Abstract Riemann Surfaces of Integral Domains and Spectral Spaces (*).

DAVID E. DOBBS (1) - RICHARD FEDDER (2) - MARCO FONTANA (3)

Sunto. – *La superficie astratta di Riemann di un dominio R, introdotta da Zariski, è uno spazio topologico $X(R)$ il cui insieme sostegno consiste di tutti i sovranelli di valutazione di R. L'applicazione canonica suriettiva $f_R\colon X(R) \to \mathrm{Spec}(R)$, $V \mapsto$ centro di V su R, è un'applicazione chiusa, dunque* Spec(R) *è uno spazio-quoziente di $X(R)$. Il teorema principale di questo lavoro è il seguente: $X(R)$ è uno spazio spettrale, nel senso di M. Hochster, e f_R è un'applicazione spettrale. Inoltre, facendo uso della cosiddetta topologia costruttibile, viene dimostrato che se R è integralmente chiuso e* Spec(R) *è uno spazio noetheriano allora f_R è un'applicazione aperta se e soltanto se R è un going-down dominio.*

1. – Introduction.

One cornerstone of modern algebraic geometry is the study of a commutative ring R by means of its set Spec (R) of prime ideals, equipped with the Zariski topology (as in [B, Definition 4, page 99]). An older topological tool of Zariski is also available in case R is an integral domain, namely the abstract Riemann surface $X(R)$ whose underlying set is the collection of all valuation overrings of R (cf. S^* in [ZS, page 113]). The purpose of this article is twofold: to study the connection between Spec (R) and $X(R)$, and to modernize our understanding of abstract Riemann surfaces via the category of spectral spaces and spectral maps (in the sense of [H]).

As Lemma 2.1 demonstrates, the tools are connected by a continuous surjection $f\colon X(R) \to \mathrm{Spec}\,(R)$. Only rarely is f a homeomorphism. Indeed, if R is integrally closed, then f is a homeomorphism if and only if R is a Prüfer domain (cf. Proposi-

(*) Entrata in Redazione il 25 marzo 1986.

(1) Supported in part by the University of Tennessee Faculty Development Grant, by Dipartimento di Matematica dell'Università di Roma «La Sapienza» and by a NATO Collaborative research Grant N. 35/85.

(2) Supported in part by a grant from the Research Council of the Graduate School of the University of Missouri, Columbia.

(3) Work done under the auspices of GNSAGA of the Consiglio Nazionale delle Ricerche, partially supported by a NATO Grant N. 35/85.

Indirizzo degli AA.: D. E. DOBBS: Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37996, U.S.A.; R. FEDDER: Department of Mathematics, University of Missouri, Columbia, Missouri 65211, U.S.A.; M. FONTANA: Dipartimento di Matematica, Università di Roma «La Sapienza», 00185 Roma, Italia.

tion 2.2). However f is always a closed map (Theorem 2.5) and, as a result, f realizes Spec(R) as a quotient space of $X(R)$ (cf. Corollary 2.6). Section 2 concludes by using f to study the passage of the « discrete Alexandroff » separation property (cf. [A, page 28]) between $X(R)$ and Spec(R).

Proposition 3.1 establishes that f is an open map if and only if R is an FTO-domain (in the sense of [Pa]). In case R is integrally closed, openness of f may be characterized using the constructible topology (Lemma 3.2(c).) One consequence (Theorem 3.3) is that an integrally closed going-down ring (in the sense of [DP]) with Noetherian spectrum is an FTO-domain. Accordingly, Remark 3.4(b) constructs an FTO-domain with exacting properties. As with most of this article's examples, this one depends on a pullback construction, and so familiarity with [F] will be assumed.

It is well-known (cf. [ZS, Theorem 40, page 113]) that $X(R)$ is always a quasi-compact T_0-space. What more can be said? Theorem 4.1 gives the answer: $X(R)$ is a spectral space. In Corollary 4.5, a functorial variant follows: $X(-)$ may be viewed as a functor from a category of integral domains to the category of spectral spaces and spectral maps which factors through the full subcategory of abstract Riemann surfaces.

Throughout, R denotes an integral domain with integral closure R' and quotient field K. Any unexplained material is standard and may be found in the texts cited as references.

2. – Relating $X(R)$ and Spec(R).

As a set, $X(R)$ is the collection of all valuation overrings of R, that is, valuation domains V such that $R \subset V \subset K$. A basis for the open sets in the canonical topology of $X(R)$ is given by the sets

$$E(x_1, \ldots, x_n) = \{V \in X(R): x_i \in V \text{ for each } i = 1, \ldots, n\}$$

as $\{x_1, \ldots, x_n\}$ ranges over the finite subsets of K. (Since

$$E(x_1, \ldots, x_n) \cap E(y_1, \ldots, y_m) = E(x_1, \ldots, x_n, y_1, \ldots, y_m)$$

one does in fact obtain a topology.) Evidently $X(R)$ is a T_0-space, and in the usual way ([H, page 53]) thus acquires the structure of a partially ordered set: $V_1 \leqslant V_2$ if and only if V_2 is in the closure of $\{V_1\}$, that is, if and only if $V_2 \subset V_1$. As recalled in the introduction, $X(R)$ is quasi-compact. Since $E(x_1, \ldots, x_n) = X(R[x_1, \ldots, x_n])$ as topological spaces, it follows that $X(R)$ has an open basis consisting of quasi-compact opens; moreover, the typical quasi-compact open subset of $X(R)$ is the union of finitely many sets of the form $E(x_1, \ldots, x_n)$.

The relation between $X(R)$ and Spec(R) is forged with the function $f = f_R: X(R) \to$ $\to$ Spec(R) defined as follows: if $V \in X(R)$ and M is the maximal ideal of V, then $f(V) = M \cap R$. In other words, $f_R(V)$ is the center of V on R.

LEMMA 2.1. – With the above notation, $f: X(R) \to \operatorname{Spec}(R)$ is surjective, continuous, order-preserving and order-reflecting.

PROOF. – By « extension of valuations », f is surjective (cf. [G_2, Theorem 19.6]). Next, to check that f is continuous, it is enough to show that $f^{-1}(X_r)$ is open, where $X_r = \{P \in \operatorname{Spec}(R): r \notin P\}$, $r \in R$, is a basic Zariski-open subset of $\operatorname{Spec}(R)$. Without loss of generality, $r \neq 0$. Then $f^{-1}(X_r) = \{(V, M) \in X(R): r \notin M\} = \{(V, M) \in X(R): r^{-1} \in M$ or r^{-1} is a unit of $V\} = E(r^{-1})$, which is open in $X(R)$, as desired. Finally, for V_1 and V_2 in $X(R)$, [G_2, Theorem 17.6] assures that $f(V_1) \subset f(V_2)$ if and only if $V_2 \subset V_1$, that is, if and only if $V_1 \leqslant V_2$. Thus, f is both order-preserving and order-reflecting, to complete the proof.

In view of Lemma 2.1, $\operatorname{Spec}(R)$ is the continuous image (via f)of a quasi-compact space, and is thus itself quasi-compact. Besides giving this amusing proof of a well-known fact, f leads to other useful information, to which we now turn.

Recall that R is called an *i-domain* in case the contraction map $\operatorname{Spec}(T) \to \operatorname{Spec}(R)$ is an injection for each overring T of R; equivalently, if and only if R'_P is a valuation domain for each $P \in \operatorname{Spec}(R)$ (cf. [Pa, Corollary 2.15]). It is well-known (cf. [G_2, Theorem 19.15]) that the integrally closed i-domains are just the Prüfer domains. As f_R is an injection whenever R is a Prüfer domain and as $X(R) = X(R')$ in general, the next result is perhaps to be expected.

PROPOSITION 2.2. – Let $f: X(R) \to \operatorname{Spec}(R)$ be the function introduced above. Then the following conditions are equivalent:

(i) f is a homeomorphism;

(ii) f is an order-isomorphism;

(iii) f is a bijection;

(iv) f is an injection;

(v) For each $P \in \operatorname{Spec}(R)$, only one valuation overring of R dominantes R_P;

(vi) R is an i-domain.

PROOF. – (i) $\Rightarrow$ (ii): Apply Lemma 2.1.

(ii) $\Rightarrow$ (iiii): Trivial.

(iii) $\Rightarrow$ (iv): Trivial.

(iv) $\Leftrightarrow$ (v): $V \in X(R)$ dominates R_P if and only if $f(V) = P$.

(v) $\Leftrightarrow$ (vi): Combine the above remarks with [G_2, Corollary 19.9] and [B, Theorem 1, page 376].

(vi) $\Rightarrow$ (i): Assume (vi). By Lemma 2.1, it is enough to prove that $Y = f(E(x_1, \ldots, x_n))$ is open in $\operatorname{Spec}(R)$ for any $x_1, \ldots, x_n \in K$. As (vi) $\Rightarrow$ (iv), $Y = \bigcap f(E(x_i))$. Thus, without loss of generality, $n = 1$; write x for x_1. Then $Y = \{P \in \operatorname{Spec}(R)$: there exists $V \in X(R)$ such that $x \in V$ and V dominates $R_P\}$

which, since (vi) $\Rightarrow$ (v), is just $\{P \in \operatorname{Spec}(R): x \in R'_P\}$. Therefore, if I denotes the ideal $\{r \in R: rx \in R'\}$ of R, we have that $Y = \{P \in \operatorname{Spec}(R): I \not\subset P\}$, which is open in $\operatorname{Spec}(R)$. This completes the proof.

The preceding result leaves open the question of what one may assert if f_R is not an injection. Corollary 2.6 will show that f permits $\operatorname{Spec}(R)$ to be obtained as an identification space from $X(R)$. First, we pause to note a more trivial way to recover the *space* $\operatorname{Spec}(R)$ from the *set* $X(R)$.

REMARK 2.3. – Let $Y(R)$ be the set $X(R)$ endowed with the coarsest topology making $f_R: Y(R) \to \operatorname{Spec}(R)$ continuous. (Since we saw in the proof of Lemma 2.1 that $E(r^{-1}) = f^{-1}(X_r)$, it follows that $E(r^{-1})$, for $0 \neq r \in R$, is a typical subbasic open set in $Y(R)$.) The T_0-space canonically associated to $Y(R)$ is $Y(R)/\tau$, where $V_1 \tau V_2$ if and only if $f(V_1) = f(V_2)$. The function $Y(R)/\tau \to \operatorname{Spec}(R)$ induced by f is a homeomorphism.

For a proof, it is enough to show that f, viewed as a map from $Y(R)$ to $\operatorname{Spec}(R)$, is open; that is, that $Z = f(\bigcap f^{-1}(X_{r_i}))$ is open in $\operatorname{Spec}(R)$ for each finite subset $\{r_i\}$ of $R \setminus \{0\}$. This is readily shown, for $Z = \{P \in \operatorname{Spec}(R)$: there exists $(V, M) \in X(R)$ such that $f(V) = P$ and $r_i \notin M$ for each $i\} = \bigcap X_{r_i}$, which is indeed open in $\operatorname{Spec}(R)$.

It was noted above that $Y(R)$ is not a T_0-space if f is not an injection. By Proposition 2.2, $X(R)$ and $Y(R)$ are thus distinct if R is not an i-domain. A good illustration of this arises in case R is the local (Noetherian) ring at the singular point of a nodal plane curve. Then $X(R) = \{V_1, V_2, K\}$ where V_1, V_2 are distinct discrete rank 1 valuation domains (such that $R' = V_1 \cap V_2$ and K is the quotient field of R). One may check that $\{V_2, K\}$ is open in the canonical topology of $X(R)$, but the only open subsets of $Y(R)$ are $\emptyset$, $Y(R)$ and $\{K\}$.

Our next major goal is to show that f_R is always closed. The following technicalities, borrowed from [ZS, pages 115-116], will help. By analogy with the construction of $X(R)$, we let $\Omega(R)$ denote the collection of quasilocal overrings of R; and topologize $\Omega(R)$ by taking as basic opens the sets $\Omega(R[x_1, \ldots, x_n])$, where $\{x_1, \ldots, x_n\}$ ranges over the finite subsets of K. By analogy with the construction of f_R, define $g = g_R: \Omega(R) \to \operatorname{Spec}(R)$ by setting $g(S) = M \cap R$ for each $(S, M) \in \Omega(R)$. Next, let $L(R) = \{R_P: P \in \operatorname{Spec}(R)\}$ with the subspace topology inherited from $\Omega(R)$, and let $h = h_R: L(R) \to \operatorname{Spec}(R)$ denote the restriction of g to $L(R)$.

LEMMA 2.4. – With the above notation, $h: L(R) \to \operatorname{Spec}(R)$ is a homeomorphism.

PROOF. – Since $h(R_P) = P$, it is clear that h is a bijection. To see that h is continuous, it is enough to show g is continuous. Consider the complement of the inverse image of a closed set. If I is an ideal of R and $V(I) = \{P \in \operatorname{Spec}(R): I \subset P\}$ is the associated closed subset of $\operatorname{Spec}(R)$, then $\Omega(R) \setminus g^{-1}(V(I)) = \{(S, M) \in \Omega(R)$: there exists $r \in I \setminus (M \cap R)\} = \bigcup \{\Omega(R[r^{-1}]): 0 \neq r \in I\}$, which is indeed open in $\Omega(R)$, as desired.

Finally, to see that h is open, we shall prove that $Y = \mathrm{Spec}(R)\setminus h\big(L(R) \cap \Omega(R[x_1, ..., x_n])\big)$ is closed in $\mathrm{Spec}(R)$ for each finite subset $\{x_1, ..., x_n\}$ of K. It will be convenient to let $\Gamma(x_1, ..., x_n)$ denote $\Omega(R)\setminus\Omega(R[x_1, ..., x_n])$. Then $Y = h\big(L(R) \cap \Gamma(x_1, ..., x_n)\big) = \bigcup h\big(L(R) \cap \Gamma(x_i)\big)$, and so we may assume that $n = 1$, with x denoting x_1. Consider the ideal $J = \{r \in R\colon rx \in R\}$ of R. For each $P \in \mathrm{Spec}(R)$, $J \subset P$ if and only if $x \notin R_P$, that is, if and only if $R_P \in \Gamma(x)$. Consequently $Y = V(J)$, which is Zariski-closed, completing the proof.

THEOREM 2.5. – $f_R\colon X(R) \to \mathrm{Spec}(R)$ is a closed map.

PROOF. – We claim that $d\colon X(R) \to L(R)$, given by $d(V) = R_{M\cap R}$ for each $(V, M) \in X(R)$, is a closed map. This follows by applying [ZS, Lemma 4, page 116] since $L(R)$ is a « complete model » in the sense that each element of $X(R)$ dominates some element of $L(R)$. (Actually, the cited result in [ZS] shows that $X(R)\setminus\{K\} \to L(R)$ is closed, but this readily yields our claim.) The theorem now follows from Lemma 2.4 since $f = hd$ is a composite of closed maps.

Define an equivalence relation $\sim$ on $X(R)$ be decreeing $V_1 \sim V_2$ if and only if $f_R(V_1) = f_R(V_2)$, and let $X(R)/\sim$ have the quotient topology. Denote the induced function $X(R)/\sim \to \mathrm{Spec}(R)$ by $\bar{f}_R$. As Lemma 2.1 and Theorem 2.5 show that f is a continuous closed surjection, we immediately infer

COROLLARY 2.6. – With the above notation, $\bar{f}_R\colon X(R)/\sim \to \mathrm{Spec}(R)$ is a homeomorphism.

Ordinary separation properties are of no interest for $X(R)$, since $X(R)$ is a T_1-space if and only if $X(R)$ is Hausdorff if and only if R is a field. The crux is that the closure of $\{K\}$ in $X(R)$ is the entire space, so that K being a closed point implies (by the existence of dominating valuation overrings) that $\mathrm{Spec}(R) = \{0\}$ and hence that R is a field. (On the other hand, one sees similarly that K is an open—that is, isolated—point in $X(R)$ if and only if K is a finite-type R-algebra, that is, if and only if R is a G-domain in the sense of [K, Theorem 18]. In a subsequent article, we shall return to an intensive study of G-domains via abstract Riemann surfaces.)

Next, recall a more exotic separation property: a *discrete Alexandroff space* is a T_0-space in which every intersection of (arbitrarily many) open subsets is open. It is well-known (cf. [Pi, Proposition 1, section 5]) that $\mathrm{Spec}(A)$ is discrete Alexandroff (with respect to the Zariski topology) if and only if A is a g-ring. Moreover, [DFP, Theorem 2.16] shows how to retopologize any spectral set $\mathrm{Spec}(A)$ so as to give a canonical discrete Alexandroff structure. We now turn to related matters involving $X(R)$.

COROLLARY 2.7. – (*a*) If $X(R)$ is a discrete Alexandroff space, then $\mathrm{Spec}(R)$ is also discrete Alexandroff (and so R is a g-ring).

(*b*) $X(R)$ is a discrete Alexandroff space if and only if each valuation overring of R is a finite-type R'-algebra.

PROOF. – (*a*) We shall show that $Y = \bigcup Y_\alpha$ is closed for each collection $\{Y_\alpha\}$ of closed subsets of Spec(R). Since f is continuous, each $f^{-1}(Y_\alpha)$ is closed in $X(R)$, and so $Z = \bigcup f^{-1}(Y_\alpha)$ is closed by hypothesis. By Theorem 2.6, $f(Z)$ is closed. However, since f is surjective, $f(Z) = Y$.

(*b*) Without loss of generality, $R = R'$. Assume that each valuation overring of R is a finite-type R-algebra. Then by [FV, Theorem 1], $R(= R')$ is a so-called strong G-domain and, in particular, both a g-ring and a Prüfer domain (cf. [Mar, Theorem 2.2]). By the above comments, Spec(R) is then discrete Alexandroff, and so the « if » assertion follows by invoking Proposition 2.2.

Conversely, assume that $X(R)$ is discrete Alexandroff. Since $R = R'$, [FV, Theorem 14] reduces our task to showing that R is a strong G-domain. However, Spec(R) is discrete Alexandroff by (*a*), and so by [Mar, Proposition 2.4], it suffices to prove that R is a Prüfer domain. To this end, let $\{V_i\}$ be the set of minimal valuation overrings of R. As $X(R)$ is discrete Alexandroff, it follows readily that each $X(V_i)$ is open in $X(R)$. Since $X(R) = \bigcup X(V_i)$ is quasi-compact, we see next that $\{V_i\}$ is finite. Thus, by [K, Theorem 107], $R = R' = \bigcap V_i$ is a Prüfer domain, completing the proof.

REMARK 2.8. – (*a*) The reference to R' in Corollary 2.7(*b*) is unavoidable. Indeed we produce next an R for which $X(R)$ is discrete Alexandroff but some valuation overring of R is not a finite-type R-algebra.

Begin with an infinite-dimensional algebraic field extension $F \subset L$, and consider the formal power series ring $V = L[\![X]\!] = L + M$, with $M = XV$. Then $R = F + M$ has the asserted properties. Indeed, $X(R) = X(R') = X(V)$ is homeomorphic to Spec(V) by Proposition 2.2, and, being a finite T_0-space, is hence discrete Alexandroff. Moreover, V is not algebra-finite over R since L is not algebra-finite over F.

(*b*) The condition alluded to in (*a*) is, however, very useful. To reiterate: [FV, Theorem 1] demonstrates that if each valuation overring of R is a finite-type R-algebra, then R' is a strong G-domain. It is interesting to note that, as in Corollary 2.7(*b*), the proof of the cited result depends on the quasi-compactness of $X(R)$.

(*c*) Pursuing an observation in the proof of Corollary 2.7(*b*), we find that $X(R)$ is a discrete Alexandroff space if and only if $X(T)$ is open in $X(R)$ for each overring T of R. The reader can thence deduce the following addendum to Corollary 2.7(*b*): $X(R)$ is discrete Alexandroff if and only if for each valuation overring V of R, there exists a finite-type R-algebra S contained between R and V such that V is the integral closure of S.

(*d*) Recall another exotic separation property: a T_0-space X is called T_D in case, for each $Y \subset X$, the set of accumulation points of Y is closed. Any discrete Alexandroff space is a T_D-space. It is not difficult to characterize when Spec(R) is a T_D-space (cf. [FM, Proposition 1]); however, we do not have an equally neat companion for Corollary 2.7(*b*) characterizing when $X(R)$ is T_D.

We can, however, show that $X(R)$ need not be a T_D-space in case Spec(R) is T_D. To illustrate this, alter the construction in (a) by taking F to be algebraically closed in the larger field L. It is easy to verify that Spec$(R) = \{0, M\}$ is a T_D-space by using the definition of Zariski topology. However, $X(R)$ is not a T_D-space since V is not an isolated point in the closure of V. To see this, assume on the contrary that $\{V\}$ is the intersection of some $E(x_1, \ldots, x_n)$ with the closure of V. Without loss of generality, each $x_i \in L \setminus F$, and so $R[x_1, \ldots, x_n] = S + M$, where $S = F[x_1, \ldots, x_n]$ is not a field (cf. [B, Corollary 3, page 354]). Taking $W \neq L$ to be a valuation ring of L containing S (cf. [G_2, Theorem 19.6]), we find that $W + M$ is in both $E(x_1, \ldots, x_n)$ and the closure of V, the desired contradiction. (A degenerate case should be noted: if $n = 0$, select $x \in L \setminus F$ and use $F[x]_{(x)}$ in place of S in the above argument.)

(e) We next give the « discrete Alexandroff » analogue of the result in (d); that is, we shall show that the converse of Corollary 2.7(a) is false. To this end, begin with a rational prime p, and let S denote the integral closure of $\mathbb{Z}_{p\mathbb{Z}}$ in the algebraic closure of $\mathbb{Q}$. As shown by Gilmer [G_1, Example 1], S is a one-dimensional Bézout (hence, $i-$) domain with infinitely many maximal ideals. In particular, S is not a g-ring and so $X(S)$ (which is homeomorphic to Spec(S)) is not a discrete Alexandroff space. Next, let $J(S)$ be the Jacobson radical of S and let

$$u\colon \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)} = \mathbb{F}_p \to S/J(S)$$

be the induced injective integral ring-homomorphism. Take R to be the pullback of the diagram

$$\begin{array}{ccc} & & S \\ & & \downarrow \\ \mathbb{F}_p & \to & S/J(S) \end{array}$$

whose horizontal (resp., vertical) map is u (resp., the canonical projection).

By appealing to the topological characterization of R in [F, Theorem 1.4], we find that R is a one-dimensional quasilocal domain. In particular, Spec(R) is a discrete Alexandroff space. By also appealing to [F, Corollary 1.5(5)], we have $R' = S$. Thus $X(R) = X(S)$ which, as we have seen, is not discrete Alexandroff.

3. – When f is open.

Recall that if $X(R)$ has the canonical topology, then $f\colon X(R) \to$ Spec(R) is closed in general (Theorem 2.5). Moreover, by retopologizing the set $X(R)$, one may also view « f » as open (Remark 2.3). We next study openness of the genuine f, that is, for $X(R)$ with the canonical topology.

It will be convenient first to recall some background material. (For additional background, see [DP] and [Pa].) R is said to be a *going-down ring* (write: R is a GD-domain) in case the extension $R \subset T$ has the going-down property for each overring T of R. Prüfer domains and one-dimensional integral domains are the natural examples of going-down rings. Following [Pa], we say similarly that R is an *open* (resp., *finite-type open*; resp., *simple open*) *domain* in case the contraction map $\operatorname{Spec}(T) \to \operatorname{Spec}(R)$ is open for each overring T of R (resp., for each such T which is a finite-type R-algebra; resp., for each T of the form $T = R[u]$, $u \in K$). Letting FTO and SO denote «finite-type open» and «simple open», respectively, we know (cf. [Pa, page 19]) that

$$\text{open domain} \Rightarrow \text{FTO-domain} \Rightarrow \text{SO-domain} \Rightarrow \text{GD-domain};$$

and that the first of these implications cannot be reversed, even if R is quasi-semilocal. It is not known whether the other two implications may be reversed in general. As Papick [Pa, Corollary 3.37] has shown, they *are* reversible if R is quasi-semilocal. We contribute another instance of reversibility in Theorem 3.3: the case of integrally closed R with Noetherian spectrum. A key step is taken in

Proposition 3.1. – f_R is an open map if and only if R is an FTO-domain.

Proof. – For each finite subset $\{x_1, \ldots, x_n\}$ of K, there is a commutative diagram

$$\begin{array}{ccc} E(x_1, \ldots, x_n) & \longrightarrow & X(R) \\ \downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec}(R[x_1, \ldots, x_n]) & \xrightarrow{\ v\ } & \operatorname{Spec}(R) \end{array}$$

in which the top horizontal map is the inclusion, v is given by the Spec functor, and the left-hand vertical map is the (surjective) restriction of f. If R is an FTO-domain, the image of any such v is open in $\operatorname{Spec}(R)$. Hence f sends each basic open subset of $X(R)$ to an open set, and so f is an open map.

Conversely, assume that f is open. To show that R is an FTO-domain, a basic fact about the Zariski topology [B, Corollary, page 101] reduces us to proving that the image, say Y, of the composite

$$\operatorname{Spec}(R[y_1, \ldots, y_m, s^{-1}]) \to \operatorname{Spec}(R[y_1, \ldots, y_m]) \to \operatorname{Spec}(R)$$

is open for each finite subset $\{y_1, \ldots, y_m\}$ of K and nonzero element $s \in R[y_1, \ldots, y_m]$. However, taking $\{x_1, \ldots, x_n\} = \{y_1, \ldots, y_m, s^{-1}\}$, we see from the above diagram that $Y = f(E(x_1, \ldots, x_n))$. By the hypothesis on f, Y is therefore open in $\operatorname{Spec}(R)$, completing the proof.

The following material will be helpful. For an ideal I of R, let $V(I) = \{P \in \operatorname{Spec}(R): I \subset P\}$ and $D(I) = \operatorname{Spec}(R) \setminus V(I)$, as usual. For $x \in K$, set $(R: x) = \{r \in R: rx \subseteq R\}$. Here is the main ingredient: an *amenable set* (*over* R) is, by definition, a subset of $\operatorname{Spec}(R)$ of the form

$$\bigcup_{i=1}^{n} \left(D(R: x_i) \cap V(R: x_i^{-1})\right)$$

arising from a finite subset $\{x_1, \ldots, x_n\}$ of $K \setminus \{0\}$. A final piece of notation: $\operatorname{Spec}(R)_C$ denotes the set $\operatorname{Spec}(R)$ endowed with the constructible topology, in the sense of [EGA]. (This coincides with the result of applying the patch topology construction [H, page 45] to the Zariski topology on $\operatorname{Spec}(R)$.)

LEMMA 3.2. – Let R be integrally closed. Then:

(*a*) For each finite subset $\{x_1, \ldots, x_n\}$ of K, the complement in $\operatorname{Spec}(R)$ of $f(E(x_1, \ldots, x_n))$ is an amenable set over R.

(*b*) Let F be the amenable set constructed via $\{x_1, \ldots, x_n\} \subset K \setminus \{0\}$. Then the following two conditions are equivalent:

(i) F is closed in $\operatorname{Spec}(R)$;

(ii) F is closed in $\operatorname{Spec}(R)_C$ and the image of $\operatorname{Spec}(R[x_1, \ldots, x_n]) \to \operatorname{Spec}(R)$ is stable under generization.

(*c*) The following five conditions are equivalent:

(i) f_R is an open map;

(ii) For each $x \in K$, the set $\{P \in \operatorname{Spec}(R): x \in PR_P\}$ is closed in $\operatorname{Spec}(R)$;

(iii) Each amenable set over R is closed in $\operatorname{Spec}(R)$;

(iv) The image of $\operatorname{Spec}(R[x_1, \ldots, x_n]) \to \operatorname{Spec}(R)$ is stable under generization for each $\{x_1, \ldots, x_n\} \subset K$ and each amenable set is a constructible set;

(v) R is a GD-domain and each amenable set is a constructible set.

PROOF. – (*a*) Without loss of generality, each x_i is nonzero and $n \geqslant 1$. Evidently, $\operatorname{Spec}(R) \setminus f(E(x_1, \ldots, x_n))$ is just

$$Y = \bigcup_{i=1}^{n} \{P \in \operatorname{Spec}(R): x_i \notin V \text{ for each } V \in X(R) \text{ such that } V \text{ dominates } R_P\}.$$

We claim that Y coincides with

$$Z = \bigcup_{i=1}^{n} \{P \in \operatorname{Spec}(R): x_i^{-1} \in PR_P\}.$$

To see this, note first that if $P \in Z \setminus Y$, then there exists an index i and a valuation ring (V, M) dominating R_P such that $x_i^{-1} \in PR_P$ and $x_i \in V$. Then $x_i^{-1} \in MV_M = M$ and $1 = x_i^{-1}x_i \in MV = M$, the desired contradiction. Conversely, suppose that $P \in Y$. Then for some index j, x_j^{-1} is in the maximal ideal of each valuation overring V that dominates R_P. By [G_2, Corollary 19.9], the intersection of all such V is R_P', which is just R_P since we have assumed that $R = R'$. As $x_j \notin R_P$ and $x_j^{-1} \in R_P$, it follows that $x_j^{-1} \in PR_P$; that is, $P \in Z$. This proves the claim.

For each $x \in K \setminus \{0\}$, the set $\{P \in \operatorname{Spec}(R): x \in PR_P\}$ may be expressed as $\{P \in \operatorname{Spec}(R): x \in R_P\} \cap \{P \in \operatorname{Spec}(R): x^{-1} \notin R_P\}$; that is, as $D\big((R: x)\big) \cap V\big((R: x^{-1})\big)$. Accordingly, Z is the amenable set constructed via $\{x_1, \ldots, x_n\}$.

(*b*) By appeal to [DFP, Lemma 2.5(*b*)], it is enough to prove that F is stable under specialization if and only if the image, say W, of $\operatorname{Spec}\big(R[x_1, \ldots, x_n]\big) \to \operatorname{Spec}(R)$ is stable under generization. The former condition is equivalent to $\operatorname{Spec}(R) \setminus F$ being stable under generization; that is, by the explicit calculation in (*a*), equivalent to $f\big(E(x_1, \ldots, x_n)\big)$ being stable under generization. However, we have seen from the commutative diagram in the proof of Proposition 3.1 that $f\big(E(x_1, \ldots, x_n)\big)$ coincides with W.

(*c*) (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii): Since $E(x_1, \ldots, x_n)$, with $\{x_1, \ldots, x_n\} \subset K \setminus \{0\}$, is a typical basic open subset of $X(R)$, the desired equivalences follow from the proof in (*a*) that $\operatorname{Spec}(R) \setminus f\big(E(x_1, \ldots, x_n)\big) = Z$ is the amenable set constructed via $\{x_1, \ldots, x_n\}$.

Next, a general observation: each amenable set F is open in $\operatorname{Spec}(R)_C$. By the nature of the closed sets in $\operatorname{Spec}(R)_C$ (cf. [DFP, page 559]), this may be seen by recalling, for F constructed via $\{x_1, \ldots, x_n\} \subset K \setminus \{0\}$, that $\operatorname{Spec}(R) \setminus F$ is the image of $\operatorname{Spec}\big(R[x_1, \ldots, x_n]\big) \to \operatorname{Spec}(R)$. Accordingly, by [EGA, 7.2.12(ii), page 337], F is a constructible set if and only if F is closed in $\operatorname{Spec}(R)_C$.

(iii) $\Leftrightarrow$ (iv): Combine the preceding observation with (*b*).

(v) $\Rightarrow$ (iv): Trivial.

(iii) $\Rightarrow$ (v): Since (iii) implies both (iv) and (i), it is enough to invoke Proposition 3.1 and the fact that each FTO-domain is a GD-domain. The proof is complete.

THEOREM 3.3. – *Let R be integrally closed, such that $\operatorname{Spec}(R)$ is a Noetherian space. Then the following conditions are equivalent:*

(i) *f_R is an open map;*

(ii) *R is a GD-domain;*

(iii) *R is an FTO-domain;*

(iv) *R is an SO-domain;*

(v) *The image of $\operatorname{Spec}\big(R[x_1, \ldots, x_n]\big) \to \operatorname{Spec}(R)$ is stable under generization for each subset $\{x_1, \ldots, x_n\}$ of K.*

PROOF. – Since Spec(R) is Noetherian, constructible sets may be characterized as the finite unions of locally closed sets ([Mat, page 39]). It is therefore clear that each amenable set is a constructible set. Lemma 3.2(c) thus yields (v) $\Rightarrow$ (i). In addition, Proposition 3.1 and the above remarks give (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (ii) $\Rightarrow$ (v). The proof is complete.

REMARK 3.4. – (a) By Proposition 3.1 and the remarks preceding it, f_R is an open map for each quasilocal going-down ring R. In particular, if R is a pseudo-valuation domain, then f_R is an open map. (Recall that an integral domain R is called a *pseudo-valuation domain* [DF] if R has a valuation overring V such that Spec(R) = Spec(V) as sets.) Thus, the ring $F + M$ introduced in Remark 2.8(a) admits an open f, although $F + M$ does not satisfy the riding hypotheses in Theorem 3.3.

(b) There exists an integral domain R such that (i) R is integrally closed; (ii) Spec(R) is Noetherian; (iii) f_R is an open map but not a homeomorphism; and (iv) R is not an open domain. To indicate such a construction, let $F \subset L$ be distinct fields, with F algebraically closed in L. Let $V = L + M$ be a valuation domain (with maximal ideal M) such that, as a partially ordered set under inclusion, Spec(V) is isomorphic to $\{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ with the natural order inherited from $\mathbb{Q}$. Then $R = F + M$ has the asserted properties.

Indeed, by the lore of the $D + M$ construction (cf. [G_2]), $R = R'$ is not a valuation domain and Spec(R) = Spec(V) as sets. In particular, R is not an i-domain and so, by Proposition 2.2, f is not a homeomorphism. Since R is a pseudo-valuation domain, (a) shows however that f is an open map. Moreover, V has ascending chain condition on prime (radical) ideals. Spec(V) is therefore a Noetherian space, as must be its homeomorphic copy Spec(R). Finally, since the partially ordered set Spec(R) is not well-ordered, (iv) follows from the criterion in [Pa, Theorem 3.16].

(c) In view of Theorem 3.3 and (b), it seems useful to record an example in which $R \neq R'$, Spec(R) is Noetherian, and f_R is open but not a homeomorphism. For this purpose, it is enough to consider the ring R in Example 2.8(e). (More mundane examples abound via, for instance, the $D + M$-construction.) Indeed, since R is one-dimensional (hence, a GD-domain) and quasilocal, it is easy to see that R is an open domain (cf. [Pa, Theorem 3.16]). In particular, R is an FTO-domain; so, by Proposition 3.1, f is open. Moreover, Proposition 2.2 assures that f is not a homeomorphism since R is not an i-domain. The remaining assertions are clear.

4. – Abstract Riemann surfaces are spectral spaces.

Following [H], we say that a topological space X is a *spectral space* in case X is homeomorphic to Spec(A), with the Zariski topology, for some commutative ring A (not necessarily an integral domain). A continuous map $X \to Y$ between spectral spaces is called a *spectral map* in case inverse images of arbitrary quasi-compact open subsets of Y are quasi-compact. We may now state our main result.

THEOREM 4.1. – *$X(R)$ is a spectral space and $f_R\colon X(R) \to \operatorname{Spec}(R)$ is a spectral map.*

The proof of Theorem 4.1 must await some definitions and preliminary results.

For each finite subset $\{x_1, \ldots, x_n\}$, let $B(x_1, \ldots, x_n)$ denote the closed subset $X(R)\setminus E(x_1, \ldots, x_n)$ of $X(R)$. For each subset S of K, let $\bigwedge(S)$ denote the closed subset $\bigcap\{B(x)\colon x \in K\setminus S\}$ of $X(R)$. For each subset Y of $X(R)$, let $G(Y)$ denote the subset $\bigcup\{V\colon V \in Y\}$ of K. Note in general that

$$Y \subset \bigwedge(G(Y)) = \Big\{W \in X(R)\colon W \subset \bigcup\{V\colon V \in Y\}\Big\}.$$

Finally, we shall say that a subset Y of $X(R)$ is *saturated* in case $\bigwedge(G(Y)) = Y$.

LEMMA 4.2. – *Let Y be an irreducible closed subset of $X(R)$. Then:*

(a) Y is saturated.

(b) Let $x, y \in K$ and set $I = \bigcap\{M_V\colon (V, M_V) \in Y\}$. Then if $xy \in I$, either $x \in I$ or $y \in I$.

PROOF. – (a) If not, then there exists $B = B(y_1, \ldots, y_n)$ such that $Y \subset B$ and $B(x) \not\subset B$ for each $x \in K\setminus G(Y)$. If $n = 1$, then $y = y_1 \in G(Y)$, there exists $V \in Y$ such that $y \in V$, and so $V \notin B(y)$, contradicting $Y \subset B$. Hence $n \geqslant 2$. By the above reasoning, $Y \not\subset B(y_i)$ for each i. Now, since $B = \bigcup B(y_i)$, we may decompose Y as $\bigcup\big(Y \cap B(y_i)\big)$, a union of finitely many proper closed subsets, contradicting irreducibility of Y.

(b) Suppose not. As $x \notin I$ and the elements of Y are valuation domains, one readily verifies that $x^{-1} \in W$ for some $W \in Y$; thus, $Y \not\subset B(x^{-1})$. Similarly, $Y \not\subset B(y^{-1})$. As $xy \in I$, each $V \in Y$ is such that either $x^{-1} \notin V$ or $y^{-1} \notin V$; that is, $Y \subset B(x^{-1}) \cup \cup B(y^{-1})$. Accordingly, Y decomposes as the union of $Y \cap B(x^{-1})$ and $Y \cap B(y^{-1})$ contradicting irreducibility, to complete the proof.

If $(V, M) \in X(R)$, then the fact that f is continuous and closed (Lemma 2.1 and Theorem 2.5) assures that f sends the closure of $\{V\}$ to precisely the closure of $\{M\}$. To some extent, this suggests

PROPOSITION 4.3. – *Each irreducible closed subset Y of $X(R)$ has a generic point.*

PROOF. – Fix $W \in Y$ and set $I = \bigcap\{M_V\colon (V, M_V) \in Y\}$. By Lemma 4.2(*b*), $S = W\setminus I$ is a multiplicative subset of W, and so $V_1 = W_S$ is a valuation overring of W. It suffices to prove that the closure of $\{V_1\}$ is Y (as Y will then have generic point V_1).

If $x \in I$ and $y \in W$, then Lemma 4.2(*b*) assures that $xy \in I$ (lest $y^{-1} \in I \subset M_W$ and $1 = y^{-1}y \in M_W$, a contradiction). Thus I is a (prime) ideal of W. Consequently, the maximal ideal of V_1 is $IW_I = I$. As $I \supset \bigcap \{M_V: V \in Y\}$ and each V is a valuation domain, one readily verifies that $V_1 \subset \bigcup \{V: V \in Y\} = G(Y)$. Put differently, $V_1 \in \bigwedge(G(Y))$, and so Lemma 4.2(*a*) yields $V_1 \in Y$. It now suffices to show that the closure of $\{V_1\}$ contains each $V \in Y$; that is, to show that $V \subset V_1$ for each $V \in Y$. Since $M_V \supset I$, this follows directly from [G_2, Theorem 17.6(*c*)], and the proof is complete.

PROOF of THEOREM 4.1. – Spectral spaces have been characterized by HOCHSTER [H, Proposition 4] as the quasi-compact T_0-spaces X such that X has a quasi-compact open basis closed under finite intersection and each irreducible closed subspace of X has a generic point. By Proposition 4.3 and the remarks in the first paragraph of section 2, $X(R)$ satisfies these conditions of Hochster and, accordingly, is a spectral space. Moreover, to see that f_R is a spectral map, it is enough to recall from the proof of Lemma 2.1 that $f^{-1}(X_{r_1} \cup ... \cup X_{r_n}) = \bigcup E(r_i^{-1})$ is a quasi-compact open, for each finite subset $\{r_1, ..., r_n\}$ of $R \setminus \{0\}$. The proof is complete.

REMARK 4.4. – (*a*) Of course, each saturated subspace of $X(R)$ is closed. In view of Lemma 4.2(*a*), it is therefore interesting to note that a saturated subspace need not be irreducible. To see this, let $\{V_1, ..., V_n\}$ be a finite collection of $n \geq 2$ pairwise incomparable valuation overrings of R. It is well-known that if $W \in X(R)$ satisfies $W \subset V_1 \cup ... \cup V_n$, then $W \subset V_i$ for some index i. (The point is that $M_W \supset \bigcap M_{V_i}$.) Consequently, if we put $Y = \bigcup \overline{\{V_i\}}$, it follows that $\bigwedge (G(Y)) = \{W \in X(R): W \subset \bigcup V_i\} = \bigcup \{W \in X(R): W \subset V_i\} = Y$; that is, Y is saturated. It is evident that Y is not irreducible.

(*b*) We next record a point of contact with the condition mentioned in Remark 2.8(*a*), (*b*). Namely, if each valuation overring of R is a finite-type R-algebra, then each closed subspace of $X(R)$ is saturated. To see this, it is enough to show, for any (possibly infinite) subset $\{V_\alpha\}$ of $X(R)$, that $Y = \bigcup \overline{\{V_\alpha\}}$ is saturated. To this end, consider any $W \in \bigwedge (G(Y))$. By hypothesis, $W = R[x_1, ..., x_n]$ for some finite subset $\{x_1, ..., x_n\}$ of K. For each i, $1 \leq i \leq n$, choose an index α_i so that $x_i \in V_{\alpha_i}$: this is possible since $W \subset \bigcup V_\alpha$. As $W \subset V_{\alpha_1} \cup ... \cup V_{\alpha_n}$, the result recalled in (*a*) supplies j, $1 \leq j \leq n$, such that $W \subset V_{\alpha_j}$. Then $W \in \overline{\{V_{\alpha_j}\}} \subset Y$, so that Y is indeed saturated, as desired.

(*c*) In view of Theorem 4.1, [H, Proposition 10] assures that $X(R)$ is (homeomorphic to) an inverse limit of finite T_0-spaces. This is striking since $X(R) \setminus \{K\}$ is the inverse limit of the complete models [ZS, Theorem 41, page 122].

(*d*) Here is an application of the full force of Theorem 4.1: by invoking [H, Proposition 15], we recover the implication (ii) $\Rightarrow$ (i) in Proposition 2.2.

Finally, we make $X(-)$ a functor and thereby obtain a categorical formulation of Theorem 4.1.

COROLLARY 4.5. – Let $\boldsymbol{D}$ be the category whose objects form the class of all integral domains and whose morphisms are the inclusion maps. Let $\boldsymbol{Z}$ be the category of all abstract Riemann surfaces of integral domains, viewed as a full subcategory of the category $\boldsymbol{S}$ of spectral spaces and spectral maps. Then:

(*a*) The object assignment $R \mapsto X(R)$ extends to a contravariant functor $X: \boldsymbol{D} \to \boldsymbol{Z}$.

(*b*) Let $I: \boldsymbol{Z} \to \boldsymbol{S}$ be the inclusion functor. Then $\{f_R: R \in \operatorname{Ob}(\boldsymbol{D})\}$ gives a natural transformation from IX to Spec, viewed as contravariant functors $\boldsymbol{D} \to \boldsymbol{S}$.

PROOF. – (*a*) Consider integral domains $R \subset T$ (where, as usual, K denotes the quotient field of R). If $V \in X(T)$, it is well-known that $V \cap K \in X(R)$. (Cf. [G_2, Theorem 19.16(*a*)]. Note that the corresponding assertion fails if one excludes K by definition from membership in $X(R)$, since easy examples exist with $K \subset V \neq$ quotient field of T.) Thus, if $i: R \to T$ is the inclusion map, we may define a function $X(i): X(T) \to X(R)$ by $V \mapsto V \cap K$. It is evident that $X(i)$ is continuous since, with self-explanatory notation, we have $X(i)^{-1}(E_R(x_1, \ldots, x_n)) = E_T(x_1, \ldots, x_n)$. As a quasi-compact open subset of an abstract Riemann surface is just a union of finitely many basic open sets, this equation also shows that $X(i)$ is a spectral map. Now (*a*) follows easily.

(*b*) We must show, in the above notation, that

$$\begin{array}{ccc} IX(T) & \xrightarrow{f_T} & \operatorname{Spec}(T) \\ {\scriptstyle IX(i)}\downarrow & & \downarrow{\scriptstyle \operatorname{Spec}(i)} \\ IX(R) & \xrightarrow[f_R]{} & \operatorname{Spec}(R) \end{array}$$

is a commutative diagram. Observe first that if $(V, N) \in X(T)$ then $N \cap K$ is the maximal ideal of $V \cap K$. Thus $f_R(IX(i))$ sends V to $(N \cap K) \cap R = N \cap R$. As $(\operatorname{Spec}(i)) f_T$ sends V to $(N \cap T) \cap R = N \cap R$, the proof is complete.

We close with a categorical remark: Spec is not invertible on the category of abstract Riemann surfaces. This means (cf. [H, pages 43-44]), in the above notation, that there is no contravariant functor F from $\boldsymbol{Z}$ to the category of commutative rings such that I is naturally equivalent to $(\operatorname{Spec})F$. For a proof, apply the criterion in [*H*, Proposition 3(*a*)] to $\boldsymbol{Z}$: it is enough to choose R as any integral domain other than a field and to observe that K, the image of $X(K) \to X(R)$, is a non-closed point.

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