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LANDAU TYPE INEQUALITIES FOR BANACH SPACE VALUED FUNCTIONS

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Abstract. Motivated by the work of George A. Anastassiou in [George A. Anastassiou, Ostrowski and Landau inequalities for Banach space valued functions, Mathematical and Computer Modelling, 55 (2012), 312–329], we derive some Landau type inequalities for Banach space valued functions without assuming the boundary conditions.

1. Introduction

We firstly recall some classical results due to Landau [8].

Let $I=\mathbb{R}_+$ or $I=\mathbb{R}$. If $f:I\to\mathbb{R}$ is twice differentiable and $f,f''\in L^p(I)$, $p\in[1,\infty]$, then $f'\in L^p(I)$. Moreover, there exists a constant $C_p(I)>0$, which is independent of the function f, such that

$$\|f'\|_{p,I}\leqslant C_p(I)\|f\|_{p,I}^{1/2}\|f''\|_{p,I}^{1/2}, \tag{1}$$

where

$$\|f\|_{p,I}=\left(\int_I |f(x)|^p dx\right)^{\frac{1}{p}}$$

if $p\in[1,\infty)$ and $\|f\|_{\infty,I}=ess\sup_{x\in I}|f(x)|$ if $p=\infty$. Landau considered the case $p=\infty$ and proved that $C_\infty(\mathbb{R}_+)=2$ and $C_\infty(\mathbb{R})=\sqrt{2}$ are the best constants in (1). In 1932, G. H. Hardy and J. E. Littlewood [7] proved (1) for $p=2$, with the best constants $C_2(\mathbb{R}_+)=\sqrt{2}$ and $C_2(\mathbb{R})=1$. In 1935, G. H. Hardy, E. Landau and J. E. Littlewood [6] showed that the best constant C_p in (1) satisfies the estimate

$$C_p(\mathbb{R}_+)\leqslant 2$$

for $p\in[1,\infty)$, which yields $C_p(\mathbb{R})\leqslant 2$ for $p\in[1,\infty)$.

From now on, we let f be a function defined on $\mathbb{R}$ taking values in a real or complex Banach space $(X,\|\cdot\|)$. The following three Theorems can be found in [4].

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THEOREM 1. *Let $f \in C^2(I,X)$. If $|||f|||_{\infty,I}, |||f''|||_{\infty,I} < \infty$, then*

$$|||f'|||_{\infty,I} \leqslant 2\sqrt{|||f|||_{\infty,I}|||f''|||_{\infty,I}}$$

where $|||f|||_{\infty,I} := \sup_{x\in I}\|f(x)\|$.

THEOREM 2. *Let $f \in C^2(I,X)$ and $p,q > 1 : \frac{1}{p}+\frac{1}{q} = 1$. If $|||f|||_{\infty,\mathbb{R}}, |||f''|||_{p,\mathbb{R}} < \infty$, then*

$$|||f'|||_{\infty,I} \leqslant 2^{\frac{1}{q+1}}\left(\frac{q+1}{q}\right)^{\frac{q}{q+1}}|||f|||_{\infty,\mathbb{R}}^{\frac{1+q}{1+2q}}\cdot|||f''|||_{p,\mathbb{R}}^{\frac{q}{1+2q}},$$

where $|||f''|||_{p,\mathbb{R}} = (\int_{\mathbb{R}}\|f''(x)\|^p dx)^{\frac{1}{p}}$.

THEOREM 3. *Let $f \in C^{n+1}([a,b],X)$, $n\in\mathbb{N}$. Assume the boundary conditions $f^{(i)}(a) = f^{(i)}(b) = 0$ for $1 \leqslant i \leqslant n-1$ when $n \geqslant 2$. Let $p,q > 1$ satisfying $\frac{1}{p}+\frac{1}{q} = 1$. Denote by $B(\cdot,\cdot)$ the beta function. Then*

$$|||f'|||_{\infty,[a,b]} \leqslant \frac{2n|||f|||_{\infty,[a,b]}}{b-a}+\frac{1}{(n-1)!}\min\Big\{|||f^{(n+1)}|||_p(B(q+1,(n-1)q+1))^{\frac{1}{q}}(b-a)^{n-\frac{1}{p}},$$
$$|||f^{(n+1)}|||_\infty\frac{(b-a)^3}{n(n+1)}, |||f^{(n+1)}|||_1\frac{(n-1)!}{n^n}(b-a)^{n-1}\Big\},$$

In this paper, we derive some Landau type inequalities for $f \in C^3(\mathbb{R},X)$ or $f \in C^3([a,b],X)$, while we do not assume the boundary conditions $f'(a) = f'(b) = f''(a) = f''(b) = 0$. For more information about Landau inequality, we refer to [1, 2, 3, 4, 5] and the references therein.

2. Main results

Recall that if $f \in C^n(\mathbb{R},X)$, then the following vector valued Taylor's formula holds (see e.g. [9]):

$$f(y)-f(x)-f'(x)(y-x)-\cdots-\frac{f^{(n-1)}(x)}{(n-1)!}(y-x)^{n-1} = \frac{\int_x^y (y-t)^{n-1}f^{(n)}(t)dt}{(n-1)!}. \quad (2)$$

Putting $n = 3$ in (2), we obtain

$$f'(x)(y-x)+\frac{f''(x)}{2}(y-x)^2 = f(y)-f(x)-\frac{\int_x^y (y-t)^2 f'''(t)dt}{2}. \quad (3)$$

We note (3) yields the following system by choosing $y = y_1$ and $y = y_2$

$$\begin{cases} f'(x)(y_1-x)+\frac{f''(x)}{2}(y_1-x)^2 = f(y_1)-f(x)-\frac{\int_x^{y_1}(y_1-t)^2 f'''(t)dt}{2} \\ f'(x)(y_2-x)+\frac{f''(x)}{2}(y_2-x)^2 = f(y_2)-f(x)-\frac{\int_x^{y_2}(y_2-t)^2 f'''(t)dt}{2}. \end{cases} \quad (4)$$

The coefficient determinant D of system (4) is $D=\frac{1}{2}(y_1-x)(y_2-x)(y_2-y_1)$. If $D\neq 0$, we have

$$f'(x)=\frac{(y_2-x)(f(y_1)-f(x))}{(y_2-y_1)(y_1-x)}-\frac{(y_1-x)(f(y_2)-f(x))}{(y_2-y_1)(y_2-x)} \\ -\frac{(y_2-x)\int_x^{y_1}(y_1-t)^2f'''(t)dt}{2(y_1-x)(y_2-y_1)}+\frac{(y_1-x)\int_x^{y_2}(y_2-t)^2f'''(t)dt}{2(y_2-x)(y_2-y_1)} \tag{5}$$

and

$$f''(x)=\frac{2(f(y_2)-f(x))}{(y_2-y_1)(y_2-x)}-\frac{2(f(y_1)-f(x))}{(y_2-y_1)(y_1-x)}-\frac{\int_x^{y_1}(y_1-t)^2f'''(t)dt}{(y_2-x)(y_2-y_1)} \\ +\frac{\int_x^{y_2}(y_2-t)^2f'''(t)dt}{(y_1-x)(y_2-y_1)}. \tag{6}$$

Now we can establish some Landau type inequalities. The first Theorem is

THEOREM 4. *Let $f\in C^3(\mathbb{R},X)$. If $|||f|||_{\infty,\mathbb{R}}, |||f'''|||_{\infty,\mathbb{R}}<\infty$, then*

$$|||f'|||_{\infty,\mathbb{R}}\leqslant\sqrt[3]{\frac{9}{8}}|||f|||_{\infty,\mathbb{R}}^{\frac{2}{3}}|||f'''|||_{\infty,\mathbb{R}}^{\frac{1}{3}};$$

$$|||f''|||_{\infty,\mathbb{R}}\leqslant\sqrt[3]{3}|||f|||_{\infty,\mathbb{R}}^{\frac{1}{3}}|||f'''|||_{\infty,\mathbb{R}}^{\frac{2}{3}}.$$

Proof. Let $h>0$. Putting $y_1=x-h$ and $y_2=x+h$ in (5), we have

$$f'(x)=\frac{f(x+h)-f(x-h)}{2h}-\frac{1}{4h}\int_{x-h}^{x}(x-h-t)^2f'''(t)dt \\ -\frac{1}{4h}\int_x^{x+h}(x+h-t)^2f'''(t)dt. \tag{7}$$

Therefore,

$$\begin{aligned}\|f'(x)\|\leqslant&\frac{\|f(x+h)-f(x-h)\|}{2h}+\frac{1}{4h}\int_{x-h}^{x}(x-h-t)^2\|f'''(t)\|dt\\&+\frac{1}{4h}\int_x^{x+h}(x+h-t)^2\|f'''(t)\|dt\\ \leqslant&\frac{|||f|||_{\infty,\mathbb{R}}}{h}+|||f'''|||_{\infty,\mathbb{R}}\left(\frac{1}{4h}\int_{x-h}^{x}(x-h-t)^2dt+\frac{1}{4h}\int_{x-h}^{x}(x-h-t)^2dt\right)\\ =&\frac{|||f|||_{\infty,\mathbb{R}}}{h}+|||f'''|||_{\infty,\mathbb{R}}\cdot\frac{h^2}{6}.\end{aligned}$$

The function

$$g(h)=\frac{|||f|||_{\infty,\mathbb{R}}}{h}+|||f'''|||_{\infty,\mathbb{R}}\cdot\frac{h^2}{6},\ \ h>0$$

attains the minimal value only for $h_{\min} = \sqrt[3]{\frac{3|||f|||_{\infty,\mathbb{R}}}{|||f'''|||_{\infty,\mathbb{R}}}}$. Consequently

$$|||f'|||_{\infty,\mathbb{R}} \leqslant g(h_{\min}) = \sqrt[3]{\frac{9}{8}}|||f|||_{\infty,\mathbb{R}}^{\frac{2}{3}}|||f'''|||_{\infty,\mathbb{R}}^{\frac{1}{3}}.$$

Similarly, Putting $y_1 = x-h$ and $y_2 = x+h$ in (6), we have

$$\begin{aligned} f''(x) =& \frac{f(x+h)+f(x-h)-2f(x)}{h^2} - \frac{1}{2h^2}\int_{x-h}^{x}(x-h-t)^2 f'''(t)dt \\ & - \frac{1}{2h^2}\int_{x}^{x+h}(x+h-t)^2 f'''(t)dt. \end{aligned} \tag{8}$$

With the same argument, we get

$$\|f''(x)\| \leqslant \frac{4|||f|||_{\infty,\mathbb{R}}}{h^2} + |||f'''|||_{\infty,\mathbb{R}}\cdot\frac{h}{3}.$$

Therefore

$$|||f''|||_{\infty,\mathbb{R}} \leqslant \frac{4|||f|||_{\infty,\mathbb{R}}}{h^2} + |||f'''|||_{\infty,\mathbb{R}}\cdot\frac{h}{3}.$$

By choosing

$$h = \sqrt[3]{\frac{24|||f|||_{\infty,\mathbb{R}}}{|||f'''|||_{\infty,\mathbb{R}}}},$$

we have

$$|||f''|||_{\infty,\mathbb{R}} \leqslant \sqrt[3]{3}|||f|||_{\infty,\mathbb{R}}^{\frac{1}{3}}|||f'''|||_{\infty,\mathbb{R}}^{\frac{2}{3}}.$$

Our second Theorem is

THEOREM 5. *Let $f \in C^3(\mathbb{R},X)$ and $p,q>1 : \frac{1}{p}+\frac{1}{q}=1$. If $|||f|||_{\infty,\mathbb{R}}, |||f'''|||_{p,\mathbb{R}} < \infty$, then*

$$|||f'|||_{\infty,\mathbb{R}} \leqslant \left(\frac{1}{2q}\right)^{\frac{q}{1+2q}}\left(\frac{1}{1+q}\right)^{\frac{1+q}{1+2q}}(1+2q)^{\frac{2q}{1+2q}}|||f|||_{\infty,\mathbb{R}}^{\frac{1+q}{1+2q}}\cdot|||f'''|||_{p,\mathbb{R}}^{\frac{q}{1+2q}}$$

and

$$|||f''|||_{\infty,\mathbb{R}} \leqslant 4(1+2q)^{\frac{2q-1}{1+2q}}\left(\frac{1}{8q}\right)^{\frac{2q}{1+2q}}|||f|||_{\infty,\mathbb{R}}^{\frac{1}{1+2q}}\cdot|||f'''|||_{p,\mathbb{R}}^{\frac{2q}{1+2q}}.$$

where

$$|||f'''|||_{p,\mathbb{R}} = \left(\int_{\mathbb{R}}\|f'''(x)\|^p dx\right)^{\frac{1}{p}}.$$

Proof. By (7), we have, for $h>0$,

$$\begin{aligned} f'(x) =& \frac{f(x+h)-f(x-h)}{2h} - \frac{1}{4h}\int_{x-h}^{x}(x-h-t)^2 f'''(t)dt \\ & - \frac{1}{4h}\int_{x}^{x+h}(x+h-t)^2 f'''(t)dt. \end{aligned} \tag{9}$$

By Hölder's inequality,

$$\begin{aligned}\left\|\int_{x-h}^{x}(x-h-t)^2 f'''(t)dt\right\| &\leqslant \int_{x-h}^{x}(x-h-t)^2\|f'''(t)\|dt \\ &\leqslant \left(\int_{x-h}^{x}\|f'''(x)\|^p dx\right)^{\frac{1}{p}}\left(\int_{x-h}^{x}(x-h-t)^{2q}dt\right)^{\frac{1}{q}} \\ &\leqslant |||f'''|||_{p,\mathbb{R}}\cdot\left(\frac{1}{2q+1}\right)^{\frac{1}{q}} h^{\frac{2q+1}{q}}.\end{aligned} \tag{10}$$

Similarly,

$$\begin{aligned}\left\|\int_{x}^{x+h}(x+h-t)^2 f'''(t)dt\right\| &\leqslant \int_{x}^{x+h}(x+h-t)^2\|f'''(t)\|dt \\ &\leqslant \left(\int_{x}^{x+h}\|f'''(x)\|^p dx\right)^{\frac{1}{p}}\left(\int_{x}^{x+h}(x+h-t)^{2q}dt\right)^{\frac{1}{q}} \\ &\leqslant |||f'''|||_{p,\mathbb{R}}\cdot\left(\frac{1}{2q+1}\right)^{\frac{1}{q}} h^{\frac{2q+1}{q}}.\end{aligned} \tag{11}$$

Therefore, by (9),

$$\begin{aligned}\|f'(x)\| &\leqslant \frac{\|f(x+h)-f(x-h)\|}{2h}+\frac{1}{4h}|||f'''|||_{p,\mathbb{R}}\cdot\left(\frac{1}{2q+1}\right)^{\frac{1}{q}} h^{\frac{2q+1}{q}}\times 2 \\ &\leqslant \frac{|||f|||_{\infty,\mathbb{R}}}{h}+\frac{1}{2}\left(\frac{1}{2q+1}\right)^{\frac{1}{q}}|||f'''|||_{p,\mathbb{R}}h^{\frac{q+1}{q}}.\end{aligned} \tag{12}$$

Choosing $h=\left(\frac{2q}{1+q}\right)^{\frac{q}{1+2q}}(1+2q)^{\frac{1}{1+2q}}\left(\frac{|||f|||_{\infty,\mathbb{R}}}{|||f'''|||_{p,\mathbb{R}}}\right)^{\frac{q}{1+2q}}$ in (12), we have

$$\|f'(x)\| \leqslant \left(\frac{1}{2q}\right)^{\frac{q}{1+2q}}\left(\frac{1}{1+q}\right)^{\frac{1+q}{1+2q}}(1+2q)^{\frac{2q}{1+2q}}|||f|||_{\infty,\mathbb{R}}^{\frac{1+q}{1+2q}}\cdot|||f'''|||_{p,\mathbb{R}}^{\frac{q}{1+2q}}.$$

Consequently

$$|||f'|||_{\infty,\mathbb{R}} \leqslant \left(\frac{1}{2q}\right)^{\frac{q}{1+2q}}\left(\frac{1}{1+q}\right)^{\frac{1+q}{1+2q}}(1+2q)^{\frac{2q}{1+2q}}|||f|||_{\infty,\mathbb{R}}^{\frac{1+q}{1+2q}}\cdot|||f'''|||_{p,\mathbb{R}}^{\frac{q}{1+2q}}.$$

Next, by (8), we have, for $h>0$,

$$\begin{aligned}\|f''(x)\| \leqslant& \frac{\|f(x+h)+f(x-h)-2f(x)\|}{h^2}+\frac{1}{2h^2}\left|\int_{x-h}^{x}(x-h-t)^2 f'''(t)dt\right| \\ &+\frac{1}{2h^2}\left|\int_{x}^{x+h}(x+h-t)^2 f'''(t)dt\right| \\ \leqslant& \frac{4|||f|||_{\infty,\mathbb{R}}}{h^2}+\left(\frac{1}{2q+1}\right)^{\frac{1}{q}}|||f'''|||_{p,\mathbb{R}}h^{\frac{1}{q}}.\end{aligned} \tag{13}$$

To get the last inequality above, we use (10) and (11). Choosing $h = (8q)^{\frac{q}{1+2q}}(2q+1)^{\frac{1}{1+2q}}\left(\frac{|||f|||_{\infty,\mathbb{R}}}{|||f'''|||_{p,\mathbb{R}}}\right)^{\frac{q}{1+2q}}$ in (13), we obtain

$$\|f''(x)\| \leqslant 4(1+2q)^{\frac{2q-1}{1+2q}}\left(\frac{1}{8q}\right)^{\frac{2q}{1+2q}} |||f|||_{\infty,\mathbb{R}}^{\frac{1}{1+2q}} \cdot |||f'''|||_{p,\mathbb{R}}^{\frac{2q}{1+2q}}.$$

Consequently

$$|||f''|||_{\infty,\mathbb{R}} \leqslant 4(1+2q)^{\frac{2q-1}{1+2q}}\left(\frac{1}{8q}\right)^{\frac{2q}{1+2q}} |||f|||_{\infty,\mathbb{R}}^{\frac{1}{1+2q}} \cdot |||f'''|||_{p,\mathbb{R}}^{\frac{2q}{1+2q}}.$$

To prove the last Theorem, we need the following Lemma.

LEMMA 1. *Let* $h > 0$ *and* $p,q > 1 : \frac{1}{p}+\frac{1}{q} = 1$. *If* $f \in C^3([x-h,x+h],X)$, *then*

$$\left\|\int_x^{x+h}(x+h-t)^2 f'''(t)dt\right\| \leqslant \begin{cases} |||f'''|||_{\infty,[x,x+h]} \cdot \frac{h^3}{3}, \\ |||f'''|||_{1,[x,x+h]} \cdot h^2, \\ |||f'''|||_{p,[x,x+h]} \cdot \left(\frac{1}{2q+1}\right)^{\frac{1}{q}} h^{\frac{2q+1}{q}}; \end{cases} \tag{14}$$

$$\left\|\int_{x-h}^{x}(x+h-t)^2 f'''(t)dt\right\| \leqslant \begin{cases} |||f'''|||_{\infty,[x-h,x]} \cdot \frac{h^3}{3}, \\ |||f'''|||_{1,[x-h,x]} \cdot h^2, \\ |||f'''|||_{p,[x-h,x]} \cdot \left(\frac{1}{2q+1}\right)^{\frac{1}{q}} h^{\frac{2q+1}{q}}. \end{cases} \tag{15}$$

Proof. We have, by Hölder's inequality,

$$\left\|\int_x^{x+h}(x+h-t)^2 f'''(t)dt\right\| \leqslant \int_x^{x+h}(x+h-t)^2\|f'''(t)\|dt$$

$$\leqslant \begin{cases} |||f'''|||_{\infty,[x,x+h]} \cdot \int_x^{x+h}(x+h-t)^2 dt, \\ |||f'''|||_{1,[x,x+h]} \cdot \max_{t\in[x,x+h]}(x+h-t)^2, \\ |||f'''|||_{p,[x,x+h]} \cdot \left(\int_x^{x+h}(x+h-t)^{2q}dt\right)^{\frac{1}{q}}. \end{cases}$$

$$= \begin{cases} |||f'''|||_{\infty,[x,x+h]} \cdot \frac{h^3}{3}, \\ |||f'''|||_{1,[x,x+h]} \cdot h^2, \\ |||f'''|||_{p,[x,x+h]} \cdot \left(\frac{1}{2q+1}\right)^{\frac{1}{q}} h^{\frac{2q+1}{q}}. \end{cases}$$

The proof of inequality (15) is similar. In fact, we have

$$\left\|\int_{x-h}^{x}(x-h-t)^2 f'''(t)dt\right\| \leqslant \int_{x-h}^{x}(x-h-t)^2\|f'''(t)\|dt$$

$$\leqslant \begin{cases} |||f'''|||_{\infty,[x-h,x]} \cdot \int_{x-h}^{x}(x-h-t)^2 dt, \\ |||f'''|||_{1,[x-h,x]} \cdot \max_{t\in[x-h,x]}(x-h-t)^2, \\ |||f'''|||_{p,[x-h,x]} \cdot \left(\int_{x-h}^{x}(x-h-t)^{2q}dt\right)^{\frac{1}{q}} \end{cases} = \begin{cases} |||f'''|||_{\infty,[x-h,x]} \cdot \frac{h^3}{3}, \\ |||f'''|||_{1,[x-h,x]} \cdot h^2, \\ |||f'''|||_{p,[x-h,x]} \cdot \left(\frac{1}{2q+1}\right)^{\frac{1}{q}} h^{\frac{2q+1}{q}}. \end{cases}$$

Now we can prove the last Theorem.

THEOREM 6. *Let* $f \in C^3([a,b],X)$ *and* $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$|||f'|||_{\infty,[a,b]} \leqslant \min\left\{ \frac{5(b-a)^2}{81}|||f'''|||_{\infty,[a,b]}, \frac{5(b-a)}{9}|||f'''|||_{1,[a,b]}, \frac{(1+2^{\frac{q+1}{q}})(b-a)^{\frac{1+q}{q}}}{[3^{2q+1}(2q+1)]^{\frac{1}{q}}}|||f'''|||_{p,[a,b]} \right\} + \frac{12}{b-a}|||f|||_{\infty,[a,b]}$$

and

$$|||f''|||_{\infty,[a,b]} \leqslant \min\left\{ \frac{17(b-a)}{18}|||f'''|||_{\infty,[a,b]}, \frac{9}{2}|||f'''|||_{1,[a,b]}, |||f'''|||_{p,[a,b]}\frac{(1+2^{\frac{1+3q}{q}})(b-a)^{\frac{1}{q}}}{2[3(2q+1)]^{\frac{1}{q}}} \right\} + \frac{36}{(b-a)^2}|||f|||_{\infty,[a,b]}.$$

Proof. Step 1. We firstly assume $x \in [a, a+\frac{1}{3}(b-a)]$. Then

$$x + \frac{1}{3}(b-a),\ x + \frac{2}{3}(b-a) \in [a,b].$$

Putting $y_1 = x + \frac{1}{3}(b-a)$ and $y_2 = x + \frac{2}{3}(b-a)$ in (5) and (6), respectively, we obtain

$$f'(x) = \frac{3(4f(y_1) - 3f(x) - f(y_2))}{2(b-a)} - \frac{1}{b-a}\int_x^{x+\frac{b-a}{3}} \left(x + \frac{b-a}{3} - t\right)^2 f'''(t)dt + \frac{1}{2(b-a)}\int_x^{x+\frac{2(b-a)}{3}} \left(x + \frac{2(b-a)}{3} - t\right)^2 f'''(t)dt$$

and

$$f''(x) = \frac{9f(y_2) + 9f(x) - 18f(y_1)}{(b-a)^2} - \frac{9}{2(b-a)^2}\int_x^{x+\frac{b-a}{3}} \left(x + \frac{b-a}{3} - t\right)^2 f'''(t)dt + \frac{9}{(b-a)^2}\int_x^{x+\frac{2(b-a)}{3}} \left(x + \frac{2(b-a)}{3} - t\right)^2 f'''(t)dt.$$

Therefore,

$$\|f'(x)\| \leqslant \frac{12}{b-a}|||f|||_{\infty,[a,b]} + \frac{1}{b-a}\left\| \int_x^{x+\frac{b-a}{3}} \left(x + \frac{b-a}{3} - t\right)^2 f'''(t)dt \right\| + \frac{1}{2(b-a)}\left\| \int_x^{x+\frac{2(b-a)}{3}} \left(x + \frac{2(b-a)}{3} - t\right)^2 f'''(t)dt \right\|.$$

By Lemma 1, we have

$$\begin{aligned}
\|f'(x)\| \leqslant &\min\left\{|||f'''|||_{\infty,[a,b]}\cdot\frac{(b-a)^2}{81}, |||f'''|||_{1,[a,b]}\frac{b-a}{9}, |||f'''|||_{p,[a,b]}\frac{(b-a)^{\frac{1+q}{q}}}{[3^{2q+1}(2q+1)]^{\frac{1}{q}}}\right\}\\
&+\min\left\{|||f'''|||_{\infty,[a,b]}\cdot\frac{4(b-a)^2}{81}, |||f'''|||_{1,[a,b]}\frac{4(b-a)}{9},\right.\\
&\left.|||f'''|||_{p,[a,b]}\frac{2^{\frac{q+1}{q}}(b-a)^{\frac{1+q}{q}}}{[3^{2q+1}(2q+1)]^{\frac{1}{q}}}\right\}+\frac{12}{b-a}|||f|||_{\infty,[a,b]}\\
=&\min\left\{\frac{5(b-a)^2}{81}|||f'''|||_{\infty,[a,b]}, \frac{5(b-a)}{9}|||f'''|||_{1,[a,b]},\right.\\
&\left.\frac{(1+2^{\frac{q+1}{q}})(b-a)^{\frac{1+q}{q}}}{[3^{2q+1}(2q+1)]^{\frac{1}{q}}}|||f'''|||_{p,[a,b]}\right\}+\frac{12}{b-a}|||f|||_{\infty,[a,b]}.
\end{aligned}$$

Also by Lemma 1, we have

$$\begin{aligned}
\|f''(x)\| \leqslant &\frac{36}{(b-a)^2}|||f|||_{\infty,[a,b]}+\frac{9}{2(b-a)^2}\left\|\int_x^{x+\frac{b-a}{3}}\left(x+\frac{b-a}{3}-t\right)^2 f'''(t)dt\right\|\\
&+\frac{9}{(b-a)^2}\left\|\int_x^{x+\frac{2(b-a)}{3}}\left(x+\frac{2(b-a)}{3}-t\right)^2 f'''(t)dt\right\|\\
\leqslant &\min\left\{|||f'''|||_{\infty,[a,b]}\cdot\frac{b-a}{18}, \frac{1}{2}|||f'''|||_{1,[a,b]}, |||f'''|||_{p,[a,b]}\frac{(b-a)^{\frac{1}{q}}}{2[3(2q+1)]^{\frac{1}{q}}}\right\}\\
&+\min\left\{|||f'''|||_{\infty,[a,b]}\cdot\frac{8(b-a)}{9}, 4|||f'''|||_{1,[a,b]}, |||f'''|||_{p,[a,b]}\frac{2^{\frac{1+2q}{q}}(b-a)^{\frac{1}{q}}}{[3(2q+1)]^{\frac{1}{q}}}\right\}\\
&+\frac{36}{(b-a)^2}|||f|||_{\infty,[a,b]}\\
=&\min\left\{\frac{17(b-a)}{18}|||f'''|||_{\infty,[a,b]}, \frac{9}{2}|||f'''|||_{1,[a,b]}, |||f'''|||_{p,[a,b]}\frac{(1+2^{\frac{1+3q}{q}})(b-a)^{\frac{1}{q}}}{2[3(2q+1)]^{\frac{1}{q}}}\right\}\\
&+\frac{36}{(b-a)^2}|||f|||_{\infty,[a,b]}.
\end{aligned}$$

Step 2. Secondly, we assume $x \in [a+\frac{1}{3}(b-a), a+\frac{2}{3}(b-a)]$. Then $x+\frac{1}{3}(b-a), x-\frac{1}{3}(b-a) \in [a,b]$. Putting $y_1 = x-\frac{1}{3}(b-a)$ and $y_2 = x+\frac{1}{3}(b-a)$ in (5), we obtain

$$
\begin{aligned}
f'(x) =& \frac{3(f(y_2)-f(y_1))}{2(b-a)} - \frac{3}{4(b-a)} \int_{x-\frac{b-a}{3}}^{x} \left(x-\frac{b-a}{3}-t\right)^2 f'''(t)dt \\
& - \frac{3}{4(b-a)} \int_{x}^{x+\frac{b-a}{3}} \left(x+\frac{b-a}{3}-t\right)^2 f'''(t)dt.
\end{aligned}
$$

Therefore, by Lemma 1,

$$
\begin{aligned}
\|f'(x)\| \leqslant& \frac{3}{b-a}|||f|||_{\infty,[a,b]} + \frac{3}{4(b-a)} \left\| \int_{x-\frac{b-a}{3}}^{x} \left(x-\frac{b-a}{3}-t\right)^2 f'''(t)dt \right\| \\
& + \frac{3}{4(b-a)} \left\| \int_{x}^{x+\frac{b-a}{3}} \left(x+\frac{b-a}{3}-t\right)^2 f'''(t)dt \right\| \\
\leqslant& \min\left\{ |||f'''|||_{\infty,[a,b]} \cdot \frac{(b-a)^2}{108}, |||f'''|||_{1,[a,b]} \frac{b-a}{12}, |||f'''|||_{p,[a,b]} \frac{(b-a)^{\frac{1+q}{q}}}{4[3^{q+1}(2q+1)]^{\frac{1}{q}}} \right\} \\
& + \min\left\{ |||f'''|||_{\infty,[a,b]} \cdot \frac{(b-a)^2}{108}, |||f'''|||_{1,[a,b]} \frac{b-a}{12}, |||f'''|||_{p,[a,b]} \frac{(b-a)^{\frac{1+q}{q}}}{4[3^{q+1}(2q+1)]^{\frac{1}{q}}} \right\} \\
& + \frac{3}{b-a}|||f|||_{\infty,[a,b]} \\
=& \min\left\{ |||f'''|||_{\infty,[a,b]} \cdot \frac{(b-a)^2}{54}, |||f'''|||_{1,[a,b]} \frac{b-a}{6}, |||f'''|||_{p,[a,b]} \frac{(b-a)^{\frac{1+q}{q}}}{2[3^{q+1}(2q+1)]^{\frac{1}{q}}} \right\} \\
& + \frac{3}{b-a}|||f|||_{\infty,[a,b]}.
\end{aligned}
$$

Similarly, putting $y_1 = x-\frac{1}{3}(b-a)$ and $y_2 = x+\frac{1}{3}(b-a)$ in (6), we obtain

$$
\begin{aligned}
f''(x) =& \frac{9(f(y_1)+f(y_2)-2f(x))}{(b-a)^2} - \frac{9}{2(b-a)^2} \int_{x-\frac{b-a}{3}}^{x} \left(x-\frac{b-a}{3}-t\right)^2 f'''(t)dt \\
& + \frac{9}{2(b-a)^2} \int_{x}^{x+\frac{b-a}{3}} \left(x+\frac{b-a}{3}-t\right)^2 f'''(t)dt.
\end{aligned}
$$

Hence, following the prove above, we have

$$\|f''(x)\| \leqslant \frac{36}{(b-a)^2}|||f|||_{\infty,[a,b]} + \frac{9}{2(b-a)^2}\left\|\int_{x-\frac{b-a}{3}}^{x}\left(x-\frac{b-a}{3}-t\right)^2 f'''(t)dt\right\|$$
$$+\frac{9}{2(b-a)^2}\left\|\int_{x}^{x+\frac{b-a}{3}}\left(x+\frac{b-a}{3}-t\right)^2 f'''(t)dt\right\|.$$
$$\leqslant \min\left\{|||f'''|||_{\infty,[a,b]}\cdot\frac{b-a}{9}, |||f'''|||_{1,[a,b]}, |||f'''|||_{p,[a,b]}\frac{(b-a)^{\frac{1}{q}}}{[3(2q+1)]^{\frac{1}{q}}}\right\}$$
$$+\frac{36}{(b-a)^2}|||f|||_{\infty,[a,b]}.$$

Step 3. Lastly, We assume $x\in[a+\frac{2}{3}(b-a),b]$. Then $x-\frac{1}{3}(b-a), x-\frac{2}{3}(b-a)\in[a,b]$. Putting $y_1=x-\frac{1}{3}(b-a)$ and $y_2=x-\frac{2}{3}(b-a)$ in (5) and (6), respectively, we obtain

$$f'(x)=\frac{3(3f(x)+f(y_2)-4f(y_1))}{2(b-a)}-\frac{1}{b-a}\int_{x-\frac{b-a}{3}}^{x}\left(x-\frac{b-a}{3}-t\right)^2 f'''(t)dt$$
$$+\frac{1}{2(b-a)}\int_{x-\frac{2(b-a)}{3}}^{x}\left(x-\frac{2(b-a)}{3}-t\right)^2 f'''(t)dt$$

and

$$f''(x)=\frac{9f(y_2)+9f(x)-18f(y_1)}{(b-a)^2}+\frac{9}{2(b-a)^2}\int_{x-\frac{b-a}{3}}^{x}\left(x-\frac{b-a}{3}-t\right)^2 f'''(t)dt$$
$$-\frac{9}{(b-a)^2}\int_{x-\frac{2(b-a)}{3}}^{x}\left(x+\frac{2(b-a)}{3}-t\right)^2 f'''(t)dt.$$

Following the prove in *Step 1*, we have

$$\|f'(x)\| \leqslant \frac{12}{b-a}|||f|||_{\infty,[a,b]} + \frac{1}{b-a}\left\|\int_{x-\frac{b-a}{3}}^{x}\left(x-\frac{b-a}{3}-t\right)^2 f'''(t)dt\right\|$$
$$+\frac{1}{2(b-a)}\left\|\int_{x-\frac{2(b-a)}{3}}^{x}\left(x-\frac{2(b-a)}{3}-t\right)^2 f'''(t)dt\right\|$$
$$\leqslant \min\left\{\frac{5(b-a)^2}{81}|||f'''|||_{\infty,[a,b]}, \frac{5(b-a)}{9}|||f'''|||_{1,[a,b]},\right.$$
$$\left.\frac{(1+2^{\frac{q+1}{q}})(b-a)^{\frac{1+q}{q}}}{[3^{2q+1}(2q+1)]^{\frac{1}{q}}}|||f'''|||_{p,[a,b]}\right\}+\frac{12}{b-a}|||f|||_{\infty,[a,b]}.$$

and

$$\|f''(x)\| \leqslant \frac{36}{(b-a)^2}|||f|||_{\infty,[a,b]} + \frac{9}{2(b-a)^2}\left\|\int_{x-\frac{b-a}{3}}^{x}\left(x-\frac{b-a}{3}-t\right)^2 f'''(t)dt\right\|$$
$$+\frac{9}{(b-a)^2}\left\|\int_{x-\frac{2(b-a)}{3}}^{x}\left(x+\frac{2(b-a)}{3}-t\right)^2 f'''(t)dt\right\|$$
$$\leqslant \min\left\{\frac{17(b-a)}{18}|||f'''|||_{\infty,[a,b]}, \frac{9}{2}|||f'''|||_{1,[a,b]}, |||f'''|||_{p,[a,b]}\frac{(1+2^{\frac{1+3q}{q}})(b-a)^{\frac{1}{q}}}{2[3(2q+1)]^{\frac{1}{q}}}\right\}$$
$$+\frac{36}{(b-a)^2}|||f|||_{\infty,[a,b]}.$$

Step 4. Combining *Step 1*, *Step 2* and *Step 3*, we have, for all $x \in [a,b]$,

$$\|f'(x)\| \leqslant \min\left\{\frac{5(b-a)^2}{81}|||f'''|||_{\infty,[a,b]}, \frac{5(b-a)}{9}|||f'''|||_{1,[a,b]},\right.$$
$$\left.\frac{(1+2^{\frac{q+1}{q}})(b-a)^{\frac{1+q}{q}}}{[3^{2q+1}(2q+1)]^{\frac{1}{q}}}|||f'''|||_{p,[a,b]}\right\} + \frac{12}{b-a}|||f|||_{\infty,[a,b]}$$

and

$$\|f''(x)\| \leqslant \min\left\{\frac{17(b-a)}{18}|||f'''|||_{\infty,[a,b]}, \frac{9}{2}|||f'''|||_{1,[a,b]}, |||f'''|||_{p,[a,b]}\frac{(1+2^{\frac{1+3q}{q}})(b-a)^{\frac{1}{q}}}{2[3(2q+1)]^{\frac{1}{q}}}\right\}$$
$$+\frac{36}{(b-a)^2}|||f|||_{\infty,[a,b]}.$$

Consequently,

$$|||f'|||_{\infty,[a,b]} \leqslant \min\left\{\frac{5(b-a)^2}{81}|||f'''|||_{\infty,[a,b]}, \frac{5(b-a)}{9}|||f'''|||_{1,[a,b]},\right.$$
$$\left.\frac{(1+2^{\frac{q+1}{q}})(b-a)^{\frac{1+q}{q}}}{[3^{2q+1}(2q+1)]^{\frac{1}{q}}}|||f'''|||_{p,[a,b]}\right\} + \frac{12}{b-a}|||f|||_{\infty,[a,b]};$$
$$|||f''|||_{\infty,[a,b]} \leqslant \min\left\{\frac{17(b-a)}{18}|||f'''|||_{\infty,[a,b]}, \frac{9}{2}|||f'''|||_{1,[a,b]},\right.$$
$$\left.|||f'''|||_{p,[a,b]}\frac{(1+2^{\frac{1+3q}{q}})(b-a)^{\frac{1}{q}}}{2[3(2q+1)]^{\frac{1}{q}}}\right\} + \frac{36}{(b-a)^2}|||f|||_{\infty,[a,b]}. \quad \square$$

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