considered here (see Eq. (31)). When the Lagrangian does not contain time explicitly (and the actual velocity is a virtual velocity), i.e., when $\partial L(t, \mathbf{q}, \dot{\mathbf{q}}) / \partial t = 0$, the Jacobi integral I given by [11]

$$I := \dot{\mathbf{q}}^T \frac{\partial L}{\partial \dot{\mathbf{q}}} - L \quad (47)$$

is conserved. The Lagrangian in Eq. (31), however, contains time explicitly, but by using the transformation

$$\mathbf{y} = \mathbf{x} e^{\frac{\gamma t}{2}} \quad (48)$$

it becomes

$$L = \frac{1}{2} \left(\dot{\mathbf{y}} - \frac{\gamma}{2} \mathbf{y} \right)^T \mathbf{M} \left(\dot{\mathbf{y}} - \frac{\gamma}{2} \mathbf{y} \right) - \frac{1}{2} \mathbf{y}^T \mathbf{K} \mathbf{y} - \mathbf{y}^T \mathbf{B} \left(\dot{\mathbf{y}} - \frac{\gamma}{2} \mathbf{y} \right). \quad (49)$$

Hence, the Jacobi integral is given by Eq. (47) as

$$I = \frac{1}{2} \left(\dot{\mathbf{y}} + \frac{\gamma}{2} \mathbf{y} \right)^T \mathbf{M} \left(\dot{\mathbf{y}} - \frac{\gamma}{2} \mathbf{y} \right) - \dot{\mathbf{y}}^T (\mathbf{B} + \mathbf{B}^T) \mathbf{y} + \frac{1}{2} \mathbf{y}^T \mathbf{K} \mathbf{y} - \frac{\gamma}{2} \mathbf{y}^T \mathbf{B} \mathbf{y}, \quad (50)$$

and since the matrix \mathbf{B} is assumed to be skew-symmetric in this paper, Eq. (50) simplifies to

$$I = \frac{1}{2} \left(\dot{\mathbf{y}} + \frac{\gamma}{2} \mathbf{y} \right)^T \mathbf{M} \left(\dot{\mathbf{y}} - \frac{\gamma}{2} \mathbf{y} \right) + \frac{1}{2} \mathbf{y}^T \mathbf{K} \mathbf{y}. \quad (51)$$

Rewriting this Jacobi integral in terms of \mathbf{x} and $\dot{\mathbf{x}}$ by using Eq. (48), we obtain the conservation law

$$I(t, \mathbf{x}, \dot{\mathbf{x}}) = e^{\gamma t} \left[\frac{1}{2} (\dot{\mathbf{x}} + \gamma \mathbf{x})^T \mathbf{M} \dot{\mathbf{x}} + \frac{1}{2} \mathbf{x}^T \mathbf{K} \mathbf{x} \right] = \text{constant}. \quad (52)$$

Thus, for an MDOF linear system whose equation of motion is given by Eq. (31) it is guaranteed that the function $I(t, \mathbf{x}, \dot{\mathbf{x}})$ in Eq. (52) is conserved at all times.

CONCLUSIONS

We have discussed here extensions of the inverse problem of the calculus of variations for non-potential forces to multi-degree-of-freedom systems. For a two-degree-of-freedom linear system with linear damping, the conditions for the existence of a Lagrangian are explicitly obtained by solving the Helmholtz conditions. Three general cases when such Lagrangians are guaranteed to exist are obtained, depending on the parameter values of the coupled linear systems. The Helmholtz conditions are near-impossible to solve for general n -degree-of-freedom systems, and though they are explicit, from a practical standpoint they provide little assistance in solving the inverse problem for such systems. By using and generalizing results for single-degree-of-freedom systems, a simple procedure that does not require the use of the Helmholtz conditions and that is

easily extended to n -degree-of-freedom linear systems, is developed. We specifically include systems that commonly arise in the theory of linear vibrations--systems with positive definite mass matrices, and symmetric stiffness and damping matrices. The method yields several new Lagrangians for linear multi-degree-of-freedom systems. Conservation laws for such dissipative MDOF systems are also obtained by finding the corresponding Jacobi integrals.

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