

## PORE FLUID EFFECTS ON SHEAR MODULUS FOR SANDSTONES WITH SOFT ANISOTROPY

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### ABSTRACT

A general analysis of poroelasticity for vertical transverse isotropy (VTI) shows that four eigenvectors are pure shear modes with no coupling to the pore-fluid mechanics. The remaining two eigenvectors are linear combinations of pure compression and uniaxial shear, both of which are coupled to the fluid mechanics. After reducing the problem to a  $2 \times 2$  system, the analysis shows in a relatively elementary fashion how a poroelastic system with isotropic solid elastic frame, but with anisotropy introduced through the poroelastic coefficients, interacts with the mechanics of the pore fluid and produces shear dependence on fluid properties in the overall mechanical system. The analysis shows, for example, that this effect is always present (though sometimes small in magnitude) in the systems studied, and can be quite large (up to a definite maximum increase of 20 per cent) in some rocks — including Spirit River sandstone and Schuler-Cotton Valley sandstone.

### INTRODUCTION

An important paper by Gassmann [1] concerns the effects of fluids on the mechanical properties of porous rock. His main result is the well-known fluid-substitution formula (that now bears his name) for the bulk modulus in undrained, isotropic poroelastic media. He also postulated that the effective shear modulus would be independent of the mechanical properties of the fluid when the medium is isotropic. That the independence of shear modulus from fluid effects is guaranteed for isotropic me-

dia at very low or quasistatic frequencies was shown recently by Berryman [2] to be tightly coupled to the original bulk modulus result of Gassmann; each result implies the other in isotropic media. It has gone mostly without discussion in the literature that Gassmann [1] also derived general results for anisotropic porous rocks in the same 1951 paper. It is not hard to see that these results imply that, contrary to the isotropic case, the overall undrained shear modulus in fact generally does depend on fluid properties in anisotropic media. However, Gassmann's paper does not remark at all on this difference in behavior between isotropic and anisotropic porous rocks. Brown and Korringa [3] also address the same class of problems, including both isotropic and anisotropic cases, but again they do not remark on the shear modulus results in either case. Norris [4] studies partial saturation in isotropic layered materials in the low-frequency regime ( $\approx 100$  Hz) and takes as a fundamental postulate that Gassmann's results hold for the low frequency shear modulus, but it seems that some justification should be provided for such an assumption, and furthermore some indication of its range of validity established.

On the other hand, Hudson [5], in his early work on cracked solids, explicitly demonstrates differences between fluid-saturated and dry cracks and relates his work to that of Walsh [6] and O'Connell and Budiansky [7], but does not make any connection to the work of either Gassmann [1], or Brown and Korringa [3]. Mukerji and Mavko [8] show numerical results based on work of Gassmann [3], Brown and Korringa [3] and Hudson [5] demonstrating the fluid dependence of shear in

anisotropic rock, but again they do not remark on these results at all. Mavko and Jizba [9] use a simple reciprocity argument to establish a direct, but approximate, connection between undrained shear response and undrained compressional response in rocks containing cracks. Berryman and Wang [10] show that deviations from Gassmann's results sufficient to produce shear modulus dependence on fluid mechanical properties require the presence of anisotropy on the microscale, thereby explicitly violating the microhomogeneous and microisotropy conditions implicit in Gassmann's original derivation. Berryman *et al.* [11] go further and make use of differential effective medium analysis to show explicitly how the undrained, overall isotropic shear modulus can depend on fluid trapped in penny-shaped cracks. Meanwhile, laboratory results (see Berryman *et al.* [12]) show conclusively that the shear modulus does depend on fluid mechanical properties for low-porosity, low-permeability rocks, and high-frequency laboratory experiments ( $f > 500$  kHz).

One thing lacking from all the preceding work is a simple example showing how the presence of anisotropy influences the shear modulus, and specifically when and how the shear modulus becomes fluid dependent. Our main purpose in the present work is therefore to demonstrate, in a set of rather simple examples, how the shear behavior becomes dependent on fluid properties in anisotropic media — even though overall shear modulus is always independent of the fluid properties in microhomogeneous isotropic media at sufficiently low frequencies, whether drained or undrained. Two other distinct but related analyses addressing this topic have been presented recently by the author [13,14]. Both of these prior papers have made explicit use of layered media, composed of isotropic poroelastic materials, together with exact results for such media based on Backus averaging [15]. In contrast, the present analysis does *not* make use of such a specific model, and is therefore believed to be about as simple as possible, while still achieving the level of understanding desired for this rather subtle technical issue. One important simplification we make here in order to separate what part is due to poroelastic effects, and what part would be present in any elastic (*i.e.*, possibly zero permeability porous medium) is to model each material as if the elastic part is entirely isotropic, while the poroelastic effects (*i.e.*, the Biot-Willis coefficients [16] for the anisotropic overall material) supply the only source of anisotropy in the system. Thus, we specifically distinguish two possible sources of anisotropy, the elastic or “hard” anisotropy that is assumed not to be present here, and the poroelastic or “soft” anisotropy that is the source of all the effects we want to study in this paper.

Our analysis for general transversely isotropic media is presented in the next three sections. In particular the “eigenvectors” section also introduces the effective undrained shear modulus relevant to our general discussion. Examples are then presented for two sandstones. The paper's results and conclusions are summarized in the final section.

## FLUID-SATURATED POROELASTIC MEDIA

In contrast to traditional elastic analysis, the presence in rock of a saturating pore fluid introduces the possibility of an additional control field and an additional type of strain variable. The pressure  $p_f$  in the fluid is a new field parameter that can be controlled. Allowing sufficient time for global pressure equilibration will permit us to consider  $p_f$  to be a constant throughout the percolating (connected) pore fluid, while restricting the analysis to quasistatic processes. The change  $\zeta$  in the amount of fluid mass contained in the pores (see Biot [17] or Berryman and Thigpen [18]) is a new type of strain variable, measuring how much of the original fluid in the pores is squeezed out during the compression of the pore volume while including the effects of compression or expansion of the pore fluid itself due to changes in  $p_f$ . It is most convenient to write the resulting equations in terms of compliances rather than stiffnesses, so the basic equation to be considered takes the following form for isotropic media:

$$\begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ -\zeta \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & s_{12} & -\beta \\ s_{12} & s_{11} & s_{12} & -\beta \\ s_{12} & s_{12} & s_{11} & -\beta \\ -\beta & -\beta & -\beta & \gamma \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ -p_f \end{pmatrix}, \quad (1)$$

where  $e_{ij}$  and  $\sigma_{ij}$  for  $i, j = 1, 2, 3$  are the components of overall strain and stress, respectively, in 3D. The constants appearing in the matrix on the right hand side will be defined in the following two paragraphs. It is important to write the equations this way rather than using the inverse relation in terms of the stiffnesses, because the compliances  $s_{ij}$  appearing in (1) are simply related to the drained elastic constants  $\lambda_{dr}$  and  $G_{dr}$  in the same way they are related in normal elasticity, whereas the individual stiffnesses obtained by inverting the equation in (1) must contain coupling terms through the parameters  $\beta$  and  $\gamma$  that depend on the pore and fluid compliances. Thus, we find that

$$s_{11} = \frac{1}{E_{dr}} = \frac{\lambda_{dr} + G_{dr}}{G_{dr}(3\lambda_{dr} + 2G_{dr})} \quad (2)$$

and

$$s_{12} = -\frac{\nu_{dr}}{E_{dr}}, \quad (3)$$

where the drained Young's modulus  $E_{dr}$  is defined by the second equality of (2) and the drained Poisson's ratio is determined by

$$\nu_{dr} = \frac{\lambda_{dr}}{2(\lambda_{dr} + G_{dr})}. \quad (4)$$

When the external stress is hydrostatic so  $\sigma = \sigma_{11} = \sigma_{22} = \sigma_{33}$ , equation (1) telescopes down to

$$\begin{pmatrix} e \\ -\zeta \end{pmatrix} = \begin{pmatrix} 1/K_{dr} & -\alpha/K_{dr} \\ -\alpha/K_{dr} & \alpha/BK_{dr} \end{pmatrix} \begin{pmatrix} \sigma \\ -p_f \end{pmatrix}, \quad (5)$$

where  $e = e_{11} + e_{22} + e_{33}$ ,  $K_{dr} = \lambda_{dr} + \frac{2}{3}G_{dr}$  is the drained bulk modulus,  $\alpha = 1 - K_{dr}/K_m$  is the Biot-Willis parameter [16] with  $K_m$  being the bulk modulus of the solid minerals present, and Skempton's pore-pressure buildup parameter  $B$  [19] is given by

$$B = \frac{1}{1 + K_p(1/K_f - 1/K_m)}. \quad (6)$$

New parameters appearing in (6) are the bulk modulus of the pore fluid  $K_f$  and the pore modulus  $K_p^{-1} = \alpha/\phi K_{dr}$  where  $\phi$  is the porosity. The expressions for  $\alpha$  and  $B$  can be generalized slightly by supposing that the solid frame is composed of more than one constituent, in which case the  $K_m$  appearing in the definition of  $\alpha$  is replaced by  $K_s$  and the  $K_m$  appearing explicitly in (6) is replaced by  $K_\phi$  (see Brown and Korringa [3], Rice and Cleary [20], Berryman and Wang [21]). This is an important additional complication [22], but — for the sake of desired simplicity — we will not pursue the matter further here.

Comparing (1) and (5), we find that

$$\beta = \frac{\alpha}{3K_{dr}} \quad (7)$$

and

$$\gamma = \frac{\alpha}{BK_{dr}}. \quad (8)$$

## RELATIONS FOR ANISOTROPY IN POROELASTIC MATERIALS

Gassmann [1], Brown and Korringa [3], and many others have considered the problem of obtaining effective constants for anisotropic poroelastic materials when the pore fluid is confined within the pores. The confinement condition amounts to a constraint that the increment of fluid content  $\zeta = 0$ , while the external loading  $\sigma$  is changed and the pore-fluid pressure  $p_f$  is allowed to respond as necessary and thus equilibrate.

To recall an elementary derivation of the Gassmann equation for anisotropic materials, we consider the anisotropic generalization of (1)

$$\begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ -\zeta \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & s_{13} & -\beta_1 \\ s_{12} & s_{22} & s_{23} & -\beta_2 \\ s_{13} & s_{23} & s_{33} & -\beta_3 \\ -\beta_1 & -\beta_2 & -\beta_3 & \gamma \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ -p_f \end{pmatrix}. \quad (9)$$

Three shear contributions have been immediately excluded from consideration since they can easily be shown not to interact mechanically with the fluid effects. This form is not completely general in that it includes only orthorhombic, cubic, hexagonal, and all isotropic systems. Also, we have assumed that the material axes are aligned with the spatial axes. But this latter assumption is not significant for the derivation that follows. Such an assumption is important when properties of laminated materials having arbitrary orientation relative to the spatial axes need to be considered, but we do not treat this more general problem here.

If the fluid is confined (or undrained on the time scales of interest to applications in high frequency wave propagation), then  $\zeta \equiv 0$  in (9) and  $p_f$  becomes a linear function of  $\sigma_{11}$ ,  $\sigma_{22}$ ,  $\sigma_{33}$ . Eliminating  $p_f$  from the resulting equations, we obtain the general expression for the strain dependence on external stress under such undrained conditions:

$$\begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \end{pmatrix} = \left[ \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{12} & s_{22} & s_{23} \\ s_{13} & s_{23} & s_{33} \end{pmatrix} - \gamma^{-1} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} (\beta_1 \ \beta_2 \ \beta_3) \right] \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{pmatrix} \\ \equiv \begin{pmatrix} s_{11}^* & s_{12}^* & s_{13}^* \\ s_{12}^* & s_{22}^* & s_{23}^* \\ s_{13}^* & s_{23}^* & s_{33}^* \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{pmatrix}. \quad (10)$$

The  $s_{ij}$ 's are fluid-drained constants, while the  $s_{ij}^*$ 's are the fluid-undrained (or fluid-confined) constants.

The fundamental result (10) was obtained earlier by both Gassmann [1] and Brown and Korringa [3], and may be written simply as

$$s_{ij}^* = s_{ij} - \frac{\beta_i \beta_j}{\gamma}, \quad \text{for } i, j = 1, 2, 3. \quad (11)$$

This expression is just the anisotropic generalization of the well-known Gassmann equation for isotropic, microhomogeneous porous media.

## EIGENVECTORS FOR TRANSVERSE ISOTROPY

The  $3 \times 3$  system (10) can be analyzed fairly easily, and in particular the eigenfunctions and eigenvalues of this system can be obtained in general. However, such general results do not provide much physical insight into the problem we are trying to study, so instead of proceeding in this direction we will now restrict attention to transversely isotropic materials. This case is relevant to many layered earth materials and also industrial systems, and it is convenient because we can immediately eliminate one of the eigenvectors from further consideration. Three mutu-

ally orthogonal (but unnormalized) vectors of interest are:

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}. \quad (12)$$

Treating these vectors as stresses, the first corresponds to a simple hydrostatic stress, the second to a planar shear stress, and the third to a pure shear stress applied uniaxially along the  $z$ -axis (also the same as the symmetry axis for the layered system). Transverse isotropy of the layered system requires:  $s_{11} = s_{22}$ ,  $s_{13} = s_{23}$ , and for the poroelastic problem  $\beta_1 = \beta_2$ . Thus, it is immediately apparent that the planar shear stress  $v_2$  is an eigenvector of the system, and furthermore it results in no contribution from the pore fluid. Therefore, this vector will be of no further interest here, and the system can thereby be reduced to  $2 \times 2$ .

### Compliance Formulation

If we define the effective compliance matrix for the system as  $S^*$  having the matrix elements given in (11), then the bulk modulus for this system is defined in terms of  $v_1$  by

$$\frac{1}{K_u} = v_1^T S^* v_1 = \frac{1}{K_{dr}} - \gamma^{-1} (2\beta_1 + \beta_3)^2, \quad (13)$$

where the  $T$  superscript indicates the transpose, and  $1/K_{dr} \equiv \sum_{i,j=1}^3 s_{ij}$ . This is the result usually quoted as Gassmann's equation for the bulk modulus of the undrained (or confined) anisotropic (VTI) system. Also, note that in general

$$\sum_{i=1}^3 \beta_i = 2\beta_1 + \beta_3 = \alpha/K_{dr}. \quad (14)$$

Thus, even though  $v_1$  is not an eigenvector of this system, it nevertheless plays a fundamental role in the mechanics. Furthermore, this role is quite well-understood. What is perhaps not so well-understood then, especially for poroelastic systems, is the role of  $v_3$ . Understanding this role will become our main focus for the remainder of this discussion.

The true eigenvectors of the subproblem of interest (*i.e.*, in the space orthogonal to the four pure shear eigenvectors already discussed) are necessarily linear combinations of  $v_1$  and  $v_3$ . We can construct the relevant contracted operator for the  $2 \times 2$  subsystem by considering:

$$\begin{pmatrix} v_1^T \\ v_3^T \end{pmatrix} S^* \begin{pmatrix} v_1 \\ v_3 \end{pmatrix} \equiv \begin{pmatrix} 9A_{11}^* & 18A_{13}^* \\ 18A_{13}^* & 36A_{33}^* \end{pmatrix} \quad (15)$$

(in all cases the \* superscripts indicate that the pore-fluid effects are included) and the reduced matrix

$$\Sigma^* = A_{11}^* v_1 v_1^T + A_{13}^* (v_1 v_3^T + v_3 v_1^T) + A_{33}^* v_3 v_3^T, \quad (16)$$

where

$$\begin{aligned} A_{11}^* &= [2(s_{11}^* + s_{12}^* + 2s_{13}^*) + s_{33}^*]/9, \\ A_{13}^* &= (s_{11}^* + s_{12}^* - s_{13}^* - s_{33}^*)/9, \\ A_{33}^* &= (s_{11}^* + s_{12}^* - 4s_{13}^* + 2s_{33}^*)/18. \end{aligned} \quad (17)$$

Providing some understanding of these connections and the implications for shear modulus dependence on fluid content is one purpose for this work.

First we remark that  $A_{11}^* = 1/9K_u$ , where  $K_u$  is again the undrained (or Gassmann) bulk modulus for the system in (13). Therefore,  $A_{11}^*$  is proportional to the undrained bulk compliance of this system. The other two matrix elements cannot be given such simple interpretations in general. To simplify the analysis we note that, at least for purposes of modeling, anisotropy of the compliances  $s_{ij}$  and the poroelastic coefficients  $\beta_i$  can be treated independently. Anisotropy displayed in the  $s_{ij}$ 's corresponds mostly to the anisotropy in the solid elastic components of the system, while anisotropy in the  $\beta_i$ 's corresponds mostly to anisotropy in the shapes and spatial distribution of the porosity. We will therefore distinguish these contributions by calling anisotropy appearing in the  $s_{ij}$ 's the "hard anisotropy," and the anisotropy in the  $\beta_i$ 's will in contrast be called the "soft anisotropy."

Now, it is clear that the eigenvectors  $f(\theta)$  for this problem (*i.e.*, for the reduced operator  $\Sigma^*$ ) necessarily take the form

$$f(\theta) = \frac{\cos \theta}{\sqrt{3}} v_1 + \frac{\sin \theta}{\sqrt{6}} v_3, \quad (18)$$

with two solutions for the rotation angle:  $\theta_-$  and  $\theta_+ = \theta_- + \frac{\pi}{2}$ , guaranteeing that the two solutions (the eigenvectors) are orthogonal. It is easily seen that the eigenvalues are given by

$$\Lambda_{\pm}^* = 3 \left[ A_{33}^* + A_{11}^*/2 \pm \sqrt{(A_{33}^* - A_{11}^*/2)^2 + 2(A_{13}^*)^2} \right] \quad (19)$$

and the rotation angles are determined by

$$\tan \theta_{\pm}^* = \frac{\Lambda_{\pm}^*/3 - A_{11}^*}{\sqrt{2}A_{13}^*} = \frac{A_{33}^* - A_{11}^*/2 \pm \sqrt{(A_{33}^* - A_{11}^*/2)^2 + 2(A_{13}^*)^2}}{\sqrt{2}A_{13}^*}. \quad (20)$$

One part of the rotation angle is due to the drained (fluid free) “hard anisotropic” nature of the rock frame material. We will call this part  $\bar{\theta}$ . The remainder is due to the presence of the fluid in the pores, and we will call this part  $\delta\theta \equiv \theta^* - \bar{\theta}$  for the “soft anisotropy.” Using a standard formula for tangents, we have

$$\delta\theta_{\pm} = \tan^{-1} \left[ \frac{\tan \theta_{\pm}^* - \tan \bar{\theta}_{\pm}}{1 + \tan \theta_{\pm}^* \tan \bar{\theta}_{\pm}} \right]. \quad (21)$$

Furthermore, definite formulas for  $\bar{\theta}_{\pm}$  are found from (20) by taking  $\gamma \rightarrow \infty$  (corresponding to air saturation of the pores).

Since

$$\tan \theta_+^* \cdot \tan \theta_-^* = -1, \quad (22)$$

it is sufficient to consider just one of the signs in front of the radical in (20). The most convenient choice for analytical purposes turns out to be the minus sign (which corresponds to the eigenvector with the larger component of pure compression). Furthermore, it is also clear from the form of (20) that often the behavior of most interest to us here occurs for cases when  $A_{13}^* \neq 0$ .

In the limit of a nearly isotropic solid frame (so the “hard anisotropy” vanishes and thus we will also call this the “quasi-isotropic” limit), it is not hard to see that

$$A_{33}^* \simeq \frac{1}{12G_{dr}} - \frac{(\beta_1 - \beta_3)^2}{9\gamma}, \quad (23)$$

where  $G_{dr}$  is the drained shear modulus of the quasi-isotropic solid frame. Similarly, the remaining coefficient

$$A_{13}^* \simeq -\frac{(\beta_1 - \beta_3)(2\beta_1 + \beta_3)}{9\gamma}, \quad (24)$$

since all the solid contributions approximately cancel in this limit.

Expanding the square root in (19), we also have

$$\Lambda_+^* = 6A_{33}^* + \Delta \quad \text{and} \quad \Lambda_-^* = 3A_{11}^* - \Delta, \quad (25)$$

where  $\Delta$  is defined consistently by either of the two preceding expressions or by  $2\Delta \equiv \Lambda_+^* - \Lambda_-^* + 3A_{11}^* - 6A_{33}^*$ .

### Stiffness formulation

The dual to the problem just studied replaces compliances everywhere with stiffnesses, and then proceeds as before. Equations (15)–(18) are replaced by

$$\begin{pmatrix} v_1^T \\ v_3^T \end{pmatrix} C^* \begin{pmatrix} v_1 \\ v_3 \end{pmatrix} \equiv \begin{pmatrix} 9B_{11}^* & 18B_{13}^* \\ 18B_{13}^* & 36B_{33}^* \end{pmatrix} \quad (26)$$

(in all cases the \* superscripts indicate that the pore-fluid effects are included) and the reduced matrix

$$(\Sigma^*)^{-1} = B_{11}^* v_1 v_1^T + B_{13}^* (v_1 v_3^T + v_3 v_1^T) + B_{33}^* v_3 v_3^T, \quad (27)$$

where

$$\begin{aligned} B_{11}^* &= [2(c_{11}^* + c_{12}^* + 2c_{13}^*) + c_{33}^*]/9, \\ B_{13}^* &= (c_{11}^* + c_{12}^* - c_{13}^* - c_{33}^*)/9, \\ B_{33}^* &= (c_{11}^* + c_{12}^* - 4c_{13}^* + 2c_{33}^*)/18. \end{aligned} \quad (28)$$

It is a straightforward exercise to check that the two reduced problems are in fact inverses of each other. We will not repeat this analysis here, as it is wholly repetitive of what has gone before. The main difference in the details is that the expressions for the  $B$ 's in terms of the  $\beta$ 's are rather more complicated than those for the compliance version, which is also why we chose to display the compliance formulation instead.

### Effective and undrained shear moduli $G_{eff}$ and $G_u$

Four shear moduli are easily defined for the anisotropic system under study. Furthermore,  $G_i = G_{dr}$  for  $i = 1, \dots, 4$ , since these are all related to the four shear eigenvectors of the systems, and these do not couple to the pore-fluid mechanics. But, the eigenvectors in the reduced  $2 \times 2$  system studied here are usually mixed in character, being quasi-compressional or quasi-shear modes. It is therefore somewhat problematic to find a proper definition for the fifth shear modulus. The author has analyzed this problem previously [13], and concluded that a sensible (though approximate) definition can be made using  $G_5 = G_{eff}$ . There are several different ways of arriving at the same result, but for the present analysis the most useful of these is to express  $G_{eff}$  in terms of the product  $\Lambda_+ \Lambda_-$  (the eigenvalue product, which is also the determinant of the  $2 \times 2$  compliance system). The result, which will be quoted here without further discussion (see [13] for details), is

$$\frac{1}{3K_u} \cdot \frac{1}{2G_{eff}} \equiv \Lambda_+ \Lambda_- = 18 [A_{11}^* A_{33}^* - (A_{13}^*)^2]. \quad (29)$$

And, since  $A_{11}^* = 1/9K_u$ , we have

$$\frac{1}{G_{eff}} = 12 [A_{33}^* - (A_{13}^*)^2 / A_{11}^*]. \quad (30)$$

To obtain one choice for an isotropic average overall undrained shear modulus, we next take the arithmetic mean of the five shear

compliances:

$$\frac{1}{G_u} \equiv \frac{1}{5} \sum_{i=1}^5 \frac{1}{G_i}. \quad (31)$$

Combining these definitions and results gives:

$$\begin{aligned} \frac{1}{G_u} - \frac{1}{G_{dr}} &= -\frac{4}{15} \frac{(\beta'_1 - \beta'_3)^2}{1 - \alpha B} \frac{\alpha B}{K_{dr}} \\ &= \frac{4}{15} \frac{(\beta'_1 - \beta'_3)^2}{1 - \alpha B} \left[ \frac{1}{K_u} - \frac{1}{K_{dr}} \right]. \end{aligned} \quad (32)$$

The  $\beta'_i$ 's are defined by  $\beta'_i = \beta_i K_{dr} / \alpha$ . The final equality is presented to emphasize the similarity of the present results to those of Mavko and Jizba [9] and Berryman *et al.* [11]. Setting  $\beta'_1 = 0$ ,  $\beta'_3 = 1$ ,  $B = 1$ , and  $\alpha \simeq 0$  recovers the expressions of Mavko and Jizba [9] for the case of a very dilute system of flat cracks.

Note that (31) is just the Reuss average (lower bound) of the shear modulus. Also note that the definition (30) of  $G_{eff}$  is actually based on the Voigt average. In terms of mathematical rigor, the result (32) therefore cannot be considered rigorous; it is neither an upper nor a lower bound. The justification for the formula comes not from absolute rigor, but instead from frequent observations that  $G_{eff}$  is in fact a very close estimate of the energy per unit volume in the fifth shear mode and from the knowledge that the Reuss average for compressional modulus tends to be much closer (than does the Voigt average) to observed results for many composite systems. So, for these reasons, the result (32) should be viewed, not as a rigorous formula (it is not), but as a good physical estimate of the undrained shear modulus.

TABLE. Elastic and poroelastic parameters of the two rock samples considered in the text. Bulk and shear moduli of the grains  $K_m$  and  $G_m$ , bulk and shear moduli of the drained porous frame  $K_{dr}$  and  $G_{dr}$ , the effective and undrained shear moduli  $G_{eff}$  and  $G_u$ , and the Biot-Willis parameter  $\alpha = 1 - K_{dr}/K_m$ . The porosity is  $\phi$ .

Elastic/Poroelastic Parameters	Schuler-Cotton Valley Sandstone	Spirit River Sandstone
$G_m$ (GPa)	36.7	69.0
$G_u$ (GPa)	17.7	12.41
$G_{dr}$ (GPa)	15.7	11.33
$G_{eff}$ (GPa)	35.8	20.11
$K_m$ (GPa)	41.8	30.0
$K_{dr}$ (GPa)	13.1	7.04
$\alpha$	0.687	0.765
$\phi$	0.033	0.052

## EXAMPLES AND DISCUSSION

It is clear from (25) that fluid effects in  $\Delta$  cannot increase the overall compliance eigenvalues simultaneously for both the quasi-bulk and the quasi-shear modes. Rather, if one increases, then the other must decrease. Furthermore, it is certainly always true that the presence of pore liquid either has no effect or else strengthens (*i.e.*, stiffens) the porous medium in compression. But this effect on the bulk modulus has been at least partially accounted for in  $A_{11}^* = 1/9K^*$  through the original contribution derived by Gassmann [1]. So presumably the contribution of  $\Delta$  to compliance cannot be so large as to negate completely the liquid effects on the undrained bulk modulus.

### Examples

To clarify the situation, we show some examples in Figures 1–4. The details of the analysis that produces these figures are summarized in the Appendix. The main point is that, for the compliance version of the analysis, the contours of constant energy are ellipses when the vector  $f$  in (18) is interpreted as a stress. Analogously, when the vector is treated as a strain, the contours of constant energy are ellipses for the dual (or stiffness) formulation. If we choose to think of these figures as diagrams in the complex plane, then we note that — while circles and lines transform to circles and lines when transforming back and forth between these two planes — the shapes of ellipses are not preserved (except, of course, in the special case — which is precisely that of isotropy — when the ellipses degenerate to circles). Eigenvectors are determined by the directions in which the points of contact of these two curves lie (indicated by red circles).

For the two sets of examples considered here, the values used for the moduli of the samples are taken from results contained in Berryman [23] wherein it was shown how certain laboratory data could be fit using an elastic differential effective medium scheme. These results are summarized in the TABLE.

Figures 1 and 2 present results for Schuler-Cotton Valley sandstone. Laboratory data on this material were also presented by Murphy [24]. The values chosen for  $\beta_1$  and  $\beta_3$  were  $\beta_1 = 0.20\alpha/K_{dr}$  and  $\beta_3 = 0.60\alpha/K_{dr}$ . The value of the energy per unit volume used for normalization was  $U \simeq 900.0$  GPa. Computed values for the effective and undrained shear moduli were  $G_{eff} = 35.8$  GPa and  $G_u = 17.7$  GPa.

Figures 3 and 4 present results for Spirit River sandstone. Laboratory data on this material were presented by Knight and Nolen-Hoeksema [25]. The values chosen for  $\beta_1$  and  $\beta_3$  were  $\beta_1 = 0.25\alpha/K_{dr}$  and  $\beta_3 = 0.50\alpha/K_{dr}$ . The value of the energy per unit volume used for normalization was  $U \simeq 900.0$  GPa. Computed values for the effective and undrained shear moduli were  $G_{eff} = 20.11$  GPa and  $G_u = 12.41$  GPa.

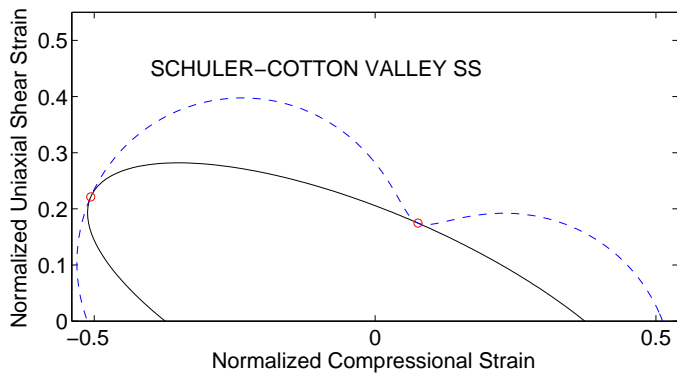


Figure 1. For Schuler-Cotton Valley Sandstone [24] having bulk modulus  $K_{dr} = 13.1$  GPa and shear modulus  $G_{dr} = 15.7$  GPa, the locus of points  $z = Re^{i\theta}$  [see equation (34)] having constant energy  $U = 900$  GPa, when the linear combination of pure compression and pure uniaxial shear is interpreted as strain field applied to the stiffness matrix (solid black line). The plot is in the complex  $z$ -plane, with the inverse of the corresponding expression for the compliance energy superposed for comparison (dashed blue line). Red circles at the two points of intersection correspond to the two eigenvectors of the system of equations. The ellipse (solid black line) in this plane corresponds to the more complex curve in Figure 2.

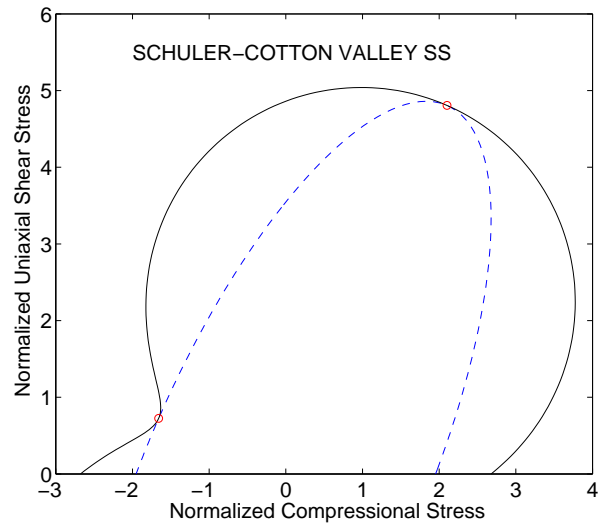


Figure 2. Same parameters as Figure 1, but the linear combination of pure compression and pure uniaxial shear is interpreted as a stress field and is applied to the compliance matrix (dashed blue line). The plot is again in the complex  $z$ -plane, with the inverse of the corresponding expression for the stiffness energy superposed for comparison (solid black line). Red circles at the two points of intersection correspond to the two eigenvectors of the system of equations. The ellipse (dashed blue line here) corresponds to the more complex curve in Figure 1.

## Discussion

We can compare the results obtained with results obtained for the same rocks using differential effective medium theory to fit data. The two characteristics that will interest us here are: (1) comparisons between the values chosen in our examples for the anisotropic  $\beta$ 's and the best fitting crack aspect ratios found in [23], and (2) comparisons between the magnitudes of changes in the overall shear moduli from their drained to undrained values.

The preferred crack aspect ratios found for Schuler-Cotton Valley sandstone and Spirit River sandstone in [23] were respectively, 0.015 and 0.0125. Here we found that  $(\beta'_1, \beta'_3)$  for the same samples were, respectively, (0.20, 0.60) and (0.25, 0.50). Clearly, these values are at least weakly correlated with those of the aspect ratios for the same samples, but no stronger conclusions can be reached at the present time concerning these values.

Similarly, the comparisons of the changes in shear modulus magnitude from drained to undrained also show a weak correlation. The increases in shear moduli observed in the measured laboratory data for Schuler-Cotton Valley sandstone and Spirit River sandstone are, respectively, about 10%, and 20%. As seen in the TABLE, the magnitude of the changes predicted here is also about 10% in both cases. In fact, we know from related work in [14] that the maximum effect on shear for any heterogeneous, saturated porous medium is a 20% increase. So the observed values of about 10% may be considered typical. Thus, agreement is good both qualitatively and semi-quantitatively in

all cases. We conclude that the theory presented here is correctly predicting the magnitudes of these shear modulus enhancements due to pore-fluid effects.

## SUMMARY AND CONCLUSIONS

The preceding discussion shows how overall shear modulus dependence on pore-fluid mechanics arises in simple anisotropic (the specific example used was transversely isotropic) media. The results demonstrate in an entirely elementary fashion how compression-to-shear coupling enters the analysis for anisotropic materials, and furthermore how this coupling leads to overall shear dependence on mechanics of fluids in the pore system.

These effects need not always be large. However, the effect can be very substantial (on the order of a 10% to 20% increase in the overall shear modulus) in cracked or fractured materials, when these pores are liquid-filled. The anisotropy and liquid stiffening effects then both come strongly into play in the results we see, such as those illustrated in Figures 1–4. In particular, if  $\beta_1 \approx \beta_3$ , then soft anisotropy does not make a significant contribution. But, if either  $\beta_1 \ll \beta_3$  or  $\beta_1 \gg \beta_3$ , then the contribution can be significant.

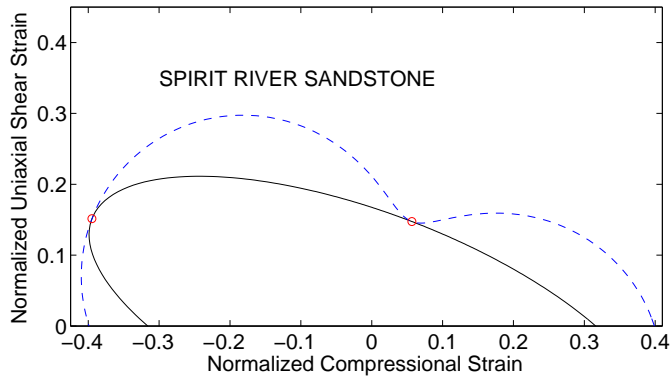


Figure 3. For Spirit River Sandstone [25] having bulk modulus  $K_{dr} = 7.04$  GPa and shear modulus  $G_{dr} = 11.33$  GPa, the locus of points  $z = Re^{i\theta}$  [see equation (34)] having constant energy  $U = 900$  GPa, when the linear combination of pure compression and pure uniaxial shear is interpreted as strain field applied to the stiffness matrix (solid black line). Otherwise the same type of plot as Figure 1. The ellipse (solid black line) in this plane corresponds to the more complex curve in Figure 4.

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## APPENDIX

The equation of an ellipse centered at the origin whose semi-major and semi-minor axes are of lengths  $a$  and  $b$  and whose angle of rotation with respect to the  $x$ -axis in the  $(x,y)$ -plane is  $\psi$  is given by

$$(x \cos \psi + y \sin \psi)^2 / a^2 + (-x \sin \psi + y \cos \psi)^2 / b^2 = 1. \quad (33)$$

For comparison, when a stress of magnitude  $r = \sqrt{x^2 + y^2}$  is applied to a poroelastic system, the energy stored in the anisotropic media of interest here [using (16) and (18)] is given by

$$U(r, \theta) = 3r^2 \left[ A_{11} \cos^2 \theta + 2\sqrt{2}A_{13} \cos \theta \sin \theta + 2A_{33} \sin^2 \theta \right] = R^2 U(r_0, \theta), \quad (34)$$

where in the second equation  $R \equiv r/r_0$ , and  $r_0$  an arbitrary number (say unity) having the dimensions of stress (*i.e.*, dimensions of Pa). It is not hard to see that, when  $U(r, \theta) = \text{const}$ ,

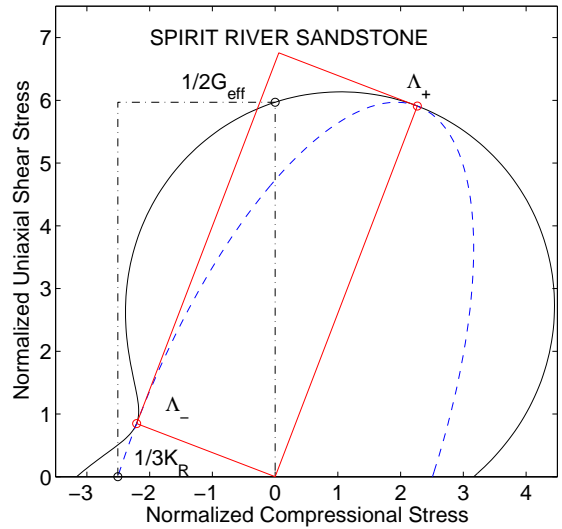


Figure 4. Same parameters as Figure 3, but the linear combination of pure compression and pure uniaxial shear is interpreted as a stress field and is applied to the compliance matrix (dashed blue line). Otherwise the same as Figure 2. The ellipse (dashed blue line here) corresponds to the more complex curve in Figure 3. Eq. (29) shows that the areas of the two rectangles displayed here are equal.

the two equations (33) and (34) have the same functional form and, therefore, that contours of constant energy in the complex ( $z = x + iy$ ) plane are ellipses. Furthermore, we can solve for the parameters of the ellipse by setting  $U = 1$  (in arbitrary units for now) in (34) and then factoring  $r^2$  out of both equations. We find that

$$\begin{aligned} 3A_{11} &= \frac{\cos^2 \psi}{a^2} + \frac{\sin^2 \psi}{b^2}, \\ 6\sqrt{2}A_{13} &= \sin 2\psi \left( \frac{1}{a^2} - \frac{1}{b^2} \right), \\ 6A_{33} &= \frac{\sin^2 \psi}{a^2} + \frac{\cos^2 \psi}{b^2}. \end{aligned} \quad (35)$$

These three equations can be inverted for the parameters of the ellipse, giving:

$$\begin{aligned} \frac{1}{a^2} &= \frac{3A_{11} \cos^2 \psi - 6A_{33} \sin^2 \psi}{\cos 2\psi}, \\ \frac{1}{b^2} &= -\frac{3A_{11} \sin^2 \psi - 6A_{33} \cos^2 \psi}{\cos 2\psi}, \\ \tan 2\psi &= \frac{2\sqrt{2}A_{13}}{A_{11} - 2A_{33}}. \end{aligned} \quad (36)$$

Although contours of constant energy are of some interest,



it is probably more useful to our intuition for the poroelastic application to think instead about contours associated with applied stresses and strains of unit magnitude, *i.e.*, for  $r = 1$  (in appropriate units) and  $\theta$  varying from 0 to  $\pi$ . We then have the important function  $U(1, \theta)$ . [Note that, when  $\theta$  varies instead between  $\pi$  and  $2\pi$ , we just get a copy of the behavior for  $\theta$  between 0 and  $\pi$ . The only difference is that the stress and strain vectors have an overall minus sign relative to those on the other half-circle. For a linear system, such an overall phase factor of unit magnitude is irrelevant to the mechanics of the problem.] Then, if we set  $U(r, \theta) = \text{const} = R^2 U(r_0, \theta)$  and plot  $z = Re^{i\theta}$  in the complex plane, we will have a plot of the ellipse of interest with  $R$  determined analytically by

$$R = \sqrt{U(r, \theta)/U(r_0, \theta)} = \sqrt{\text{const}/U(r_0, \theta)}. \quad (37)$$

We call  $R$  the magnitude of the normalized stress (*i.e.*, normalized with respect to  $r_0$ ).

The analysis just outlined can then be repeated for the stiffness matrix and applied strain vectors. The mathematics is completely analogous to the case already discussed, so we will not repeat it here. Since strain is already a dimensionless quantity, the factor that plays the same role as  $r_0$  above can in this case be chosen to be unity if desired, as the main purpose of the factor  $r_0$  above was to keep track of the dimensions of the stress components.

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