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Resonance Response of Nonlinear Circular Plates Subjected to Uniform Static Load

The axisymmetric resonant frequency response of static pressure loaded, nonlinear clamped, circular plates have been investigated analytically and experimentally. In the analysis, the set of nonlinear partial differential equations of motion and compatibility are solved by applying Galerkin's method with Dini-Bessel and Fourier-Bessel series expansions for the assumed solution forms. These expansions satisfy the boundary conditions exactly. The equations are solved to yield a second approximation for the first mode response and a first approximation for the second mode response. Experiments were conducted on a number of circular plates. The analytical and experimental results are found to be in good agreement for plates with $\beta \leqslant 57.3$. The agreement was only fair for plates with $\beta > 57.3$.

Introduction

In this paper the axisymmetric nonlinear resonant frequency response of clamped, thin, circular plates subjected to uniform static pressure resulting in large deflections and small oscillations was considered, Fig. 1. The small oscillations were induced by harmonic vibration of the clamped boundary; the displacement of the boundary being identical with that of an electrodynamic shaker head.

Some resonant responses of clamped circular plates have been investigated previously. The linear solutions are well known and appear in textbooks by Timoshenko [1].[1] The dynamic post-buckling response of a plate has been treated by Herzog and Masur [2] among others. Yamaki [3] analytically determined the influence of large amplitudes on the first mode, zero static pressure response.

Reismann [4] and his associates are conducting related work at the State University at Buffalo.

A linear analysis does not yield a solution to the problem stated, and thus the nonlinear equations of motion and compatibility must be solved. An approximate method is used. A method that has been successfully applied to static [5] and

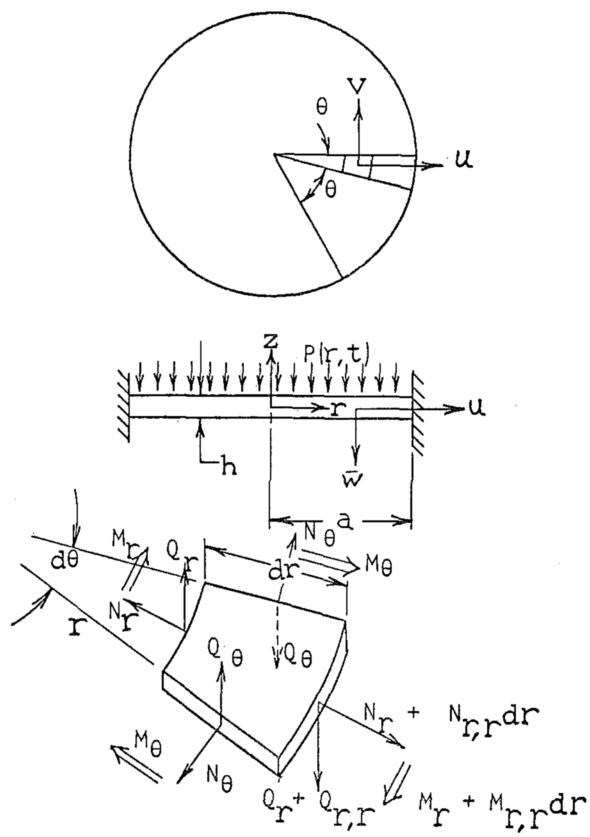

Fig. 1 Plate configuration and boundary conditions

[1] Numbers in brackets designate References at end of paper.

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dynamic [6, 7] problems in the stability of thin shallow spherical shells is that of expressing the unknown functions in terms of truncated Fourier and Dini-Bessel series expansions. In this paper, a similar method was applied to the clamped circular plate.

Experiments were conducted to test the validity of the theory, and to describe the experimentally observed phenomenon, if any, which may be absent from the behavior of the analytical model.

Analysis

Basic Equations and Boundary Conditions

The equations governing the nonlinear axisymmetric response of thin circular plates subjected to uniform pressure P_0^* which result in large deflections are

$$\nabla^4 F = \frac{-Eh\bar{w}_{,rr}}{r}\bar{w}_{,r} \tag{1a}$$

$$D\nabla^4\bar{w} - \frac{1}{r}(F_{,r}\bar{w}_{,r})_{,r} + \rho h\bar{w}_{,tt} - P_0^* = 0 \tag{1b}$$

where

$$\nabla^2 = \frac{1}{r}(r(\ )_{,r})_{,r}$$

N_r and N_θ are the membrane forces per unit length, and u the radial displacement of the plate at the middle surface; thus

$$N_r = \frac{1}{r}F_{,r} = \frac{Eh}{1-\nu^2}\left(u_{,r} + \frac{1}{2}\bar{w}_{,r} + \frac{\nu}{r}u\right) \tag{2a}$$

$$N_\theta = F_{,rr} = \frac{Eh}{1-\nu^2}\left(\frac{u}{r} + \nu u_{,r} + \frac{\nu}{2}\bar{w}_{,r}^2\right) \tag{2b}$$

The clamped boundary conditions are, Fig. 1:

$$\bar{w} = 0 \quad \text{for} \quad r = a \tag{3a}$$

$$\bar{w}_{,r} = 0 \quad \text{for} \quad r = a \tag{3b}$$

$$\bar{w}_{,r} = 0 \quad \text{for} \quad r = 0 \tag{3c}$$

$$u = 0 \quad \text{for} \quad r = a \tag{3d}$$

Boundary condition (3d) with equations (2a) and (2b) yields a boundary condition in terms of the stress function, $F(r, t)$

$$F_{,rr} - \frac{\nu}{a}F_{,r} = 0, \qquad r = a \tag{4}$$

Introducing the following dimensionless quantities:

$$x = \frac{r}{a};\quad \tau = \omega t;\quad w = \frac{\bar{w}}{h};\quad f = \frac{F}{Eh^3}$$

$$C = \frac{1}{12(1-\nu^2)};\quad \lambda^2 = \frac{\rho a^4\omega^2}{Eh^2};\quad P_0 = \frac{P_0^* a^4}{Eh^4}$$

The differential equations (1a) and (1b) become

$$x\bar{\nabla}^4 f = -w_{,x}w_{,xx} \tag{5a}$$

$$Cx\bar{\nabla}^4 w - (f_{,x}w_{,x})_{,x} + \lambda^2 x w_{,\tau\tau} - P_0 x = 0 \tag{5b}$$

where

$$\bar{\nabla}^2 = \frac{1}{x}(x(\ )_{,x})_{,x}$$

with the boundary conditions (3a)–(3c) and (4) rewritten as follows:

$$w = 0 \quad \text{for} \quad x = 1 \tag{6a}$$

$$w_{,x} = 0 \quad \text{for} \quad x = 1 \tag{6b}$$

$$w_{,x} = 0 \quad \text{for} \quad x = 0 \tag{6c}$$

$$f_{,xx} - \nu f_{,x} = 0 \quad \text{for} \quad x = 1 \tag{6d}$$

Integrating (5a) and (5b) once with respect to x, and using boundary condition (6c), we obtain

$$x(\bar{\nabla}^2 f)_{,x} + \tfrac{1}{2}(w_{,x})^2 = 0 \tag{7a}$$

$$cx(\bar{\nabla}^2 w)_{,x} - \frac{x^2}{2}P_0 - f_{,x}w_{,x} + \lambda^2\int_0^x s w_{,\tau\tau}ds = 0 \tag{7b}$$

Nomenclature

P_0 = dimensionless static pressure, $P_0 = \dfrac{P_0^* a^4}{Eh^4}$

N_r, N_θ = in-plane forces

$\bar{w}(r, t)$ = transverse midsurface displacement

$u(r, t)$ = in-plane midsurface displacement

$F(r, t)$ = stress function

$w(x, \tau)$ = dimensionless transverse displacement: $w(x, \tau) = \dfrac{\bar{w}(r, t)}{h}$

$f(x, \tau)$ = dimensionless stress function: $f(x, \tau) = \dfrac{F(r, t)}{Eh^3}$

$a_m(\tau), b_n(\tau)$ = coefficients of expansion of $w_{,x}(x, \tau)$ and $f_{,x}(x, \tau)$

f_{2E}, f_{22t} = second mode axisymmetric natural frequencies, experimental, two-term theoretical, respectively

$\lambda_{2E}\lambda_{22t}$ = second-mode dimensionless axisymmetric natural frequencies, experimental, two-term theoretical, respectively

A_{m0}, A_{m1} = coefficients of time expansions of $a_m(\tau)$

β = dimensionless geometry parameter $\beta = a/h$

x = dimensionless radius: $x = r/a$

τ = dimensionless time: $\tau = \omega t$

α_m = zeros of Bessel function $J_1(\alpha_m) = 0$

γ_n = zeros of $\gamma_n J_{1,x}(\gamma_n) = \nu J_1(\gamma_n)$

f_{1E} = first-mode, experimental, axisymmetric, natural frequency

f_{11t} = first-mode, one-term theoretical, axisymmetric, natural frequency

f_{12t} = first-mode, two-term theoretical, axisymmetric, natural frequency

λ = dimensionless natural frequency, $\lambda^2 = \dfrac{\rho a^4\omega^2}{386Eh^2}$

λ_{1E} = first-mode, experimental, dimensionless natural frequency

λ_{11t} = first-mode, one-term, theoretical, dimensionless natural frequency

λ_{12t} = first-mode, two-term, theoretical, dimensionless natural frequency

$(\ )_{,x}, (\ )_{,r}$ = partial derivatives with respect to radius

$(\ )_{,tt}, (\ )_{,}$ = partial derivatives with respect to time

$G_{ni}(A_{m0}, A_{m1}, P_0, P_1)$ = algebraic forms in resonant responses

C = $1/12(1-\nu^2)$

Solution

Assume solutions for equations (7a) and (7b) of the form:

$$w_{,x}(x,\tau) = \sum_{m=1}^{\infty} a_m(\tau)J_1(\alpha_m x) \quad (8a)$$

$$f_{,x}(x,\tau) = \sum_{n=1}^{\infty} b_n(\tau)J_1(\gamma_n x) \quad (8b)$$

where J_1 is Bessel function and α_m are zeros of Bessel functions, i.e.,

$$J_1(\alpha_m) = 0 \quad (9a)$$

and γ_n are zeros of the following expression,

$$\gamma_n J_{1,x}(\gamma_n) - \nu J_1(\gamma_n) = 0 \quad (9b)$$

Equation (8a) defines a Fourier-Bessel series [5] on a segment [0, 1], where

$$a_m(\tau) = \frac{2\int_0^1 sJ_1(\alpha_m s)w_{,s}(s,\tau)ds}{J_2^2(\alpha_m)}$$

Equation (8b) defines a Dini-Bessel series [5] on a segment [0, 1], where

$$b_n(\tau) = \frac{2\gamma_n^2\int_0^1 sJ_1(\gamma_n s)f_{,s}(s,\tau)ds}{(\gamma_n^2 - 1)J_1^2(\gamma_n) + \gamma_n^2 J_{1,x}^2(\gamma_n)}$$

In this paper, M-term approximations are used in the Fourier-Bessel (8a) and Dini-Bessel (8b) series solutions. Thus the unknown functions are represented as

$$w_{,x}(x,\tau) = \sum_{m=1}^{M} a_m(\tau)J_1(\alpha_m x) \quad (10a)$$

$$f_{,x}(x,\tau) = \sum_{n=1}^{M} b_n(\tau)J_1(\gamma_n x) \quad (10b)$$

Using known differential relationships of Bessel-functions [5] and substituting (10a) and (10b) into (7a) and (7b) we get

$$\sum_{n=1}^{M} b_n(\tau)\gamma_n^2 xJ_1(\gamma_n x) - \frac{1}{2}\sum_{i=1}^{M}\sum_{q=1}^{M} a_i(\tau)a_q(\tau) \times J_1(\alpha_q x)J_1(\alpha_i x) = 0 \quad (11a)$$

$$C\sum_{m=1}^{M} a_m(\tau)\alpha_m^2 xJ_1(\alpha_m x) + \frac{x^2}{2}P_0 + \sum_{n=1}^{M}\sum_{m=1}^{M} a_m(\tau)b_n(\tau)J_1(\alpha_m x)J_1(\gamma_n x) - \frac{\lambda^2}{2}\sum_{m=1}^{M}\frac{a_m(\tau)_{,\tau\tau}}{\alpha_m}x^2J_0(\alpha_m) + \lambda^2\sum_{m=1}^{M}\frac{a_m(\tau)_{,\tau\tau}}{\alpha_m^2}xJ_1(\alpha_m x) = 0 \quad (11b)$$

Applying Galerkin's method, that is, multiplying equation (11a) by $J_1(\gamma_j x)dx$, $j = 1, 2; .. M$; equation (11b) by $J_1(\alpha_k x)dx$, $k = 1, 2, \ldots M$; and integrating from 0 to 1; we obtain nonlinear temporal equations of the form

$$\sum_{n=j=1}^{M} b_j(\tau)H_4'' \frac{1}{2}\sum_{i=1}^{M}\sum_{q=1}^{M}\sum_{n=j=1}^{M} a_i(\tau)a_q(\tau)H_5 \quad (12a)$$

$$\left(C\sum_{m=1}^{M} a_m(\tau)\alpha_m^2 + \lambda^2\sum_{n=1}^{M}\frac{a_m(\tau)_{,\tau\tau}}{\alpha_m^2}\right)H_1 + \sum_{m=1}^{M}\sum_{n=1}^{M} a_m(\tau)b_n(\tau)H_2 = \left(\sum_{m=1}^{M}\frac{\lambda^2J_0(\alpha_m)}{2\alpha_m}a_m(\tau)_{,\tau\tau} - \frac{P_0}{2}\right)H_3 \quad (12b)$$

where the coefficients $(H_1 \ldots H_5)$ are defined as integrals of Bessel functions.

Substituting (12a) in (12b) we have second-order, nonlinear, temporal differential equations of the form

$$C\sum_{m=1}^{M} a_m(\tau)\alpha_m^2 H_1 + \lambda^2\sum_{m=1}^{M}\frac{a_m(\tau)_{,\tau\tau}}{\alpha_m^2}H_1 + \frac{1}{2}\sum_{m=1}^{M}\sum_{i=1}^{M}\sum_{q=1}^{M}\sum_{n=j=1}^{M} a_m(\tau)a_i(\tau)a_q(\tau)\frac{H_2H_5}{H_4} + \frac{P_0}{2}H_3 = \frac{\lambda^2}{2}\sum_{m=1}^{M}\frac{a_m(\tau)_{,\tau\tau}}{\alpha_m}J_0(\alpha_m)H_3 = 0 \quad (13)$$

Since we are concerned with a static deformation and periodic external excitation, the approximate steady-state responses $a_m(\tau)$ are assumed to be of the form $a_m(\tau) = A_{m0} + A_{m1}\cos\tau$, where A_{m0} is the static response and $A_{m1}\cos\tau$ is the dynamic response. A_{m1} is considered small especially in comparison with the static term A_{m0}, and thus the higher-order terms are omitted.

Substituting $a_m(\tau)$ into equation (13) and applying the principle of harmonic balance we have

$$G_{n0}(A_{m0}, 0, P_0) = 0 \quad (14a)$$

$$G_{n1}(A_{m0}, A_{m1}, 0) = 0 \quad (14b)$$

where

$$G_{n0}(A_{m0}, 0, P_0) = C\sum_{m=1}^{M}\alpha_m^2 H_1A_{m0} + \frac{1}{2}\sum_{i=1}^{M}\sum_{m=1}^{M}\sum_{q=1}^{M}\sum_{j=n=1}^{M} A_{i0}A_{m0}A_{q0}\frac{H_2H_5}{H_4} + \frac{P_0}{2}H_3 = 0$$

$$G_{n1}(A_{m0}, A_{m1}, 0) = C\sum_{m=1}^{M}\alpha_m^2H_1A_{m1} - \lambda^2\sum_{m=1}^{M}\frac{A_{m1}}{\alpha_m^2}H_1 + \frac{1}{2}\sum_{i=1}^{M}\sum_{m=1}^{M}\sum_{q=1}^{M}\sum_{j=n=1}^{M}[A_{q0}A_{i0}A_{m1} + A_{m0}A_{q0}A_{i1} + A_{m0}A_{i0}A_{q1}]\frac{H_2H_5}{H_4} + \sum_{m=1}^{M}\frac{\lambda^2J_0(\alpha_m)}{2\alpha_m}A_{m1}H_3 = 0$$

Equations (14a) and (14b) constitute the set of equations relating the static pressure to the natural frequency in the clamped circular plate.

In this paper, solutions were obtained for the one and two-term approximations of $w_{,x}$ and $f_{,x}$.

The one-term solution gives a first approximation to the first-mode resonant response, λ_{11}, while the two-term solution yields a second approximation to the first-mode resonant response, λ_{12},

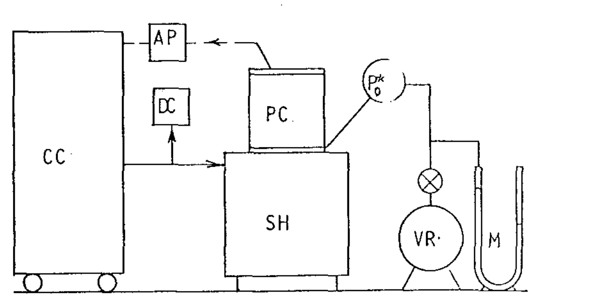

Fig. 2 Schematic of experimental apparatus and setup

and a first approximation on the second-mode resonant response, λ_{22}. Results are shown in Figs. 4–9.

The values of α_1 and α_2, γ_1 and γ_2 for $\nu = 1/3$ are found to be

$$\alpha_1 = 3.8317, \alpha_2 + 7.0156$$

$$\gamma_1 = 1.545, \quad \gamma_2 = 5.2665$$

Experimental Work

The experimental setup is shown in Figs. 2 and 3. The plate specimen (PL) positioned between two mounting rings was subjected to a uniform static pressure $P_0{}^*$ within the pressure chamber (PC). The pressure $P_0{}^*$ was controlled by a valve on the vacuum reservoir (VR) and monitored by a water manometer (M).

The plate was boundary driven, through the mounting rings and chamber walls, by the shaker head (SHH).

The digital counter (DC) monitored the oscillator frequency, which was the sinusoidal input signal to the shaker. Grains of fine sand were placed on the plate specimen (PL) to indicate the plate response and modal shapes.

The specimens were prepared from rigid polyvinyl chloride plastic sheets, with Poisson's ratio, $\nu = 1/3$ and a Young's modulus, E, ranging from 4.8 to 5.3 $\times$ 10^5 psi. *PVC* was chosen because of workability and a relatively high proportional limit. The vinyl sheets were of different ages and the laboratory temperatures varied slightly from day to day; consequently Young's modulus was determined for each specimen immediately after the experiment was completed. The modulus was found to be a function of age, temperature, and to a slight extent, frequency. A simple cantilever beam analysis [8] was performed on a beam segment of the actual specimen with the aid of a small shaker. The modulus variation with frequency is shown for a typical specimen in Fig. 10.[2] This small variation was ignored in the calculations; its inclusion would improve the agreement between experimental and theoretical resonant frequencies.

The boundary conditions were executed by a number of different methods. The initial method was that of clamping rings. The clamping rings were faced off on the lathe to insure parallel clamping surfaces. However it was noted that the clamping bolt torque influenced the plate response. The clamping rings caused the plate to deform slightly, due to the induced radial compressive stress. This resulted in a lower resonant frequency.

Another method attempted was that of casting the boundaries with a Hysol plastic. This method did not prove satisfactory because the plastic contracted as it cured, inducing a radial stress at the boundary and deforming the plate.

The method finally adopted was that of securing the plate to the upper clamping ring with epoxy to prevent plate slippage under pressure. Then, with the aid of a sealing silicon grease on the lower ring, the two rings were bolted together with a torque just sufficient to insure a seal. The resulting boundary, while not completely stress free, was superior to the other methods attempted.

The experimental procedure consisted of sweeping the frequency scale at each static pressure and noting the first and second mode axisymmetric resonant frequencies. The excitation amplitude was then decreased and each resonant frequency was determined at least three times. The repeatability of this average measuring method was found to be within 1 percent, after some patient practice. This repeatability indicated that material creep was not a significant factor in the experiment.

[2] In some materials, the modulus is highly dependent upon frequency. The modulus of acrylic plastic (Plexiglas), for instance, ranges from $E = 5 \times 10$ psi at 20 cps to $E = 7.5 \times 10$ psi at 1000 cps. This phenomenon was experimentally demonstrated in the summer of 1967 in the Applied Mechanics Laboratory at Syracuse University by Dr. C. Stevens of Ohio State University.

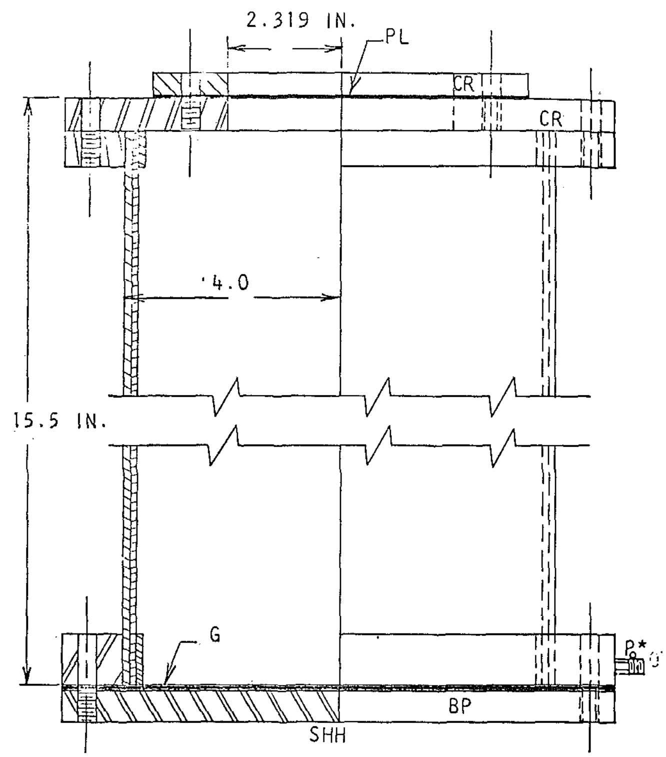

Fig. 3 Pressure chamber

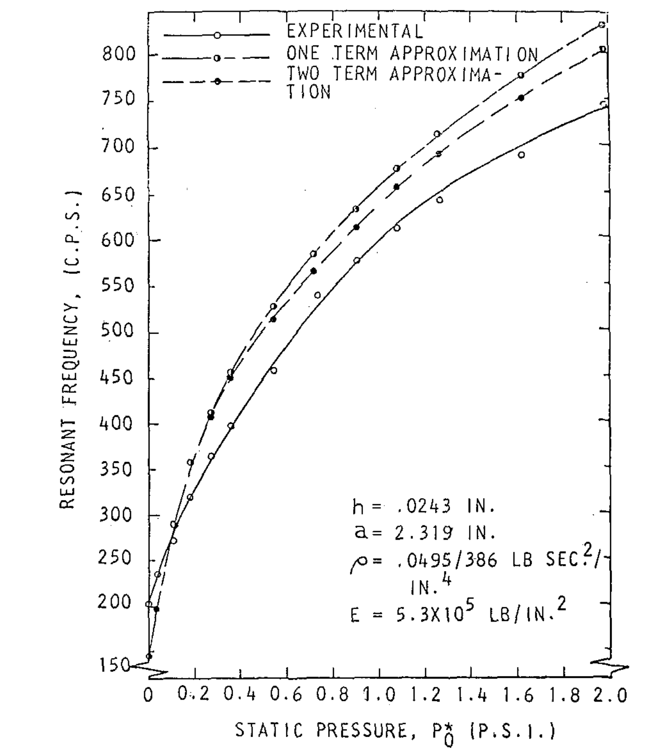

Fig. 4 Comparison of experimental and theoretical first-mode resonant frequencies

During the course of the experiments it became evident that an unpredictable interaction between the plate and the pressure chamber existed. The existence of this interaction was indicated by plate responses unrelated to plate geometric parameters. The pressure chamber was shown to be the cause of these interactions by holding other factors constant while changing the chamber's geometric configuration. The effects of this interaction were greatly reduced by making the pressure chamber sufficiently large.

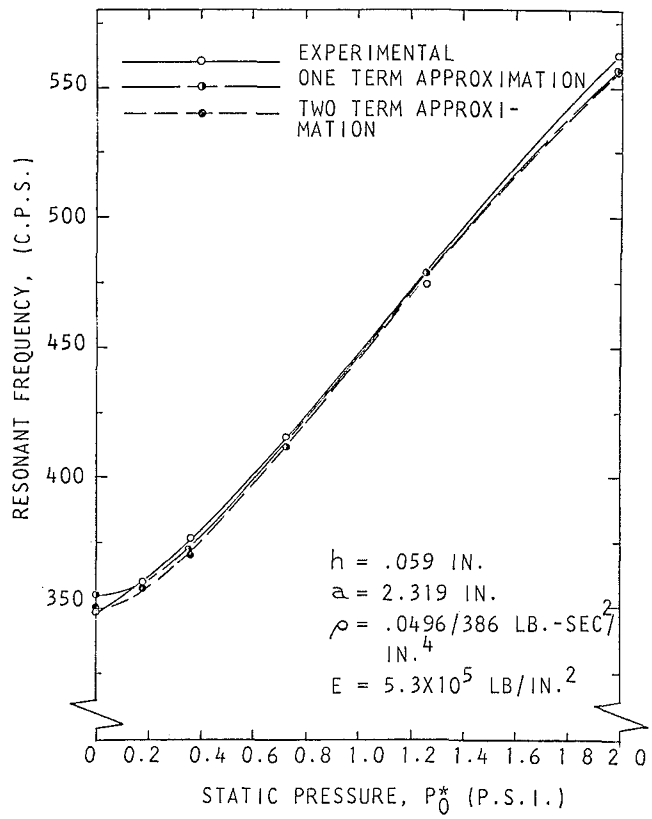

Fig. 5 Comparison of experimental and theoretical first-mode resonant frequencies

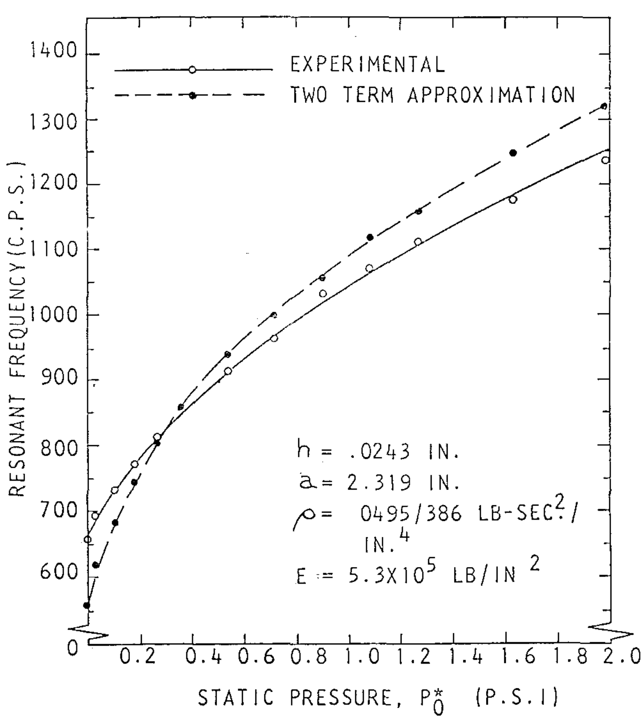

Fig. 6 Comparison of experimental and theoretical second-mode resonant frequencies

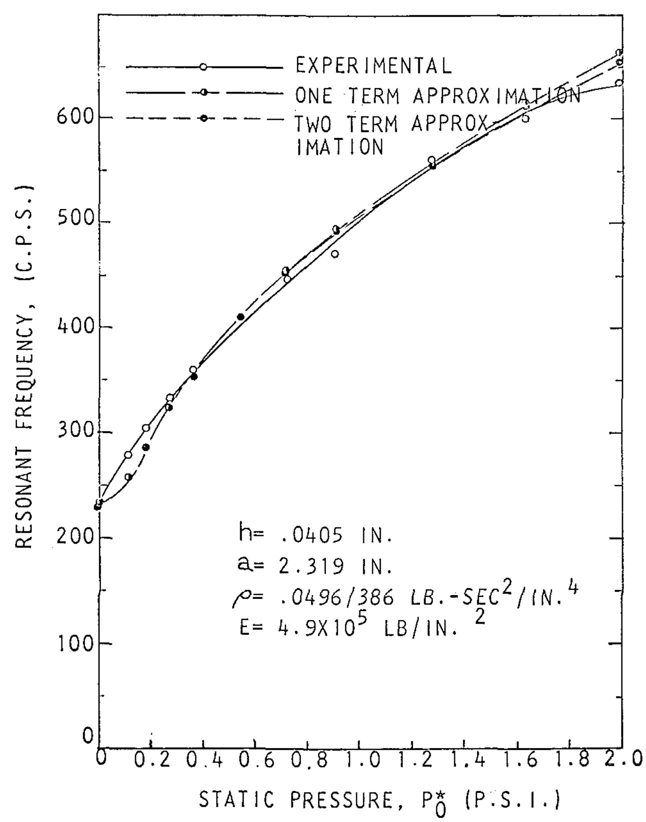

Fig. 7 Comparison of experimental and theoretical second-mode resonant frequencies

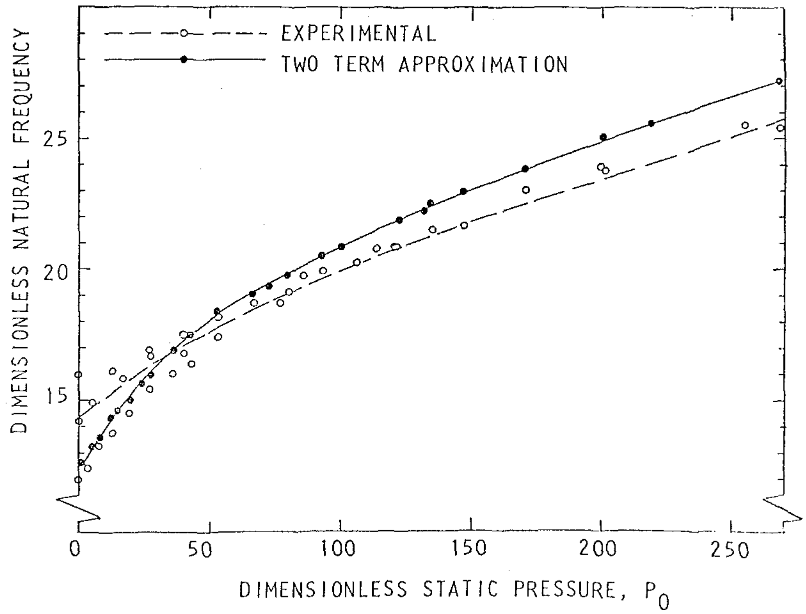

Fig. 8 Comparison of experimental and theoretical first-mode dimensionless resonant frequencies

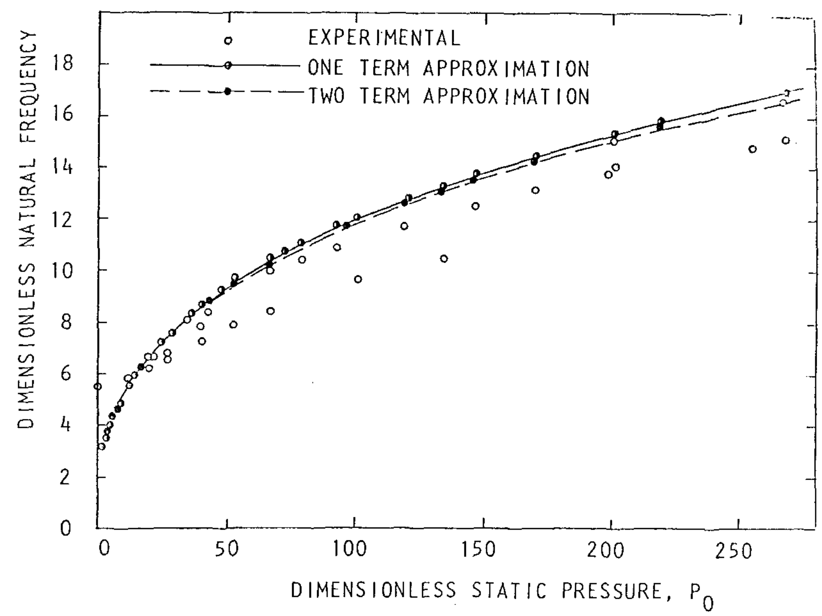

Fig. 9 Comparison of experimental and theoretical second-mode dimensionless resonant frequencies

Results

Theoretical Results

Representative analytical results for the effect of static pressure P_0^* on clamped, circular plate resonance are presented in Figs. 4–9. In Figs. 4 and 5, the one and two-term theoretical solution predictions for the first-mode axisymmetric-resonant frequencies, as a function of the static pressure P_0^*, are presented. Figs. 6 and 7 depict the effects of static pressure on the theoretical second spatial mode axisymmetric natural frequencies.

The first-mode dimensionless theoretical natural frequencies are presented as a function of the dimensionless pressure P_0 in Fig. 8. Fig. 9 depicts the theoretical dimensionless second-mode responses versus the dimensionless pressure P_0.

Experimental Results

Figs. 4 and 5 present the first-mode experimental resonant frequencies as a function of the static pressure, P_0^*. Figs. 6 and 7 depict the effects of static pressure on the experimental second-mode natural frequencies.

The first-mode dimensionless experimental natural frequencies are presented as a function of the dimensionless pressure P_0 in Fig. 8. Fig. 9 depicts the experimental dimensionless second-mode resonance versus the dimensionless static pressure P_0.

Conclusions

The following conclusions can be drawn from the analysis and the experiments:

1 Analytical method used yields results which are in good agreement with experimental results. The one and two-term solutions yield good approximate predictions of the first and second-mode resonant responses for $\beta \approx 120$. They yield excellent agreement for $\beta \leqslant 57.3$.

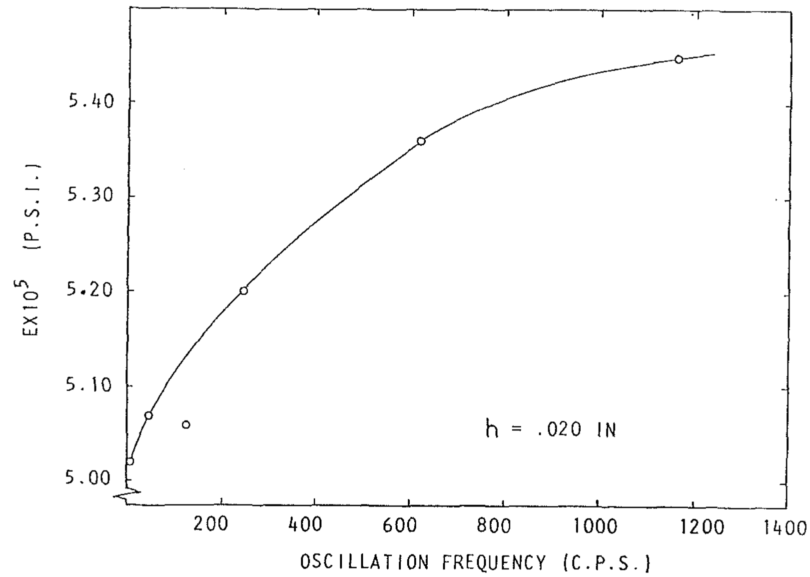

Fig. 10 Young's modulus variation of vinyl polyethylene with oscillation frequency

2 Zero static pressure (linear problem) of the approximate theoretical solutions compare well with exact solutions; see Timoshenko [1].

3 Experimental observations suggest that thin plates (large) β are extremely sensitive to initial imperfections.

4 The theoretical analysis is applicable for the higher modes. The inclusion of additional terms in an assumed solution forms should result in bettter agreement with the experimental data.

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