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On Double Continuous Monotonic Decomposition of Graphs

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ABSTRACT

Let G = (ν, ε) be a connected simple graph of order *p* and size *q*. If $H_1, H_2, \ldots, H_k$, $k \in \mathbb{N}$ are edge-disjoint subgraphs of G $\ni$ $\varepsilon(G) = \varepsilon(H_1) \cup \varepsilon(H_2) \cup \ldots \cup \varepsilon(H_k)$, then $H_1, H_2, \ldots H_k$ is said to be a *decomposition* of G. Ascending Subgraph Decomposition (ASD) is a decomposition of G into subgraph H_i (not necessarily connected) $\ni$ $|\varepsilon(H_i)| = i$ and is isomorphic to a proper subgraph of H_{i+1}. A decomposition, $\{H_1, H_2, \ldots H_k, \forall k \in \mathbb{N}\}$, is said to be a Continuous Monotonic Decomposition (CMD) if each H_i is connected and $|\varepsilon(H_i)| = i$ for each $i \in \mathbb{N}$. A connected graph G is said to accept Double Continuous Monotonic Decomposition (DCMD), if the graph can be decomposed, into two copies of $\{H_1, H_2, \ldots H_k\} \forall k \in \mathbb{N}$, of a graph G, if each H_i is connected and $|\varepsilon(H_i)| = i \; \forall i \in \mathbb{N}$. DCMD of Tensor Product of a Complete Graph and K_2.

Key words : Graph Theory, Complete Tripartite Graph, Continuous Monotonic Decomposition, Tensor Product of Graphs, Double Continuous Monotonic Decomposition, Triangular Numbers.

INTRODUCTION

An undirected graph with the property that there is a path between every pair of vertices is known as a *connected graph.* A graph G, referred to here is an undirected connected graph without loops or multiple edges. A graph G is called *n*-regular graph if *deg*(*v*) = $n \in \mathbb{N}$, $\forall \nu \in V(G)$. A complete graph with vertices $n \in \mathbb{N}$, denoted by K_n, is a connected simple graph with every vertex is connected with every other vertex by an edge.

A complete *m*-partite graph $G = K_{n1, n2,.....nm}$ $\forall$ $n_1, n_2,...n_m \in \mathbb{N}$ is a graph whose vertex set ν can be partitioned into *m* subsets $\nu_1, \nu_2,..., \nu_m$ such that every edge of G joins every vertex of ν_i with every vertex of ν_j where $i \neq j$ and $|\nu_i| = i$. When m = 2, G is a complete bipartite graph and m = 3, G is a complete tripartite graph. Terms not defined here are used in the sense of Harary[1].

Decomposition of Graphs

Let $G = (\nu, \varepsilon)$ be a connected simple graph of order *p* and size *q*. If $H_1, H_2,...H_k$ $\forall$ $k \in \mathbb{N}$ are edge-disjoint subgraphs of G $\ni$ $\varepsilon(G) = \varepsilon(H_1) \cup \varepsilon(H_2) \cup ..... \cup \varepsilon(H_k)$, then $H_1, H_2, ... H_k$ is said to be a ***decomposition*** of G. Different types of decomposition of G have been studied in the literature by imposing suitable conditions on the subgraphs H_i.

Continuous Monotonic Decomposition of Graphs

Alavi *et al.*[2], introduced Ascending Subgraph Decomposition (ASD) as a decomposition of G into subgraph H_i (not necessarily connected) $\ni |\varepsilon(H_i)| = i$ and is isomorphic to a proper subgraph of H_{i+1}. Gnana Dhas and Paulraj Joseph introduced a new concept known as continuous monotonic decomposition of graphs[3]. A decomposition, $\{H_1, H_2, ... H_k\}$ $\forall$ $k \in \mathbb{N}$, is said to be a **Continuous Monotonic Decomposition** (CMD) if each H_i is connected and $|\varepsilon(H_i)| = i$ $\forall$ $i \in \mathbb{N}$. If G admits a CMD, $\{H_3, H_4, ... H_k\}$ $\forall$ $k \in \mathbb{N}$, where each H_i is a cycle of length *i* in G, then we say that G admits **Continuous Monotonic Cycle Decomposition** (CMCD)[4]. A CMD in which each H_i is a star is said to be a **Continuous Monotonic Star Decomposition** (CMSD) and a CMD in which each H_i is a path is said to be a **Continuous Monotonic Path Decomposition** (CMPD)[3].

Example 1

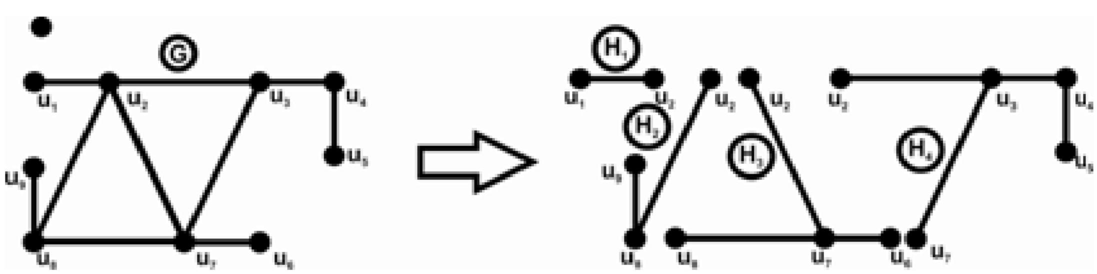

Fig. 1 Continuous Monotonic Decomposition of G into H_1, H_2, H_3, and H_4

Triangular number is a natural number that is the sum consecutive natural numbers, beginning with 1. Pythagoras found that number is triangular if and only if it is of the form $\frac{n(n+1)}{2}$ for some $n \geq 1$. Plutarch stated that n is a triangular number if and only if $8n+1$ is a perfect square. The square of any integer is either of the form 3k or 3k + 1 for some $k \in \mathbb{N}$.

Euler identified that if n is a triangular number, then so are $9n+1$, $25n+3$ and $49n+6$. If t_n denotes the n^{th} triangular number, then $t_n = {}^{(n+1)}C_2$. All these number theory results are used in the sense of David M. Burton[6].

Continuous Monotonic Decomposition of a wide variety of graphs had been studied by Gnana Dhas and Paulraj Joseph, and Navaneetha Krishnan and Nagarajan[3-5]. If a graph G admits a CMD $\{H_1, H_2, \ldots H_k\}$ $\forall k \in \mathbb{N}$ if and only if $q = {}^{(n+1)}C_2$ [3]. But we know that for any positive integer n, ${}^{(n+1)}C_2$ is a triangular number. Hence, if we are able to find out the number of the edges of any connected graph, it is easy for us to conclude whether it admits CMD or not. Joseph Varghese and A Antonysamy presented the necessary and sufficient condition for a collection of complete tripartite graphs to admit CMD[6].

The tensor product of the graphs G and H has vertex set V(G)xV(H) in which two vertices (g_1, h_1) and (g_2, h_2) are adjacent whenever $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. The tensor product is commutative and distributive over edge disjoint union of graphs. i.e., if $G = H_1 \oplus H_2 \oplus \ldots\ldots \oplus H_k$, then $G x H = (H_1 x H) \oplus (H_2 x H) \oplus \ldots \oplus (H_k x H)$.

Theorem 1 A complete graph K_m accepts Continuous Monotonic Decomposition of $H_1, H_2, \ldots, H_{m-1}$ $\forall m \in \mathbb{N}$.

Proof. As K_m has m(m-1)/2 edges and since it is a connected graph it can be decomposed into connected subgraphs $H_1, H_2, \ldots H_{m-1}$ whose sum of the edges is 1 + 2 + (m-1) = (m-1)m/2, the result is obvious.

Lemma: Tensor product of a Complete Graph K_m with K_2, produces an (m-1)-regular bipartite graph with 2m vertices.

Proof. Let the vertices of K_m and K_2 be $\{u_1, u_2, \ldots, u_m\}$ and $\{v_1, v_2\}$, respectively. Vertices of the Tensor Product K_m x K_2 are $\{u_iv_j, 1 \leq i \leq m, 1 \leq j \leq 2\}$.

i.e., $\nu (K_m \text{ x } K_2) = i.j$, $1 \leq i \leq m$, $1 \leq j \leq 2 = 2m$

Now, in the Tensor Product,

(u_iv_j, u_kv_j) is not an edge $\forall$ $1 \leq i, k \leq m$ and for a fixed j, since (v_j, v_j) is not an edge in K_2 for fixed j. These are 'm' in number.

Also, (u_iv_1, u_iv_2) is not an edge for fixed i, since (u_i, u_i) is not an edge in K_m for a fixed i. For a fixed i, there is one vertex like this.

All the other vertices of the Tensor Product are connected to each other.

i.e., if we enlist the 2m vertices as $\{u_1v_1, u_2v_1, \ldots\ldots, u_mv_1, u_1v_2, u_2v_2, \ldots\ldots, u_mv_2\}$, each vertex is therefore connected to m-1 vertices by an edge. Hence, the Tensor Product is an (m-1)-regular graph.

Also, we can notice that the vertices can be partitioned into $\{u_1v_1, u_2v_1, \ldots\ldots, u_mv_1\}$ and $\{u_1v_2, u_2v_2, \ldots\ldots, u_mv_2\}$ such that none of the vertices in each set is connected to any other vertex in the same set by an edge.

Hence the Tensor Product of a Complete Graph K_m with K_2 is also a bipartite graph.

Double Continuous Monotonic Decomposition

***Definition**:* A connected graph G is said to accept **Double Continuous Monotonic Decomposition,** if the graph can be decomposed, into two copies of $\{H_1, H_2, \ldots\ldots H_k\}$ $\forall$ $k \in \mathbb{N}$, of a graph G, if each H_i is connected and $|\varepsilon(H_i)| = i$ $\forall$ $i \in \mathbb{N}$.

Theorem 2 Tensor product of a Complete Graph K_m with K_2, accepts Double Continuous Monotonic Decomposition for all $m > 1$.

Proof. By **Lemma**, we have seen that Tensor product of a Complete Graph K_m with K_2, produces an (m-1)-regular bipartite graph with 2m vertices.

Therefore, the total number of edges in a Tensor Product of a Complete Graph K_m with K_2, is m(m-1). *i.e.*, when $m > 1$, this is nothing but twice a triangular number.

But a graph with the number of edges as a triangular number accepts a Continuous Monotonic Decomposition. Hence a Tensor Product of a Complete Graph K_m with K_2 accepts a Double Continuous Monotonic Decomposition.

The Decomposition is as follows:

For $m > 1$, $K_m \times K_2$, can be decomposed into $\{H_1, H_2, \ldots\ldots H_{(m-1)}\}$ and $\{H_1, H_2, \ldots\ldots H_{(m-1)}\}$.

Example 2

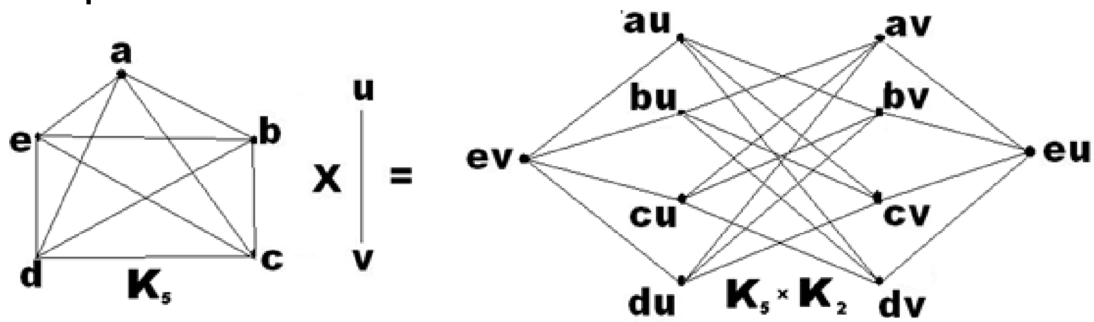

Fig 2: Tensor Product K_5 x K_2

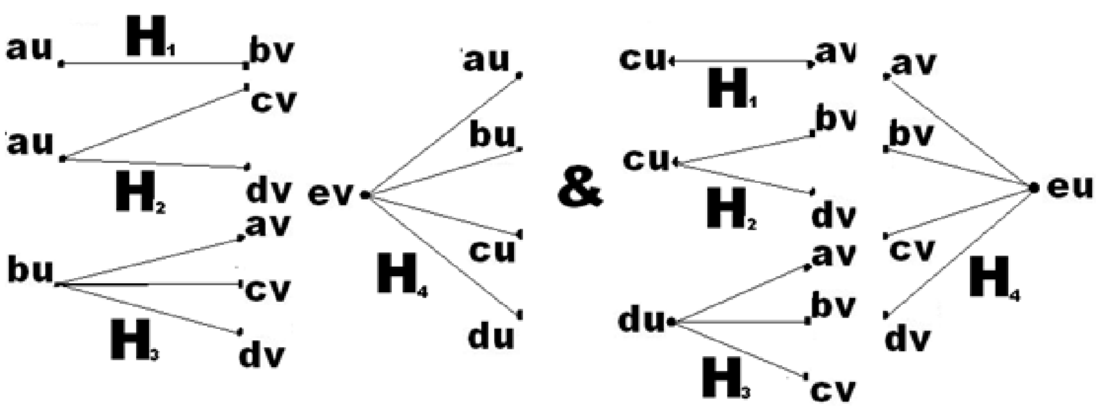

Fig 3: Double Continuous Monotonic Decomposition of K_5 x K_2

REMARK

Though a Tensor Product of a Complete Graph K_m with K_2 accepts a Double Continuous Monotonic Decomposition for all $m > 1$, it could be easily seen that some it accepts a Continuous Monotonic Decomposition for some m. This is found out be analyzing the triangular number series we obtain in the Tensor Product.

In fact, in the Tensor Product of a Complete Graph K_m with K_2 we obtain the edge series as twice of the triangular number series. But some in some cases, twice a triangular is also a triangular number.

Hence, for m = 3, 15, 85, 493, 2871, 16731 and 97513 the Tensor Product of a Complete Graph K_m with K_2 accepts Continuous Monotonic Decomposition. These are the only values for $1 < m < 500000$, where the Tensor Product of a Complete Graph K_m with K_2 accepts Continuous Monotonic Decomposition.

m	Tensor Product	CMD
3	K_3xK_2	$H_1, H_2, \ldots, H_3$
15	$K_{15}xK_2$	$H_1, H_2, \ldots, H_{20}$
85	$K_{85}xK_2$	$H_1, H_2, \ldots, H_{119}$
493	$K_{493}xK_2$	$H_1, H_2, \ldots, H_{696}$
2871	$K_{2871}xK_2$	$H_1, H_2, \ldots H_{4059}$
16731	$K_{16731}xK_2$	$H_1, H_2, \ldots H_{23660}$
97513	$K_{97513}xK_2$	$H_1, H_2, \ldots H_{137903}$

List of K_mxK_2's which accept CMD for $1 < m < 500,000$ and their Decomposition.

CONCLUSION

The result described above list classes of Tensor Product of Graphs which accept Double Continuous Monotonic Decomposition. Finding the size of the graph is the major task in the process. The study can be extended to different products and tensor products.

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