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ELLIPTICAL CONTACT AREA BETWEEN ELASTIC BODIES BOUNDED BY HIGH ORDER SURFACES

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ABSTRACT

Hertz theory fails when contacting surfaces are expressed by a sum of even polynomials of higher powers than two. An alternative analytical solution implies the knowledge of contact area. In the case of elliptical domains, there are some published proposals for the correlation between pressure distribution and surface normal displacement. This paper identifies the class of high order surfaces which lead to elliptical contact domains and solves a contact between fourth order surfaces.

INTRODUCTION

One of Hertz theory assumptions accepts that bounding surfaces are approximated by elliptical paraboloids around initial contact point. This hypothesis leads to an elliptical contact domain. The surfaces described by even monomials of higher power than two violate this assumption and Hertz theory fails. Therefore, classical theory of elastic contacts must be improved. The prior guess of contact area shape is a major difficulty when finding an analytical solution. Nevertheless, Shtaerman [1], Galin [2] and Gladwell [3] advanced some elements of solution for elliptical contact domains. They found various correlations between a certain pressure applied over an elliptic domain sited in the bounding plane of an elastic half-space and normal displacement generated in the points of this domain. Shtaerman's correlation seems to be incorrect and Galin's theorem has a qualitative feature only. Gladwell's results are based on Legendre function which is difficult to apply. By using the results of Shtaerman and Galin, this author advanced previously an explicit correlation between generalized Hertz pressure and resulting polynomial surface displacement inside an elliptical domain [4]. This is used now to identify the class of punch high order surfaces which lead to elliptical contact domains, to validate the result by using Hertz theory and to analyze the contact between fourth order surfaces.

CONTACT OF HOMOGENEOUS SURFACES

A pressure expressed as a product between typical Hertz square root and an even polynomial, applied over an ellipse of half-axes a and b, is called generalized Hertz pressure. This is given by following equation:

$$p(x,y) = p_0 \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}} \sum_{i=0}^{n-1} c_i \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^i , \tag{1}$$

where p_0 is central pressure and $c_0=1$. This pressure generates following normal displacement in the points of contact domain:

$$w(x,y) = \frac{1}{\pi}\eta\, p_0 b \sum_{i=0}^{n} \left[c_i \sum_{k=0}^{i} (-1)^k C_i^k \, I_k \right], \tag{2}$$

where I_k represents following definite integral [4]:

$$I_k = \frac{\pi(2k+1)!!}{2^{k+1}(k+1)!} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\left[1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{\left(\beta\frac{x}{a}\sin\gamma+\frac{y}{b}\cos\gamma\right)^2}{1-e^2\sin^2\gamma}\right]^{k+1}}{\sqrt{1-e^2\sin^2\gamma}} d\gamma \tag{3}$$

β=b/a being the aspect ratio of the ellipse and *e* its eccentricity.

The punch surface which produces the generalized Hertz pressure is found from classical interference condition:

$$z(x,y) = \delta - \frac{1}{\pi}\eta\, p_0 b \sum_{i=0}^{n} \left[c_i \sum_{k=0}^{i} (-1)^k C_i^k \, I_k \right], \tag{4}$$

in which δ represents central penetration of half-space.

Equations (1) to (4) define an explicit, bi-univocal correspondence between generalized Hertz pressure distribution and punch surface. If pressure distribution is the product between typical Hertz square root and an even order polynomial of order 2n-2 with unitary free term, the punch surface is expressed by an even order polynomial of degree 2n with respect to co-ordinates x and y, with no free term.

Supplementary, force balance equation holds:

$$\iint_A p(x,y)\,dx\,dy = Q , \tag{5}$$

where Q is normal load.

A contact problem on elliptical area possesses n+1 unknowns: central pressure p_0, normal approach δ, ellipse half-axes and coefficients of pressure distribution. Equation (5) yields p_0 and remaining unknowns satisfy Eq.(4). As z(0,0)=0, δ results imposing the free term in right side member of Eq.(4) be zero. There are n-1 unknown coefficients in pressure equation, including those equal to zero, and n+1 unknowns in the problem. Surface equation has $s=0.5(n^2+3n)$ coefficients. From these, $0.5(n^2+n-2)$ are zero and n+1 nonzero. Only n+1 surface coefficients are useful to solve the problem (2 nonzero and n-1 equal to zero), whereas $0.5(n^2-n)$ zero coefficients are redundant. The problem is determined and the domain is elliptical only if n=1, which represents a Hertz contact. Usually, the problem is undetermined and contact domain is non-elliptical. To force contact area be elliptical, surface polynomial must contain only n+1 independent coefficients, from which only two are different of zero. The remaining n-1 nonzero coefficients must comply with right side member of Eq.(4). Only homogeneous polynomial surfaces satisfying this restriction yield an elliptical contact area.

HERTZ POINT CONTACT

In Hertz point contacts n=1 and punch equation is:

$$z(x,y) = A\,x^2 + B\,y^2\,. \tag{6}$$

According to Eq.(1), the pressure takes the known Hertz form:

$$p(x,y) = p_0\,\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\quad. \tag{7}$$

After some calculations, Eq.(4) takes following form:

$$\delta - Ax^2 - By^2 = \eta\,p_0 b\left[K-\frac{x^2}{a^2}D-\frac{y^2}{b^2}(K-D)\right], \tag{8}$$

K and E being complete elliptic integrals of eccentricity e, and $D=\left[K-E\right]/e^2$. The identification of coefficients in Eq.(8) and force balance Eq.(5) yield all known Hertz formulae.

HOMOGENEOUS FOURTH ORDER SURFACES

Following forth order surface equation:

$$z(x,y) = A\,x^4 + B\,y^4 + C\,x^2y^2 + D\,x^2 + E\,y^2 \tag{9}$$

is homogeneous if D=E=0. As n=2 s=5 contact area is elliptical if Eq.(9) contains only 2 independent nonzero coefficients, let them be A and B. The coefficient C cannot be imposed priory, its value resulting from Eq.(4) by coefficients identification. After performing implied calculations, pressure distribution is:

$$p(x,y) = p_0\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\left[1+2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\right], \tag{10}$$

whereas contact elements are:

$$a = n_a\sqrt[5]{\frac{5\,\eta\,Q}{12\,A}}\,,\; n_a = \sqrt[5]{\frac{1}{\pi\,e^4}\left[\left(2+e^2\right)K-2\left(1+e^2\right)E\right]}\,; \tag{11}$$

$$b = n_b\sqrt[5]{\frac{5\,\eta\,Q}{12\,B}}\,,\; n_b = \sqrt[5]{\frac{\beta}{\pi e^4}\left[\left(2-3e^2\right)\beta^2K-2\left(1-2e^2\right)E\right]}\,; \tag{12}$$

$$p_0 = \frac{n_p}{\pi}\sqrt[5]{\frac{125}{54}\frac{A\,B\,Q^3}{\eta^2}}\,,\; n_p = \frac{1}{n_a n_b}\,; \tag{13}$$

$$\delta = \frac{n_\delta}{2}\sqrt[5]{\frac{625}{648}A\,\eta^4Q^4}\,,\; n_\delta = 3\frac{K}{\pi\,n_a}\,. \tag{14}$$

The eccentricity of contact ellipse yields from equation:

$$\frac{A}{B} = \left(1-e^2\right)^2\frac{(2+e^2)K(e)-2(1+e^2)E(e)}{(3e^4-5e^2+2)K(e)-2(1-2e^2)E(e)}\,. \tag{15}$$

Coefficient C of surface equation must have following value:

$$C = \frac{3\,\eta\,p_0 b}{e^4a^2b^2}\left[(2-e^2)\,E-2(1-e^2)\,K\right]. \tag{16}$$

The pressure distribution given by Eq.(10) is shown in dimensionless coordinates in Fig.1, for an aspect ratio $\beta = 0.5$.

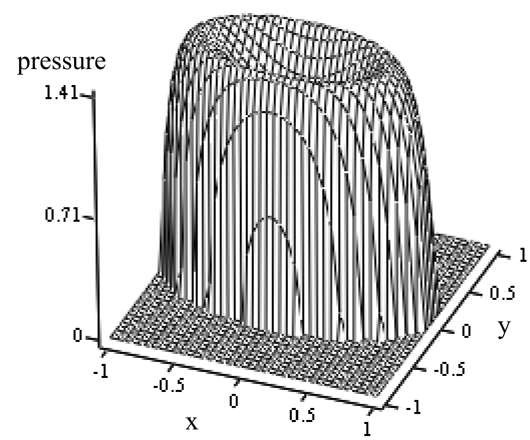

Figure 1. Pressure distribution between fourth order surfaces

CONCLUSIONS

A punch bounded by a surface described by a high order homogeneous equation, pressed against an elastic half-space, generates an elliptical contact domain if it contains only two independent non-zero coefficients. The remaining coefficients result from identification of coefficients in Eq. (4).

Hertz equations for point contacts yield as a particular case of advanced formulae, thus validating the theory advanced.

Homogeneous fourth order polynomial surfaces lead to an elliptical contact area if they possess two independent coefficients and the third is given by Eq.(16). Contact elements are found by aid of Eqs. (11) to (15).

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