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ON A SUBSTRUCTURAL GENTZEN SYSTEM, ITS EQUIVALENT VARIETY SEMANTICS AND ITS EXTERNAL DEDUCTIVE SYSTEM

Abstract

It was shown in [1] that the Gentzen system $\mathcal{G}_{LJ^*\backslash c}$, the deductive system $IPC^*\backslash c$ and the equational system $\langle \mathcal{L}, \models_{\mathsf{RL}} \rangle$ associated with the variety of residuated lattices are equivalent in the sense of [8] and [9]. In this paper we show that if we delete the rules for the implication connective $\rightarrow$ from the sequent calculus $LJ^*\backslash c$, then the Gentzen system obtained from this sequent calculus, denoted by $\mathcal{G}_{\wedge,\vee,\odot}$, is algebraizable and the variety of bounded commutative integral l-monoids is its equivalent algebraic semantics. As a consequence of this result we obtain that the contraction rule is not derivable in $\mathcal{G}_{\wedge,\vee,\odot}$, so we can say that $\mathcal{G}_{\wedge,\vee,\odot}$ is a substructural Gentzen system. It is also shown that there is no deductive system equivalent to $\mathcal{G}_{\wedge,\vee,\odot}$ or to $\langle \mathcal{L}, \models_{\mathsf{BCILM}} \rangle$ and that the variety of BCILM is an algebraic semantics for the external deductive system $S^0_{\mathcal{G}_{\wedge,\vee,\odot}}$ associated with $\mathcal{G}_{\wedge,\vee,\odot}$, with defining equation $p \approx 1$. Finally we prove that this deductive system $S^0_{\mathcal{G}_{\wedge,\vee,\odot}}$ is not protoalgebraic.

Let us recall the definition of the sequent calculus $LJ^*\backslash c$.

Definition 1. ([1, definition 2], cf. [7]) Let $\mathcal{L} = \{\wedge, \vee, \odot, \rightarrow, 0, 1\}$ be a propositional language of type $(2, 2, 2, 2, 0, 0)$. Let Γ, Π be finite sequences of $\mathcal{L}$-formulas and φ, ψ, ξ be $\mathcal{L}$-formulas. The sequent calculus $LJ^*\backslash c$ is defined by the following axioms and rules:

$$\varphi \Rightarrow \varphi \quad \text{(Axiom 1)} \qquad 0 \Rightarrow \varphi \quad \text{(Axiom 2)} \quad \emptyset \Rightarrow 1 \quad \text{(Axiom 3)}$$

$$\frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \xi}{\Gamma, \psi, \varphi, \Pi \Rightarrow \xi} \ (i \Rightarrow) \qquad\qquad \frac{\Gamma \Rightarrow \xi}{\Gamma, \varphi \Rightarrow \xi} \ (w \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \varphi \quad \varphi, \Pi \Rightarrow \xi}{\Gamma, \Pi \Rightarrow \xi} \ (cut)$$

$$\frac{\Gamma \Rightarrow \varphi \quad \psi, \Pi \Rightarrow \xi}{\varphi \to \psi, \Gamma, \Pi \Rightarrow \xi} \ (\to\Rightarrow) \qquad \frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi} \ (\Rightarrow\to)$$

$$\frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi} \ (\Rightarrow \vee 1)$$

$$\frac{\Gamma, \varphi \Rightarrow \xi \quad \Gamma, \psi \Rightarrow \xi}{\Gamma, \varphi \vee \psi \Rightarrow \xi} \ (\vee \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi} \ (\Rightarrow \vee 2)$$

$$\frac{\Gamma, \psi \Rightarrow \xi}{\Gamma, \varphi \wedge \psi \Rightarrow \xi} \ (\wedge 1 \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi} \ (\Rightarrow \wedge)$$

$$\frac{\Gamma, \varphi \Rightarrow \xi}{\Gamma, \varphi \wedge \psi \Rightarrow \xi} \ (\wedge 2 \Rightarrow)$$

$$\frac{\Gamma, \varphi, \psi \Rightarrow \xi}{\Gamma, \varphi \odot \psi \Rightarrow \xi} \ (\odot \Rightarrow) \qquad \frac{\Gamma \Rightarrow \varphi \quad \Pi \Rightarrow \psi}{\Gamma, \Pi \Rightarrow \varphi \odot \psi} \ (\Rightarrow \odot).$$

Definition 2. Let $LJ^*_{\{\wedge,\vee,\odot,0,1\}}\backslash c$ be the sequent calculus obtained by deleting the rules for the implication connective $\to$ from $LJ^*\backslash c$ and let $\mathcal{G}_{\wedge,\vee,\odot} = \left\langle \mathcal{L}, \vdash_{LJ^*_{\{\wedge,\vee,\odot,0,1\}}\backslash c} \right\rangle$ be the Gentzen system of type $(\omega, \{1\})$ determined by the sequent calculus $LJ^*_{\{\wedge,\vee,\odot,0,1\}}\backslash c$.

Let us recall the definition of the variety of bounded commutative integral l-monoids.

Definition 3. (cf. [2]) Let $\mathbf{A} = \langle A, \wedge, \vee, \odot, 0, 1\rangle$ be an algebra of type $(2, 2, 2, 0, 0)$, $\mathbf{A}$ is a bounded commutative integral l-monoid if the following conditions are satisfied:

1. $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded lattice where 1 and 0 are the maximum and minimum elements, respectively;
2. $\langle A, \odot, 1\rangle$ is a commutative monoid, i.e., for every $x, y, z \in A$,

(a) $(x \odot y) \odot z = x \odot (y \odot z)$,
(b) $x \odot y = y \odot x$,
(c) $1 \odot x = x$;

3. For all $x, y, z \in A$,
$(x \vee y) \odot z = (x \odot z) \vee (y \odot z)$.

The class of bounded commutative integral l-monoids will be denoted by BCILM.

It is easy to see that BCILM satisfies the property of monotonicity of $\odot$, i.e., for every $x, y, z \in A$, if $x \leq y$ implies $(x \odot z) \leq (y \odot z)$.

Notice that if we add the residuation property

$$x \leq y \to z \text{ iff } x \odot y \leq z$$

to the definition of BCILM, then we obtain a residuated lattice.

The following lemma will be used to show that $\mathcal{G}_{\wedge,\vee,\odot}$ is algebraizable.

Lemma 4. *([9, Lemma 2.5]) Let $\mathcal{G}$ be a Gentzen system of type (α, β), let K be a quasivariety.*

If there is a translation τ from $\mathcal{G}$ to K and a translation ρ from K to $\mathcal{G}$ such that:

1. $\Gamma \Rightarrow \Delta \dashv\vdash_{\mathcal{G}} \rho\tau(\Gamma \Rightarrow \Delta)$ *for all* $\Gamma \Rightarrow \Delta \in Seq_{\mathcal{L}}^{(\alpha,\beta)}$,
2. $(\varphi, \psi) \dashv\models_{\mathsf{K}} \tau\rho(\varphi, \psi)$ *for all* $(\varphi, \psi) \in Fm_{\mathcal{L}}^2$,
3. *for every* $\mathbf{A} \in \mathsf{K}$, *the set* $R = \left\{(X, Y) \in A^m \times A^n : m \in \alpha, n \in \beta, \tau^{\mathbf{A}}(X, Y) \subseteq \Delta_{\mathbf{A}}\right\}$ *is a $\mathcal{G}$-filter,*
4. *for all* $T \in Th\mathcal{G}$, $\theta_T = \left\{(\varphi, \psi) \in Fm_{\mathcal{L}}^2 : \rho(\varphi, \psi) \subseteq T\right\} \in Con_{\mathsf{K}} Fm_{\mathcal{L}}$,

then $\mathcal{G}$ is algebraizable with equivalent algebraic semantics K.

Theorem 5. *$\mathcal{G}_{\wedge,\vee,\odot}$ is algebraizable with its equivalent algebraic semantics, the variety BCILM, and with the translations τ from $\mathcal{G}_{\wedge,\vee,\odot}$ to BCILM and ρ from BCILM to $\mathcal{G}_{\wedge,\vee,\odot}$ defined in the following way:*

$$\tau(p_0, \ldots, p_{m-1} \Rightarrow q_0) = \begin{cases} \left(\bigodot\limits_{i<m} p_i\right) \wedge q_0 \approx \bigodot\limits_{i<m} p_i, & \text{if } m \geq 1 \\ 1 \approx q_0, & \text{if } m = 0 \end{cases}$$

$$\rho(p_0 \approx p_1) = \{p_0 \Rightarrow p_1;\ p_1 \Rightarrow p_0\}.$$

In order to simplify the notation we use $\bigodot_{i<m} p_i$ *as an abbreviation for* $p_0 \odot (p_1 \odot (\ldots \odot (p_{m-2} \odot p_{m-1}) \ldots))$.

PROOF: To prove this result we will use Lemma 4.

1. Let us prove condition $(i) : \Gamma \Rightarrow \psi \;\dashv\vdash_{LJ^*_{\{\wedge,\vee,\odot,0,1\}}\backslash c}\; \rho\tau\,(\Gamma \Rightarrow \psi)$.
   If $\Gamma = \emptyset$ we have $\rho\tau\,(\emptyset \Rightarrow \psi) = \{1 \Rightarrow \psi,\ \psi \Rightarrow 1\}$ then by using the axiom 3, the rules $(w \Rightarrow)$ and (cut) we obtain the condition (i).
   If $\Gamma = \gamma_0, \ldots, \gamma_{n-1}$,
   and we will denote the formula $\gamma_0 \odot (\gamma_1 \odot (\ldots \odot (\gamma_{m-2} \odot \gamma_{m-1}) \ldots))$ as $\bigodot \Gamma$, we have

$$\rho\tau\,(\Gamma \Rightarrow \psi) = \left\{ \bigodot \Gamma \Rightarrow \left(\bigodot \Gamma\right) \wedge \psi,\ \left(\bigodot \Gamma\right) \wedge \psi \Rightarrow \bigodot \Gamma \right\}.$$

   Note that the sequent $(\bigodot \Gamma) \wedge \psi \Rightarrow \bigodot \Gamma$ is derivable in $\mathcal{G}_{\wedge,\vee,\odot}$ by using the rule $(\wedge 2 \Rightarrow)$.
   So we have to show that $\Gamma \Rightarrow \psi \;\dashv\vdash_{LJ^*_{\{\wedge,\vee,\odot,0,1\}}\backslash c} \bigodot \Gamma \Rightarrow (\bigodot \Gamma) \wedge \psi$:
   $\vdash$)

$$\dfrac{\bigodot \Gamma \Rightarrow \bigodot \Gamma \qquad \dfrac{\Gamma \Rightarrow \psi}{\bigodot \Gamma \Rightarrow \psi}\,(\odot \Rightarrow)^{n-1}}{\bigodot \Gamma \Rightarrow (\bigodot \Gamma) \wedge \psi}\,(\Rightarrow \wedge)$$

   $\dashv$)

$$\dfrac{\gamma_{n-2} \Rightarrow \gamma_{n-2} \quad \gamma_{n-1} \Rightarrow \gamma_{n-1}}{\gamma_{n-2}, \gamma_{n-1} \Rightarrow \gamma_{n-2} \odot \gamma_{n-1}}\,(\Rightarrow \odot)$$

$$\dfrac{\dfrac{\vdots}{\Gamma \Rightarrow \bigodot \Gamma}\,(\Rightarrow \odot) \qquad \dfrac{\bigodot \Gamma \Rightarrow (\bigodot \Gamma) \wedge \psi \quad \dfrac{\psi \Rightarrow \psi}{(\bigodot \Gamma) \wedge \psi \Rightarrow \psi}\,(\wedge 1 \Rightarrow)}{\bigodot \Gamma \Rightarrow \psi}\,(cut)}{\Gamma \Rightarrow \psi}\,(cut)$$

1. Condition $(ii) : \varphi \approx \psi \dashv\models_{\mathsf{BCILM}} \{\varphi \wedge \psi \approx \varphi,\ \psi \wedge \varphi \approx \psi\}$ is trivial.
2. Let us prove condition (iii) that is, for all $\mathbf{A} \in \mathsf{BCILM}$, the set

$$R = \left\{ (X, a) \in A^m \times A : m \in \omega,\ \tau^{\mathbf{A}}(X, a) \subseteq \Delta_{\mathbf{A}} \right\}$$

   is a $\mathcal{G}_{\wedge,\vee,\odot}$-filter.

Notice that

$$\begin{aligned} R &= \left\{(X,a) \in A^m \times A : m \in \omega\backslash\{0\}\,,\ \bigodot X \leq a\right\} \cup \\ & \quad \cup \{(\emptyset, a) \in \{\emptyset\} \times A : 1 \leq a\}\,. \end{aligned}$$

This set contains all the pairs (a, a), $(0, a)$ and $(\emptyset, 1)$ with $a \in A$, so R contains the interpretation of the axioms. Now we will show that R is closed under the interpretation of the inference rules of $LJ^*_{\{\wedge,\vee,\odot,0,1\}}\backslash c$.

Let us check, for example, the following:

- *Right introduction rule for* $\odot$:

$$\frac{\Gamma \Rightarrow \varphi \quad \Pi \Rightarrow \psi}{\Gamma, \Pi \Rightarrow \varphi \odot \psi}.$$

If $\Gamma = \emptyset$ and $\Pi = \emptyset$, we have to show that if $(\emptyset, a) \in R$ and $(\emptyset, b) \in R$, then $(\emptyset, a \odot b) \in R$. We have that $1 \leq a$, $1 \leq b$ and by using the fact that 1 is the maximum element, we obtain that $a = b = 1$, by applying the property (2)-(c) we have $1 = a \odot b$, so $(\emptyset, a \odot b) \in R$.
If $\Gamma = \emptyset$ and $\Pi \neq \emptyset$ (analogous for $\Gamma \neq \emptyset$ and $\Pi = \emptyset$), we have to show that if $(\emptyset, a) \in R$ and $(Y, b) \in R$, then $(Y, a \odot b) \in R$. We have that $1 \leq a$ and $\bigodot Y \leq b$ and by using the fact that 1 is the maximum element, we obtain that $a = 1$, then by applying the property (2)-(c) we have $b = a \odot b$, we obtain $\bigodot Y \leq b$, so $(Y, a \odot b) \in R$.
If $\Gamma \neq \emptyset$ and $\Pi \neq \emptyset$, we have to show that if $(X, a) \in R$ and $(Y, b) \in R$, then $((X, Y), a \odot b) \in R$. We have that $\bigodot X \leq a$ and $\bigodot Y \leq b$ by applying the monotonicity of $\odot$ we obtain $(\bigodot X) \odot (\bigodot Y) \leq a \odot b$, and by the associative property of $\odot$ we have $((X, Y), a \odot b) \in R$.

- *Left introduction rule for* $\wedge$:

$$\frac{\Gamma, \psi \Rightarrow \xi}{\Gamma, \varphi \wedge \psi \Rightarrow \xi}.$$

If $\Gamma = \emptyset$, we have to show that if $(b, c) \in R$, then $(a \wedge b, c) \in R$. We have that $b \leq c$, and by using the property $a \wedge b \leq b$ we obtain $a \wedge b \leq c$, so that $(a \wedge b, c) \in R$.

If $\Gamma \neq \emptyset$, we have to show that if $((X, b), c) \in R$, then $((X, a \wedge b), c) \in R$. We have that $(\bigodot X) \odot b \leq c$, and by using the property $a \wedge b \leq b$ and the monotonicity of $\odot$ we obtain $(\bigodot X) \odot (a \wedge b) \leq (\bigodot X) \odot b$; so that $(\bigodot X) \odot (a \wedge b) \leq c$, that is $((X, a \wedge b), c) \in R$.

- *Left introduction rule for* $\vee$:

$$\frac{\Gamma, \varphi \Rightarrow \xi \quad \Gamma, \psi \Rightarrow \xi}{\Gamma, \varphi \vee \psi \Rightarrow \xi}.$$

If $\Gamma = \emptyset$, we have to show that if $(a, c) \in R$ and $(b, c) \in R$, then $(a \vee b, c) \in R$. We have that $a \leq c$ and $b \leq c$, and by using the fact that $\vee$ is the supremum, we obtain $a \vee b \leq c$, so that $(a \vee b, c) \in R$.
If $\Gamma \neq \emptyset$, we have to show that if $((X, a), c) \in R$ and $((X, b), c) \in R$, then $((X, a \vee b), c) \in R$. We have that $(\bigodot X) \odot a \leq c$ and $(\bigodot X) \odot b \leq c$, and by using the distributive property (3) we have $(\bigodot X) \odot (a \vee b) = ((\bigodot X) \odot a) \vee ((\bigodot X) \odot b)$ and by using the fact that $\vee$ is the supremum, we obtain $(\bigodot X) \odot (a \vee b) \leq c$, so $((X, a \vee b), c) \in R$.

The other rules are left to the reader.

3. Let us prove condition (iv), that is for all $T \in Th\mathcal{G}_{\wedge,\vee,\odot}$,

$$\theta_T = \left\{(\varphi, \psi) \in Fm^2_{\{\wedge,\vee,\odot,0,1\}} : \rho(\varphi, \psi) \subseteq T\right\} \in Con_{\mathsf{BCILM}} \boldsymbol{Fm}_{\{\wedge,\vee,\odot,0,1\}}.$$

It is easy to show that θ_T is a congruence.
Now let $\varphi \approx \psi$ be one of the equations which define the BCILM. We have to show that $\rho(\varphi, \psi) \subseteq T$ for all $T \in Th\mathcal{G}_{\wedge,\vee,\odot}$. It suffices to show that $\emptyset \vdash_{LJ^*_{\{\wedge,\vee,\odot,0,1\}} \backslash c} \varphi \Rightarrow \psi$ and $\emptyset \vdash_{LJ^*_{\{\wedge,\vee,\odot,0,1\}} \backslash c} \psi \Rightarrow \varphi$.
Let us check, for example, the following:

- $x \wedge x = x$:
  We have to show that $\emptyset \vdash_{LJ^*_{\{\wedge,\vee,\odot,0,1\}} \backslash c} \varphi \wedge \varphi \Rightarrow \varphi$ and $\emptyset \vdash_{LJ^*_{\{\wedge,\vee,\odot,0,1\}} \backslash c} \varphi \Rightarrow \varphi \wedge \varphi$. By using the rules $(\wedge 1 \Rightarrow)$ and $(\Rightarrow \wedge)$, respectively, the result follows.
- $x \odot y = y \odot x$:
  We have to show that $\emptyset \vdash_{LJ^*_{\{\wedge,\vee,\odot,0,1\}} \backslash c} \varphi \odot \psi \Rightarrow \psi \odot \varphi$. It is straightforward by using the rules $(\Rightarrow \odot)$, $(i \Rightarrow)$ and $(\odot \Rightarrow)$.

- $1 \odot x = x$:
  We have to show that $\emptyset \vdash_{LJ^*_{\{\wedge,\vee,\odot,0,1\}}\backslash c} 1 \odot \varphi \Rightarrow \varphi$ and $\emptyset \vdash_{LJ^*_{\{\wedge,\vee,\odot,0,1\}}\backslash c} \varphi \Rightarrow 1 \odot \varphi$. Consider the following proofs:

$$\dfrac{\dfrac{\varphi \Rightarrow \varphi}{1, \varphi \Rightarrow \varphi}(w \Rightarrow)}{1 \odot \varphi \Rightarrow \varphi}(\odot \Rightarrow) \qquad \dfrac{\emptyset \Rightarrow 1 \quad \varphi \Rightarrow \varphi}{\varphi \Rightarrow 1 \odot \varphi}(\Rightarrow \odot)$$

- $(x \vee y) \odot z = (x \odot z) \vee (y \odot z)$:
  We have to show that $\emptyset \vdash_{LJ^*_{\{\wedge,\vee,\odot,0,1\}}\backslash c} (\varphi \vee \psi) \odot \xi \Rightarrow (\varphi \odot \xi) \vee (\psi \odot \xi)$. Consider the following proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi \quad \xi \Rightarrow \xi}{\varphi, \xi \Rightarrow \varphi \odot \xi}(\Rightarrow\odot)}{\varphi, \xi \Rightarrow (\varphi \odot \xi) \vee (\psi \odot \xi)}(\Rightarrow \vee 2) \quad \dfrac{\dfrac{\psi \Rightarrow \psi \quad \xi \Rightarrow \xi}{\psi, \xi \Rightarrow \psi \odot \xi}(\Rightarrow\odot)}{\psi, \xi \Rightarrow (\varphi \odot \xi) \vee (\psi \odot \xi)}(\Rightarrow \vee 1)}{\varphi \vee \psi, \xi \Rightarrow (\varphi \odot \xi) \vee (\psi \odot \xi)}(\vee \Rightarrow)}{(\varphi \vee \psi) \odot \xi \Rightarrow (\varphi \odot \xi) \vee (\psi \odot \xi)}(\odot \Rightarrow)$$

  Now we have to prove that $\emptyset \vdash_{LJ^*_{\{\wedge,\vee,\odot,0,1\}}\backslash c} (\varphi \odot \xi) \vee (\psi \odot \xi) \Rightarrow (\varphi \vee \psi) \odot \xi$. Consider the following proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi \Rightarrow \varphi \vee \psi}(\Rightarrow \vee 2) \quad \xi \Rightarrow \xi}{\varphi, \xi \Rightarrow (\varphi \vee \psi) \odot \xi}(\Rightarrow \odot)}{\varphi \odot \xi \Rightarrow (\varphi \vee \psi) \odot \xi}(\odot \Rightarrow) \quad \dfrac{\dfrac{\dfrac{\psi \Rightarrow \psi}{\psi \Rightarrow \varphi \vee \psi}(\Rightarrow \vee 1) \quad \xi \Rightarrow \xi}{\psi, \xi \Rightarrow (\varphi \vee \psi) \odot \xi}(\Rightarrow \odot)}{\psi \odot \xi \Rightarrow (\varphi \vee \psi) \odot \xi}(\odot \Rightarrow)}{(\varphi \odot \xi) \vee (\psi \odot \xi) \Rightarrow (\varphi \vee \psi) \odot \xi}(\vee \Rightarrow)$$

□

As a consequence of this theorem, we have that the equational theory of BCILM is decidable.

THEOREM 6. *The contraction rule is not derivable in $\mathcal{G}_{\wedge,\vee,\odot}$.*

PROOF: Assume that the contraction rule is derivable, that is:

$$\Gamma, \varphi, \varphi, \Pi \Rightarrow \xi \vdash_{LJ^*_{\{\wedge,\vee,\odot,0,1\}}\backslash c} \Gamma, \varphi, \Pi \Rightarrow \xi$$

then by the theorem 5 we have that

$$\left(\bigodot \Gamma\right) \odot \varphi \odot \varphi \odot \left(\bigodot \Pi\right) \leq \xi \models_{\mathsf{BCILM}} \left(\bigodot \Gamma\right) \odot \varphi \odot \left(\bigodot \Pi\right) \leq \xi \qquad (1)$$

Consider the $\langle \wedge, \vee, \odot, 0, 1 \rangle$-reduct of the MV-algebra of three-elements, that is $\mathbf{A} = \left\langle \left\{0, \frac{1}{2}, 1\right\}, \wedge, \vee, \odot, 0, 1 \right\rangle$, where $0 < \frac{1}{2} < 1$ and $x \odot y = \neg(x \to \neg y)$. As $\frac{1}{2} \odot \frac{1}{2} = \neg(\frac{1}{2} \to \neg\frac{1}{2}) = \neg(\frac{1}{2} \to \frac{1}{2}) = \neg 1 = 0$, we have that there is an interpertation of the antecedent of (1) that is true ($0 \leq 0$) but the interpretation of the consequent of (1) is false. So the contraction rule is not derivable in $\mathcal{G}_{\wedge,\vee,\odot}$. $\square$

THEOREM 7. *The variety* BCILM *is not the equivalent algebraic semantics for any deductive system.*

PROOF: In [6, proposition 2.1] it is proved that for the four-element distributive lattice $\mathbf{A} = \langle \{0, a, b, 1\}, \wedge, \vee \rangle$, with $0 < b < a < 1$, the Leibniz operator $\Omega_{\mathbf{A}}$ cannot be an isomorphism between the filters of an arbitrary deductive system and the congruences of the algebra $\mathbf{A}$. Consider $\mathbf{A}' = \langle \{0, a, b, 1\}, \wedge, \vee, \odot, 0, 1 \rangle$, where $\odot = \wedge$, then $\mathbf{A}' \in$ BCILM. It is easy to check that the proof of [6, proposition 2.1] applies in our case to show that BCILM is not the equivalent algebraic semantics of any deductive system. $\square$

Now we will consider the external deductive system associated with $\mathcal{G}_{\wedge,\vee,\odot}$.

DEFINITION 8. Let $\mathcal{L} = \{\wedge, \vee, \odot, 0, 1\}$. Let $S^0_{\mathcal{G}_{\wedge,\vee,\odot}} = \left\langle Fm_{\mathcal{L}}, \vdash_{S^0_{\mathcal{G}_{\wedge,\vee,\odot}}} \right\rangle$ be the external deductive system associated with $\mathcal{G}_{\wedge,\vee,\odot}$; that is, $\vdash_{S^0_{\mathcal{G}_{\wedge,\vee,\odot}}}$ is the finitary and structural consequence relation on the set $Fm_{\mathcal{L}}$ defined by:

$$\Gamma \vdash_{S^0_{\mathcal{G}_{\wedge,\vee,\odot}}} \varphi \text{ iff } \{\emptyset \Rightarrow \gamma : \gamma \in \Gamma\} \vdash_{LJ^*_{\mathcal{L}} \backslash c} \emptyset \Rightarrow \varphi, \text{ for all } \Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}.$$

THEOREM 9. $S^0_{\mathcal{G}_{\wedge,\vee,\odot}}$ *and* $\mathcal{G}_{\wedge,\vee,\odot}$ *are not equivalent.*

PROOF: This theorem is a consequence of the fact that $\mathcal{G}_{\wedge,\vee,\odot}$ is algebraizable with its equivalent algebraic semantics, the variety BCILM, (theorem 5) and the fact that BCILM is not the equivalent algebraic semantics for any deductive system (theorem 7). $\square$

THEOREM 10. *The variety* BCILM *is an algebraic semantics of the deductive system* $S^0_{\mathcal{G}_{\wedge,\vee,\odot}}$ *with defining equation* $\varphi \approx 1$, *that is, for every* $\Gamma \cup \{\varphi\} \subseteq Fm_{\{\wedge,\vee,\odot,0,1\}}$,

$$\Gamma \vdash_{S^0_{\mathcal{G}_{\wedge,\vee,\odot}}} \varphi \textit{ iff } \{1 \approx \varphi_i : \varphi_i \in \Gamma\} \models_{\mathsf{BCILM}} 1 \approx \varphi.$$

PROOF: By the definition of $S^0_{\mathcal{G}_{\wedge,\vee,\odot}}$ and theorem 5, we have that $\Gamma \vdash_{S^0_{\mathcal{G}_{\wedge,\vee,\odot}}} \varphi$ iff $\{\emptyset \Rightarrow \varphi_i : \varphi_i \in \Gamma\} \vdash_{LJ^*_{\{\wedge,\vee,\odot,0,1\}}\backslash c} \emptyset \Rightarrow \varphi$ iff $\{1 \approx \varphi_i : i \in I\} \models_{\mathsf{BCILM}} 1 \approx \varphi$. □

THEOREM 11. *$S^0_{\mathcal{G}_{\wedge,\vee,\odot}}$ is not protoalgebraic.*

PROOF: To show this result we will check that the Leibniz operator is not order-preserving on the $S^0_{\mathcal{G}_{\wedge,\vee,\odot}}$-filters of the algebra $\mathbf{A} = \langle\{0, a, 1\}, \wedge, \vee, \odot, 0, 1\rangle$ of type $(2, 2, 2, 0, 0)$, where $0 < a < 1$, $\wedge$ and $\vee$ are infimum and supremum, respectively, and $\wedge = \odot$.

First we prove that the $S^0_{\mathcal{G}_{\wedge,\vee,\odot}}$-filters of $\mathbf{A}$ are exactly the lattice filters of $\mathbf{A}$; that is, the $S^0_{\mathcal{G}_{\wedge,\vee,\odot}}$-filters of $\mathbf{A}$ are: $\{1\}$, $\{1, a\}$ and $\{1, a, 0\}$.

Indeed,

- Suppose that F is a lattice filter of $\mathbf{A}$. We have to show that F is a $S^0_{\mathcal{G}_{\wedge,\vee,\odot}}$-filter; that is, if $\Gamma \vdash_{S^0_{\mathcal{G}_{\wedge,\vee,\odot}}} \varphi$, $h \in Hom(Fm_{\mathcal{L}}, \mathbf{A})$ and $h(\Gamma) \subseteq F$, then $h(\varphi) \in F$. Consider the Gentzen system $\mathcal{G}_{LJ^*_{\{\wedge,\vee,\odot,0,1\}}}$ which is obtained by adding the contraction rule to $\mathcal{G}_{\wedge,\vee,\odot}$. It is easy to see that the external deductive system $S^0_{LJ^*_{\{\wedge,\vee,\odot,0,1\}}}$ associated with $\mathcal{G}_{LJ^*_{\{\wedge,\vee,\odot,0,1\}}}$ coincides with the deductive system S obtained from $\mathcal{G}_{LJ^*_{\{\wedge,\vee,\odot,0,1\}}}$ by means of $\Gamma \vdash_S \varphi$ iff $\emptyset \vdash_{LJ^*_{\{\wedge,\vee,\odot,0,1\}}} \Gamma \Rightarrow \varphi$. This deductive system S coincides with the fragment without implication of the Intuitionistic Propositional Calculus. It is also known that the filters of any distributive lattice $\mathbf{A}$ are the S-filters. Let us suppose that $\Gamma \vdash_{S^0_{\mathcal{G}_{\wedge,\vee,\odot}}} \varphi$, then $\Gamma \vdash_S \varphi$ and as F is a S-filter we obtain that $h(\varphi) \in F$.
- Suppose that F is a $S^0_{\mathcal{G}_{\wedge,\vee,\odot}}$-filter. We have to see that F is a lattice filter of $\mathbf{A}$. We have that $\{\varphi, \psi\} \vdash_{S^0_{\mathcal{G}_{\wedge,\vee,\odot}}} \varphi \odot \psi$, since $\{\emptyset \Rightarrow \varphi, \emptyset \Rightarrow \psi\} \vdash_{LJ^*_{\{\wedge,\vee,\odot,0,1\}}\backslash c} \emptyset \Rightarrow \varphi \odot \psi$ and, therefore, if $\alpha, \beta \in A$ and $\alpha \in F$, $\beta \in F$, we have that $\alpha \odot \beta \in F$.
  We have that $\varphi \vdash_{S^0_{\mathcal{G}_{\wedge,\vee,\odot}}} \varphi \vee \psi$, since $\emptyset \Rightarrow \varphi \vdash_{LJ^*_{\mathcal{L}}\backslash c} \emptyset \Rightarrow \varphi \vee \psi$ and, therefore, if $\alpha, \beta \in A$ and $\alpha \leq \beta$ and $\alpha \in F$, we have that $\alpha \vee \beta \in F$, that is, $\beta \in F$. So, F is a lattice filter of $\mathbf{A}$.

Finally let us see that the Leibniz operator is not order-preserving on the $S^0_{\mathcal{G}_{\wedge,\vee,\odot}}$-filters of the algebra $\mathbf{A}$. Consider the $S^0_{\mathcal{G}_{\wedge,\vee,\odot}}$-filters $F_1 = \{1\}$ and $F_2 = \{1, a\}$ then we have that $(0, a) \in \Omega_{\mathbf{A}}(\{1\})$, but $(0, a) \notin \Omega_{\mathbf{A}}(\{1, a\})$. So $\Omega_{\mathbf{A}}(\{1\}) \not\subseteq \Omega_{\mathbf{A}}(\{1, a\})$. □

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